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Extremal dynamic errors in linear dynamic systems

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Abstract. Two different analytical methods of determining extremal dynamic errors in linear dynamic systems are presented. The main idea of these methods is based on finding certain additional equations. These additional equations are obtained due to the assumption that an extremal point τ obtained from the necessary condition $\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=\tau} = 0$, is also an extremum point with respect to initial conditions, that is, $\frac{\mathrm{d}\tau}{\mathrm{d}c_i} = 0$, $i = 1, \ldots, n$.

Key words: extremal dynamic errors, linear dynamic systems.

1. Introduction

In many dynamic processes the maximal dynamic error is the most important criterion. In the chemical processes and in the driving systems such criterion plays an important role. The maximal error $x_e(\tau)$ characterises the attainable accuracy and the time τ, the velocity of the rise of the transients [1–3].

2. Statement of the problem

Let us consider the differential equation describing the transient error in the linear control system of the n-th order with lumped and constant parameters:

$$\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} + a_1 \frac{\mathrm{d}^{n-1} x(t)}{\mathrm{d}t^{n-1}} + \ldots + a_{n-1} \frac{\mathrm{d}x(t)}{\mathrm{d}t} + a_n x(t) = 0 \tag{1}$$

with the initial conditions

$$x^{(i-1)}(0) = c_i \neq 0 \qquad \text{for} \quad i = 1, 2, \ldots, n \ .$$

The characteristic equation for Eq. (1) is:

$$s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0 \ . \tag{2}$$

We assume that the roots of the Eq. (2) are simple and real that is, $s_j \neq s_i$ for $j \neq i$.

The solution of Eq. (1) takes the following form

$$x(t) = \sum_{k=1}^{n} A_k e^{s_k t} \ . \tag{3}$$

The necessary condition for the transient error $x(t)$ to attain an extremal value at $t = \tau$ is given by the relation

$$x^{(1)}(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{k=1}^{n} s_k A_k e^{s_k t} = 0. \tag{4}$$

The coefficients A_k for $k = 1, 2, \ldots, n$ in the explicit form are

$$A_k = \frac{c_n - \sum\limits_{\substack{v=1,\\ v\neq k}}^{n} s_v c_{n-1} + \sum\limits_{\substack{v=1,\\ v\neq k}}^{n} s_v s_k c_{n-2} + \ldots + (-1)^{n-1} \prod\limits_{\substack{v=1,\\ v\neq k}}^{n} s_v c_1}{\prod\limits_{v=1, v\neq k}^{n} (s_v - s_k)}. \tag{5}$$

It is worth noticing that for particular c_i we have in (5) symmetrical functions of s_v without one s_k. The extremal values of $x(t)$ depend linearly on c_i, but extremum points τ depend nonlinearly on c_i. In order to obtain analytic formulae for the extremal values of $x(t)$ we will use the additional equations

$$\frac{\mathrm{d}x^{(1)}(\tau, c_1, \ldots, c_n)}{\mathrm{d}c_i} = 0 \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{6}$$

Exactly speaking

$$\frac{\mathrm{d}x^{(1)}}{\mathrm{d}c_i} = \frac{\partial x^{(1)}}{\partial c_i} + \frac{\partial x^{(1)}}{\partial \tau}\frac{\partial \tau}{\partial c_i} = 0. \tag{7}$$

We assume that $\frac{\partial x^{(1)}}{\partial \tau} \neq 0$.

We will limit our investigation to the case when τ attains its extremum with respect to initial condition c_i. In this case we will use the necessary condition that

$$\frac{\partial \tau}{\partial c_i} = 0.$$

In this way Eq. (7) will be reduced to Eq. (6).

For the equation of order n it is necessary to use $(n-2)$ equations from the set of Eq. (6). These $(n-2)$ equations together with the basic equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = 0$$

give $(n-1)$ equations for determination of the unknowns $e^{(s_i - s_n)\tau}$, $i = 1, 2, \ldots, n-1$.

We stress that the time τ must be positive $0 \leq \tau \leq \infty$, and for maintaining asymptotic stability conditions it is required

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that Re $s_j < 0$. According to this, the exponential functions $0 \le e^{(s_n - s_i)\tau} \le 1$, where $s_n < s_{n-1} < \ldots < s_2 < s_1 < 0$, $\tau \ge 0$.

3. Solution to the problem. (Basic results)

First method. In order to determine the n exponential terms $e^{s_i\tau}$, $i = 1, \ldots, n$ we have one equation (4) and we take $(n-2)$ equations (6), for example for $i = n, n-1, \ldots, 2$.

The solutions of this set of linear homogeneous equations, where A_k in (4) are defined by the relations (5) are as follows:

$$\left.\begin{aligned} e^{s_1\tau} &= \frac{s_n(s_1c_1 - c_2)}{s_1(s_nc_1 - c_2)}\, e^{s_n\tau} \\ e^{s_2\tau} &= \frac{s_n(s_2c_1 - c_2)}{s_2(s_nc_1 - c_2)}\, e^{s_n\tau} \\ \ldots &= \ldots\ldots\ldots\ldots \\ e^{s_{n-1}\tau} &= \frac{s_n(s_{n-1}c_1 - c_2)}{s_{n-1}(s_nc_1 - c_2)}\, e^{s_n\tau} