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Regular design equations for the discrete reduced-order Kalman filter

PETER HIPPE

In the presence of white Gaussian noises at the input and the output of a system Kalman filters provide a minimum-variance state estimate. When part of the measurements can be regarded as noise-free, the order of the filter is reduced. The filter design can be carried out both in the time domain and in the frequency domain. In the case of full-order filters all measurements are corrupted by noise and therefore the design equations are regular. In the presence of noise-free measurements, however, they are not regular so that standard software cannot readily be applied in a time-domain design. In the frequency domain the spectral factorization of the non-regular polynomial matrix equation causes no problems. However, the known proof of optimality of the factorization result requires a regular measurement covariance matrix. This paper presents regular (reduced-order) design equations for the reduced-order discrete-time Kalman filter in the time and in the frequency domains so that standard software is applicable. They also allow to formulate the conditions for the stability of the filter and to prove the optimality of the existing solutions.

Key words: optimal estimation, polynomials, multivariable systems, discrete-time systems

1. Introduction

In the absence of disturbances a state observer generates an estimate $\hat{x}$ asymptotically approaching the real state x of the system. Disturbances, however, cause persistent observation errors. In the presence of Gaussian white noise Kalman filters generate a state estimate $\hat{x}$ so that the observation error $\hat{x}-x$ has the smallest mean square in the stationary case [1, 12]. If all measurements are corrupted by noise, the order of the filter coincides with the order n of the system.

If κ of the measurements are not corrupted by noise, the order of the optimal filter is reduced to $n-\kappa$, provided that a certain covariance matrix is regular, which is related to the random signals disturbing an artificial output, namely the noisy measurements and the time derivatives of the undisturbed outputs. This problem was originally investigated in [3] and later applied to the discrete-time case (see [2, 14, 6, 16, 9]), and the references therein). The time-domain design of the reduced-order filter amounts to solving

The author is with Lehrstuhl für Regelungstechnik, Universität Erlangen-Nürnberg, Cauerstr. 7, D-91058 Erlangen, Germany. Fax: +49-9131-8528715, email: peter.hippe@rt.eei.uni-erlangen.de

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a discrete-time algebraic Riccati equation (DARE). Different from the continuous-time case, two different state estimates can be obtained from a discrete-time filter, namely the *a priori* state estimate $\hat{x}$ and the *a posteriori* state estimate $\hat{x}^{+}$. The former is obtained from the noisy measurements up to the time instant $k-1$, whereas the latter results from also using the noisy measurements at time k in an optimal way.

In the frequency domain the reduced-order Kalman filter is parameterized by the polynomial matrices $\tilde{\bar{D}}(z)$ and $\tilde{\bar{D}}^{+}(z)$. The optimal matrix $\tilde{\bar{D}}(z)$ can be obtained from a polynomial matrix equation by spectral factorization, and the optimal $\tilde{\bar{D}}^{+}(z)$ can subsequently be computed from these results. To obtain an equivalent frequency-domain version of the optimal filter, a special form of the DARE is required which was, *e.g.*, introduced in [17] and [4]. This DARE is formulated for a $n \times n$ covariance matrix $\bar{P}$ which is singular in the case of noise-free measurements. So far, the conditions for obtaining a stable filter when using this DARE have not been presented. There exist solutions for a regular time-domain design of the reduced-order Kalman filter, but the corresponding DAREs cannot be used to derive an equivalent frequency-domain solution.

The DARE required for a frequency-domain formulation of the filter is not in a standard form to be solved for $\bar{P}$. By an appropriate reformulation one can obtain an equation which is solvable by standard software. In the continuous-time case, this form of the Riccati equation still causes numerical problems since the Hamiltonian related to this system has zeros at $s = 0$. Here in the discrete-time case, the Hamiltonian also has eigenvalues at $z = 0$, but since this is inside the stability region, it causes no problems. Nevertheless, also a regular reduced-order system description will be derived in the discrete-time case, because it allows to obtain a regular frequency-domain design of the reduced-order filter and yields the conditions which guarantee a stable filter.

After a formulation of the underlying problem in the time domain in Section 2 the existing solution for the optimal filter is presented. Deriving a Riccati equation for a modified filtering problem, which only yields the *a priori* state estimate, one obtains a form a the DARE which can be solved for the rank deficient $\bar{P}$. After an adequate transformation of the state equations of the system one can subdivide this DARE into a regular part and a vanishing part. The regular part characterizes a full-order filter problem for a reduced-order system of the order $n-\kappa$ which can be solved by standard software. This full-order filtering problem also allows to derive conditions for the optimal reduced-order filter to be stable, and it is shown how these conditions translate into conditions on the original system.

The known polynomial matrix equation for the design of the reduced-order Kalman filter in the frequency domain is based on the left MFD of the full-order system whereas the polynomial matrix $\tilde{\bar{D}}(z)$, resulting from the spectral factorization of this polynomial matrix characterizes a system of reduced order. This is a consequence of the rank deficient measurement covariance matrix multiplying the denominator matrix of the system. Unfortunately, proofs of optimality of the spectral factor are only known in the case, where the measurement covariance is not singular. In [9] it has been observed that, on the one hand, optimality of the result can only be checked by computing the corresponding time-domain results and, on the other hand, that all examples investigated so far have shown that the resulting $\tilde{\bar{D}}(z)$ is indeed optimal.

In Section 3 it is demonstrated, that the polynomial matrix $\tilde{\bar{D}}(z)$ resulting from the non-regular polynomial matrix equation can also be obtained from a regular polynomial matrix equation. This regular polynomial matrix equation is derived from the reduced regular DARE in the time domain. As an additional result, the conditions on the MFD of the system are presented which guarantee the stability of the reduced-order filter resulting from the frequency-domain design.

Concluding remarks are presented in Section 4.

2. The filter design in the time domain

Considered are linear time-invariant discrete-time systems of the order n, with p inputs u and q stochastic inputs w

$$x(k+1) = Ax(k) + Bu(k) + Gw(k) \tag{1}$$

and it is assumed that the pair (A,G) is controllable. Part of the m outputs y are ideal measurements, so that one has

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x(k) + \begin{bmatrix} v_1(k) \\ 0 \end{bmatrix} \tag{2}$$

where $y_2 \in \mathbb{R}^{\kappa}$, $0 < \kappa \leqslant m$, is the undisturbed measurement and $y_1 \in \mathbb{R}^{m-\kappa}$ is the disturbed measurement with $v_1 \in \mathbb{R}^{m-\kappa}$ the measurement noise. Defining

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = C \tag{3}$$

it is assumed that the pair (A,C) is observable. The stochastic inputs $\{w(k)\}$ and $\{v_1(k)\}$ are independent, zero-mean, stationary Gaussian white noises with

$$\mathrm{E}\{w(k)w^T(\ell)\} = \bar{Q}\delta_{k\ell} \tag{4}$$

$$\mathrm{E}\{v_1(k)v_1^T(\ell)\} = \bar{R}_1\delta_{k\ell} \tag{5}$$

where

$$\delta_{k\ell} = \begin{cases} 1, & k=\ell, \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

It is assumed that the covariance matrices $\bar{Q}$ and $\bar{R}_1$ are real and symmetric, where $\bar{Q}$ is positive-semidefinite and $\bar{R}_1$ is positive-definite. The initial state $x(0) = x_0$ is not correlated with the disturbances, *i.e.*, $\mathrm{E}\{x_0 w^T(k)\} = 0$ and $\mathrm{E}\{x_0 v_1^T(k)\} = 0$ for all $k \geqslant 0$.

It is assumed that the covariance matrix

$$\Phi = C_2 G \bar{Q} G^T C_2^T \tag{7}$$

is positive definite. It characterizes the influence of the input noise on $y_2(k+1) = C_2 x(k+1)$. Thus the covariance matrix

$$\bar{R} = \begin{bmatrix} \bar{R}_1 & 0 \\ 0 & \Phi \end{bmatrix} \tag{8}$$

with respect to $y_1(k)$ and $y_2(k+1)$ is positive definite. This is a standing assumption in the design of reduced-order Kalman filters (see, *e.g.*, [1, 10, 13]), because it assures a filter order $n-\kappa$, which is required to obtain a minimum-variance state estimate $\hat{x}$.

Consider the $n-\kappa$ linear combinations

$$\zeta(k) = Tx(k) \tag{9}$$

and the κ ideal measurements y_2 which can be used to represent the state x of the system as

$$x(k) = \begin{bmatrix} C_2 \\ T \end{bmatrix}^{-1} \begin{bmatrix} y_2(k) \\ \zeta(k) \end{bmatrix} = \Psi_2 y_2(k) + \Theta\zeta(k) \tag{10}$$

This shows that the relations

$$\Psi_2 C_2 + \Theta T = I_n \tag{11}$$

and

$$C_2\Psi_2 = I_\kappa, \;\; C_2\Theta = 0, \;\; T\Theta = I_{n-\kappa}, \;\; T\Psi_2 = 0 \tag{12}$$

are satisfied. Then the reduced-order Kalman filter for such systems is described by

$$\hat{\zeta}(k+1) = T(A - L_1^+ C_1)\Theta\hat{\zeta}(k) + \left[TL_1^+ \;\; T(A - L_1^+ C_1)\Psi_2\right] \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} + TBu(k) \tag{13}$$

(see [4, 9]). The optimal estimate $\hat{\zeta}$ results if the matrices L_1^+ and Ψ_2 are chosen such that

$$L_1^+ = A\bar{P}C_1^T \hat{R}^{-1} \tag{14}$$

and

$$\Psi_2 = \widetilde{P} C_2^T X^{-1} \tag{15}$$

In (14) and (15) the abbreviations

$$\hat{R} = \bar{R}_1 + C_1 \bar{P} C_1^T \tag{16}$$

$$\widetilde{P} = A\bar{P}A^T + G\bar{Q}G^T - A\bar{P}C_1^T \hat{R}^{-1} C_1 \bar{P} A^T \tag{17}$$

and

$$X = C_2 \widetilde{P} C_2^T \tag{18}$$

have been used. By assumption, $\bar{R}_1$ is positive-definite and this implies that $\hat{R}$ in (16) is also positive-definite if $\bar{P}$ is positive-semidefinite. It can further be shown that if Φ in (7) is positive-definite and $\bar{P}$ is positive-semidefinite then X is also positive-definite (see Problem 1.2 in [1]). Consequently, the inverse matrices in (14) and (15) exist. The stationary value $\bar{P}$ of the error covariance

$$\bar{P}(k) = \mathrm{E}\left\{(x(k) - \hat{x}(k))(x(k) - \hat{x}(k))^T\right\} \tag{19}$$

is the positive-semidefinite matrix satisfying the DARE

$$\bar{P} = A\bar{P}A^T + G\bar{Q}G^T - \begin{bmatrix} L_1^+ & \Psi_2 \end{bmatrix} \begin{bmatrix} \hat{R} & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} (L_1^+)^T \\ \Psi_2^T \end{bmatrix} \tag{20}$$

(see [13, 10]). Since the measurements $y_2 = C_2 x$ are ideal, it is obvious that

$$C_2 \bar{P} = 0 \tag{21}$$

The version (20) of the DARE can be used as a starting point for deriving the equivalent frequency-domain solution of the filter (see Section 3). This DARE, however, is not in a standard form to be solved for $\bar{P}$.

The optimal minimum-variance *a priori* state estimate $\hat{x}$ can be obtained as

$$\hat{x}(k) = \Theta\hat{\zeta}(k) + \Psi_2 y_2(k) \tag{22}$$

This estimate uses the disturbed measurements y_1 up to the time $k-1$. Here in the discrete-time case an improved state estimate can be obtained when also using the actual measurement y_1 at time k in an optimal way. This yields the *a posteriori* state estimate

$$\hat{x}^+(k) = \hat{x}(k) + \Lambda(y_1(k) - C_1\hat{x}(k)) \tag{23}$$

where the optimal Λ has the form

$$\Lambda = \bar{P} C_1^T \hat{R}^{-1} \tag{24}$$

(see, *e.g.*, [9]), so that $L_1^+ = A\Lambda$. The *a priori* state estimate can also be obtained from an estimator (13) where L_1^+ is substituted by

$$L_1 = \Theta T L_1^+ = (I - \Psi_2 C_2) L_1^+ \tag{25}$$

and also the frequency-domain solution of the filter problem is based on an estimator representation with L_1^+ substituted by L_1 [9]. As with the reduced-order Kalman Filter in the continuous-time case the relation

$$C_2 L_1 = 0 \tag{26}$$

is satisfied [9], whereas $C_2L_1^+ \neq 0$ in general.

To obtain a regular DARE for the reduced-order Kalman filter, we first generate a DARE containing L_1 instead of L_1^+. From (25) and (14) follows

$$L_1^+ = L_1 + \Psi_2 C_2 A\bar{P}C_1^T \hat{R}^{-1} \tag{27}$$

Substituting this in (20) and observing (17) and (18) the DARE obtains the form

$$\bar{P} = A\bar{P}A^T + G\bar{Q}G^T - \begin{bmatrix} L_1 & \Psi_2 \end{bmatrix} \widetilde{X} \begin{bmatrix} L_1^T \\ \Psi_2^T \end{bmatrix} \tag{28}$$

where the abbreviation

$$\widetilde{X} = \begin{bmatrix} \hat{R} & C_1\bar{P}A^T C_2^T \\ C_2 A\bar{P}C_1^T & C_2(A\bar{P}A^T + G\bar{Q}G^T)C_2^T \end{bmatrix} \tag{29}$$

has been used. Multiplying (28) from the left by $I-\Gamma$ and from the right by $I-\Gamma^T$ with

$$\Gamma = G\bar{Q}G^T C_2^T \Phi^{-1} C_2 \tag{30}$$

it obtains the form

$$\bar{P} = \widetilde{A}\bar{P}\widetilde{A}^T + G\widetilde{Q}G^T - \widetilde{A}\bar{P}\widetilde{C}^T \widetilde{X}^{-1} \widetilde{C}\bar{P}\widetilde{A}^T \tag{31}$$

where the quantities in (31) are defined by

$$\widetilde{A} = A - G\bar{Q}G^T C_2^T \Phi^{-1} C_2 A \tag{32}$$

$$\widetilde{Q} = \bar{Q} - \bar{Q}G^T C_2^T \Phi^{-1} C_2 G\bar{Q} \tag{33}$$

and

$$\widetilde{C} = \begin{bmatrix} C_1 \\ C_2 A \end{bmatrix} \tag{34}$$

as in the continuous-time case [8]. The manipulations to obtain the form (31) use the facts that $(I-\Gamma)\bar{P}(I-\Gamma^T) = \bar{P}$ (see (21)), that $(I-\Gamma)\bar{Q}(I-\Gamma^T) = G\widetilde{Q}G^T$ (see (7) and (33)) and that

$$(I-\Gamma)[L_1 \quad \Psi_2]\widetilde{X} = \widetilde{A}\bar{P}\widetilde{C}^T \tag{35}$$

To show (35), it is helpful to write (29) as

$$\widetilde{X} = \begin{bmatrix} \hat{R} & \hat{R}L_1^{+T} C_2^T \\ C_2 L_1^+ \hat{R} & C_2(\widetilde{P} + L_1^+ \hat{R} L_1^{+T})C_2^T \end{bmatrix} \tag{36}$$

and for notational convenience we use the abbreviation

$$\Sigma = G\bar{Q}G^T C_2^T \tag{37}$$

Using (26), (25) and (12) the left hand side $S_L = (I - \Gamma)[L_1 \;\; \Psi_2]\widetilde{X}$ of (35) can be written as

$$S_L = \left[L_1^+ - \Psi_2 C_2 L_1^+ \;\; \vdots \;\; \Psi_2 - \Sigma\Phi^{-1} \right] \widetilde{X} \tag{38}$$

or with (36)

$$S_L = \left[L_1^+ \hat{R} - \Sigma\Phi^{-1} C_2 L_1^+ \hat{R} \;\; \vdots \right. \tag{39}$$
$$\left. L_1^+ \hat{R} L_1^{+T} C_2^T + \Psi_2 C_2 \widetilde{P} C_2^T - \Sigma\Phi^{-1} C_2 \widetilde{P} C_2^T - \Sigma\Phi^{-1} C_2 L_1^+ \hat{R} L_1^{+T} C_2^T \right]$$

Now observing $\Psi_2 C_2 \widetilde{P} C_2^T = \widetilde{P} C_2^T$ (which follows from (18) and (15)) and inserting $\widetilde{P} = A\bar{P}A^T + G\bar{Q}G^T - L_1^+ \hat{R} L_1^{+T}$ one obtains

$$S_L = \left[A\bar{P}C_1^T - \Sigma\Phi^{-1} C_2 A\bar{P}C_1^T \;\; \vdots \;\; A\bar{P}A^T C_2^T - \Sigma\Phi^{-1} C_2 A\bar{P}A^T C_2^T \right] \tag{40}$$

when using (14) and (7) and this is exactly the result of the right hand side of (35).

The DARE (31) is in the standard form with a regular $\bar{R} > 0$ (see (8)). Due to the rank deficient $\bar{P}$ the Hamiltonian of the DARE has eigenvalues at $z = 0$. In the continuous-time case eigenvalues at $s = 0$ cause problems when applying the MATLAB® function *lqe* to solve the corresponding ARE (see [8]). The function *dlqe*, however, directly yields the solution $\bar{P}$, since $z = 0$ is inside the stability region. Therefore, the derivation of a reduced DARE is not a necessity for obtaining $\bar{P}$ by standard software. However, in view of deriving a regular frequency-domain solution for the filter (see Section 3) the reduced version of the DARE is also considered here.

By a regular state transformation $\bar{x}(k) = \bar{T}x(k)$ with

$$\bar{T} = \begin{bmatrix} * \\ C \end{bmatrix} \tag{41}$$

the state equations (1)–(2) of the system can always be transformed into

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k) + \bar{G}w(k) \tag{42}$$

$$y(k) = \bar{C}\bar{x}(k) + \begin{bmatrix} v_1(k) \\ 0 \end{bmatrix} \tag{43}$$

with

$$\bar{A} = \bar{T}A\bar{T}^{-1},\; \bar{B} = \bar{T}B,\; \bar{G} = \bar{T}G,\; \begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \end{bmatrix} = C\bar{T}^{-1} = \begin{bmatrix} 0 & I_{m-\kappa} & 0 \\ 0 & 0 & I_\kappa \end{bmatrix} \tag{44}$$

or in components

$$\begin{bmatrix} \bar{x}_1(k+1) \\ \bar{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1(k) \\ \bar{x}_2(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} w(k) \tag{45}$$

$$y_1(k) = \begin{bmatrix} \check{C}_1 & 0 \end{bmatrix} \bar{x}(k) + v_1(k) \tag{46}$$

$$y_2(k) = \begin{bmatrix} 0 & I_\kappa \end{bmatrix} \bar{x}(k) \tag{47}$$

with $\bar{x}_1 \in \mathbb{R}^{n-\kappa}$, $0 < \kappa \leqslant m$, $\bar{x}_2 \in \mathbb{R}^{\kappa}$ and $\check{C}_1 = [0 \quad I_{m-\kappa}]$.

If the transformed matrices (44) are inserted in (31)–(34) the solution $\bar{P}_{\bar{x}} = \bar{T}\bar{P}\bar{T}^T$ of the DARE has the form

$$\bar{P}_{\bar{x}} = \begin{bmatrix} \bar{P}_r & 0 \\ 0 & 0_\kappa \end{bmatrix} \tag{48}$$

and the DARE (31) then consists of a regular (upper left) part

$$\bar{P}_r = A_r \bar{P}_r A_r^T + G_r \widetilde{Q} G_r^T - A_r \bar{P}_r C_r^T \widetilde{X}^{-1} C_r \bar{P}_r A_r^T \tag{49}$$

while the rest is vanishing. The matrices in (49) are defined by

$$A_r = A_{11} - G_1 \bar{Q} G_2^T \Phi^{-1} A_{21} \tag{50}$$

$$G_r = G_1 \tag{51}$$

and

$$C_r = \begin{bmatrix} \check{C}_1 \\ A_{21} \end{bmatrix} \tag{52}$$

so that the reduced-order Kalman filter can be regarded as a regular full-order filter for the reduced system (A_r, G_r, C_r). The feedback matrix L_r is defined by

$$L_r = A_r \bar{P}_r C_r^T \widetilde{X}^{-1} \tag{53}$$

It is known that the full-order Kalman filter for the system (A_r, G_r, C_r) is stable if the pair $(A_r, G_r \widetilde{Q}_0)$ has no uncontrollable eigenvalues on the unit circle, where

$$\widetilde{Q} = \widetilde{Q}_0 \widetilde{Q}_0^T \tag{54}$$

([5]). Introducing

$$\bar{Q} = \bar{Q}_0 \bar{Q}_0^T \tag{55}$$

and

$$\hat{Q} = I - \bar{Q}_0^T G_2^T \Phi^{-1} G_2 \bar{Q}_0 \tag{56}$$

it is easy to show that

$$\widetilde{Q}_0 = \bar{Q}_0 \hat{Q} \tag{57}$$

when taking

$$C_2 G = \bar{C}_2 \bar{G} = G_2 \tag{58}$$

into account. Given the above condition for a stable filter in terms of A_r and G_r, it is of interest to know the corresponding condition for the non-reduced system $(\bar{A}, \bar{G}, \bar{C})$. The answer is contained in the following lemma.

Lemma 10 *If the system*

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{G}\bar{Q}_0 w(k) \tag{59}$$

$$y_2(k) = \begin{bmatrix} 0 & I_\kappa \end{bmatrix} \bar{x}(k) \tag{60}$$

does not have zeros which are located on the unit circle, then the pair $(A_r, G_r\widetilde{Q}_0)$ *has no uncontrollable eigenvalues on the unit circle and vice versa.*

Proof If $z = z_i$ is a non-controllable eigenvalue of the pair $(A_r, G_r\widetilde{Q}_0)$ then

$$\text{rank} \begin{bmatrix} z_i I - A_r \vdots & G_r \widetilde{Q}_0 \end{bmatrix} < n - \kappa \tag{61}$$

(see, *e.g.*, [11]).

Now define the system matrix

$$P(z) = \begin{bmatrix} zI_{n-\kappa} - A_{11} & -A_{12} & G_1\bar{Q}_0 \\ -A_{21} & zI_\kappa - A_{22} & G_2\bar{Q}_0 \\ 0 & -I_\kappa & 0 \end{bmatrix} \tag{62}$$

which characterizes the zeros of the system (59)–(60) (see [15]). If the system (59)–(60) has a zero at $z = z_i$, then rank $P(z_i) < n + \kappa$.

Using the unimodular matrix

$$U_L = \begin{bmatrix} I_{n-\kappa} & -G_1\bar{Q}G_2^T\Phi^{-1} & 0 \\ 0 & I_\kappa & 0 \\ 0 & 0 & I_\kappa \end{bmatrix} \tag{63}$$

and the unimodular matrix

$$U_R = \begin{bmatrix} I_{n-\kappa} & 0 & 0 \\ 0 & I_\kappa & 0 \\ \bar{Q}_0^T G_2^T \Phi^{-1} A_{21} & 0 & I_q \end{bmatrix} \tag{64}$$

one obtains

$$U_L P(z_i) U_R = \begin{bmatrix} z_i I - A_r & * & G_r \widetilde{Q}_0 \\ 0 & * & G_2 \bar{Q}_0 \\ 0 & -I_\kappa & 0 \end{bmatrix} \tag{65}$$

Since it has been assumed that rank $G_2\bar{Q}_0 = \kappa$ (see (7)) the result (65) shows that the system (59)–(60) has a zero at $z = z_i$ if and only if $z = z_i$ is an uncontrollable eigenvalue in the pair $(A_r, G_r\widetilde{Q}_0)$ and *vice versa*. This is, of course, not only true for the transformed system (59)–(60) but also for the original system $(A, G\bar{Q}_0, C_2)$.

3. The filter design in the frequency domain

In the frequency domain, the system (1)–(2) or (42)–(43) is described by

$$y(z) = F_w(z)w(z) + \begin{bmatrix} v_1(z) \\ 0 \end{bmatrix} \tag{66}$$

with

$$F_w(z) = C(zI - A)^{-1}G = \bar{C}(zI - \bar{A})^{-1}\bar{G} \tag{67}$$

Given the left coprime MFD

$$F_w(z) = \bar{D}^{-1}(z)\bar{N}_w(z) \tag{68}$$

the reduced-order Kalman filter related to the *a priori* state estimate $\hat{x}$ is parameterized by the polynomial matrix $\widetilde{\bar{D}}(z)$ resulting by spectral factorization of the right hand side of

$$\widetilde{\bar{D}}(z)\widetilde{X}\widetilde{\bar{D}}^T(z^{-1}) = \bar{D}(z)\begin{bmatrix} \bar{R}_1 & 0 \\ 0 & 0 \end{bmatrix}\bar{D}^T(z^{-1}) + \bar{N}_w(z)\bar{Q}\bar{N}_w^T(z^{-1}) \tag{69}$$

where

$$\Gamma_r\left[\widetilde{\bar{D}}(z)\right] = \Gamma_r\left[\bar{D}_\kappa(z)\right] \tag{70}$$

with the row-reduced polynomial matrix

$$\bar{D}_\kappa(z) = \Pi\left\{\bar{D}(z)\begin{bmatrix} I_{m-\kappa} & 0 \\ 0 & z^{-1}I_\kappa \end{bmatrix}\right\} \tag{71}$$

(see [9, 10]). Here, $\Gamma_r[\cdot]$ denotes the highest row-degree-coefficient matrix and $\Pi[\cdot]$ taking the polynomial part. How the polynomial matrix $\widetilde{\bar{D}}^+(z)$ related to the *a posteriori* state estimate $\hat{x}^+$ is obtained from $\widetilde{\bar{D}}(z)$ is described in [9, 10].

In [9] the solution (69)–(71) is presented without rigorous proof, because the polynomial matrix equation (69) contains a singular measurement covariance matrix on the right hand side, and the known proofs of optimality of $\tilde{\bar{D}}(z)$ are formulated for full-order filters with regular measurement covariance matrices.

The polynomial matrix equation (69) was derived on the basis of the DARE (20). As shown in Section 2, the reduced-order Kalman filter can also be designed on the basis of the regularized DARE (49) with a regular measurement covariance matrix $\bar{R}$, *i.e.*, one can design the reduced-order Kalman filter as a regular full-order filter for the reduced system (A_r, C_r, G_r).

Introducing the left coprime MFD of

$$F_r(z) = C_r(zI - A_r)^{-1} G_r \tag{72}$$

namely

$$F_r(z) = \bar{D}_r^{-1}(z) \bar{N}_{wr}(z) \tag{73}$$

and the polynomial matrix $\tilde{\bar{D}}_r(z)$ parameterizing the full-order Kalman filter related to the parameters $(A_r, G_r, C_r, \bar{P}_r)$ according to

$$\bar{D}_r^{-1}(z) \tilde{\bar{D}}_r(z) = C_r(zI - A_r)^{-1} L_r + I_m \tag{74}$$

(see [9, 7]), the Riccati equation (49) can be transformed into the polynomial matrix equation

$$\tilde{\bar{D}}_r(z) \tilde{X} \tilde{\bar{D}}_r^T(z^{-1}) = \bar{D}_r(z) \bar{R} \bar{D}_r^T(z^{-1}) + \bar{N}_{wr}(z) \tilde{Q} \bar{N}_{wr}^T(z^{-1}) \tag{75}$$

by similar steps as in the derivation of (69) from (20) in [9]. This is a regular polynomial matrix equation with $\bar{R} > 0$ and consequently the polynomial matrix $\tilde{\bar{D}}_r(z)$ obtained by spectral factorization of the right hand side of (75) with

$$\Gamma_r \left[\tilde{\bar{D}}_r(z) \right] = \Gamma_r \left[\bar{D}_r(z) \right] \tag{76}$$

(see [9, 7]) parameterizes the optimal full-order Kalman filter for the reduced-order system (72) in the frequency domain.

If this $\tilde{\bar{D}}_r(z)$ is identical with $\tilde{\bar{D}}(z)$ obtained from the spectral factorization of (69), it follows that the solution procedure presented in [9] yields indeed the optimal results.

Given the transformed system description (42), (43) and the MFD (68), *i.e.*, a denominator matrix $D(z)$ such that $\bar{D}_\kappa(z)$ as defined in (71) is row reduced. Then define the MFD

$$\bar{C}(zI - \bar{A})^{-1} = \bar{D}^{-1}(z) \bar{N}_x(z) \tag{77}$$

with $\bar{D}(z)$ as in (68) and $\bar{N}_x(z)$ partitioned according to

$$\bar{N}_x(z) = \begin{bmatrix} \bar{N}_{x1}(z) & \bar{N}_{x2}(z) \end{bmatrix} \tag{78}$$

where $\bar{N}_{x1}(z)$ has $n - \kappa$ columns and $\bar{N}_{x2}(z)$ has κ columns.

Theorem 3 *The polynomial matrix $\tilde{\bar{D}}_r(z)$ resulting from (75) is identical with $\tilde{\bar{D}}(z)$ resulting from (69) if the polynomial matrices in the MFD (73) are chosen as*

$$\bar{N}_{wr}(z) = \bar{N}_{x1}(z)G_1 \tag{79}$$

and

$$\bar{D}_r(z) = [\bar{N}_{x1}(z)\ \ \bar{N}_{x2}(z)] \begin{bmatrix} 0_{n-\kappa,m-\kappa} & G_1\bar{Q}G_2^T\Phi^{-1} \\ 0_{\kappa,m-\kappa} & I_\kappa \end{bmatrix} + \bar{D}(z) \begin{bmatrix} I_{m-\kappa} & 0 \\ 0 & 0_\kappa \end{bmatrix} \tag{80}$$

The polynomial matrix $\tilde{\bar{D}}(z) = \tilde{\bar{D}}_r(z)$ parameterizes a stable filter if the pair

$$\left(\bar{D}(z) \begin{bmatrix} I_{m-\kappa} & 0 \\ 0 & 0_\kappa \end{bmatrix},\ \bar{N}_w(z)\bar{Q}_0 \right) \tag{81}$$

has no greatest common left devisor with zeros on the unit circle.

Proof Comparing (72) and (73) and observing (79) and (51) one can conclude that for arbitrary G_1

$$C_r(zI - A_r)^{-1} = \bar{D}_r^{-1}(z)\bar{N}_{x1}(z) \tag{82}$$

Writing (77) in the modified form $\bar{D}(z)\bar{C} = \bar{N}_x(z)(zI - \bar{A})$ and using (45) – (47) one obtains

$$\bar{D}(z) \begin{bmatrix} \breve{C}_1 \\ 0 \end{bmatrix} = \bar{N}_{x1}(z)(zI - A_{11}) - \bar{N}_{x2}(z)A_{21} \tag{83}$$

This allows to show that $\bar{N}_{x1}(z)(zI - A_r) = \bar{D}_r(z)C_r$ which then proves that the pair (79) and (80) constitutes a left MFD of (72). Inserting (79) and (80) in (75) it is straightforward to show, that the right hand sides of the polynomial equations (69) and (75) coincide, so that $\tilde{\bar{D}}(z)\tilde{X}\tilde{\bar{D}}^T(z^{-1}) = \tilde{\bar{D}}_r(z)\tilde{X}\tilde{\bar{D}}_r^T(z^{-1})$. Since $\tilde{X}$ is positive definite, this shows that $\tilde{\bar{D}}(z) = \tilde{\bar{D}}_r(z)$. This proves the first part of the theorem.

Since (75) represents a regular full-order filter problem for the reduced system (A_r, G_r, C_r), the filter parameterized by $\tilde{\bar{D}}_r(z)$ is optimal and stable if the pair

$$\left(\bar{D}_r(z)\ ,\ \bar{N}_{wr}(z)\tilde{Q}_0 \right) \tag{84}$$

has no common greatest left devisor $U_L(z)$ with zeros on the unit circle ([5]).

Two polynomial matrices are relatively left coprime if they satisfy the Bezout identity. If they contain a non-unimodular greatest common left devisor $U_L(z)$, the identity matrix is replaced by $U_L(z)$ ([11]).

If the pair (84) contains a non-unimodular greatest common left devisor $U_L(z)$ there exist solutions $\bar{Y}_{0r}(z)$ and $\bar{X}_{0r}(z) = \begin{bmatrix} \bar{X}_{0r1}(z) \\ \bar{X}_{0r2}(z) \end{bmatrix}$ of the Diophantine equation

$$\bar{N}_{x1}(z)G_1\widetilde{Q}_0\bar{Y}_{0r}(z) + \bar{D}_r(z)\begin{bmatrix} \bar{X}_{0r1}(z) \\ \bar{X}_{0r2}(z) \end{bmatrix} = U_L(z) \tag{85}$$

(see, *e.g.*, [9]).

If, on the other hand, the pair (81) contains a non-unimodular greatest common left devisor $U_L(z)$ there exist solutions $\bar{Y}_0(z)$ and $\bar{X}_0(z) = \begin{bmatrix} \bar{X}_{01}(z) \\ \bar{X}_{02}(z) \end{bmatrix}$ of the Diophantine equation

$$\begin{bmatrix} \bar{N}_{x1}(z)G_1 + \bar{N}_{x2}(z)G_2 \end{bmatrix} \bar{Q}_0\bar{Y}_0(z) + \bar{D}(z)\begin{bmatrix} I & 0 \\ 0 & 0_\kappa \end{bmatrix}\begin{bmatrix} \bar{X}_{01}(z) \\ \bar{X}_{02}(z) \end{bmatrix} = U_L(z) \tag{86}$$

where the fact has been exploited, that $\bar{N}_w(z) = \bar{N}_x(z)\bar{G}$ (compare (77) with (67) and (68)). Given the solutions $\bar{Y}_{0r}(z)$ and $\bar{X}_{0r}(z)$ of (85) the polynomial matrices

$$\bar{X}_{01}(z) = \bar{X}_{0r1}(z) \tag{87}$$

$$\bar{X}_{02}(z) = 0 \tag{88}$$

and

$$\bar{Y}_0(z) = \hat{Q}\bar{Y}_{0r}(z) + \bar{Q}_0^T G_2^T \Phi^{-1}\bar{X}_{0r2}(z) \tag{89}$$

solve the equation (86).

Given the solutions $\bar{Y}_0(z)$ and $\bar{X}_0(z)$ of (86) the polynomial matrices

$$\bar{X}_{01r}(z) = \bar{X}_{01}(z) \tag{90}$$

$$\bar{X}_{0r2}(z) = G_2\bar{Q}_0\bar{Y}_0(z) \tag{91}$$

and

$$\bar{Y}_{0r}(z) = \hat{Q}\bar{Y}_0(z) \tag{92}$$

solve the equation (85). This shows that if the pair (84) does not contain a greatest common left devisor with zeros on the unit circle, then also the pair (81) does not contain such a greatest common left devisor and *vice versa*. This proves the second part of the theorem.

4. Conclusions

Due to the singular measurement covariance matrix standard software cannot be used to solve the DARE of the reduded-order Kalman filter for the rank deficient covariance matrix $\bar{P}$ of the estimation error. This DARE, however, can be reformulated so that standard software becomes applicable. By applying an appropriate state transformation to the original system, a modified form of the DARE results which can be subdivided into a regular part, yielding a regular $\bar{P}_r$, and a vanishing part. The regular part defines a reduced-order system such that the full-order filter for it coincides with the reduced-order filter for the original system. This regular part also characterizes the conditions which guarantee a stable filter. These conditions for the parameters of the reduced-order system can be used to define the conditions for the original full-order system guaranteeing a stable reduced-order filter.

The polynomial matrix equation defining the parameterizing polynomial matrix $\tilde{\bar{D}}(z)$ of the reduced-order filter in the frequency domain also contains a singular measurement covariance matrix. This does not cause problems when applying spectral factorization to obtain $\tilde{\bar{D}}(z)$. However, neither a proof of optimality nor the conditions for the stability of the filter were known so far. Based on the reduced-order model of the system in the time domain, a regular full-order filter design also becomes possible in the frequency domain. This allows to prove the optimality of the results obtained by the known non-regular factorization and it also allows to formulate the conditions on the MFD of the original full-order system which guarantee a stable filter.

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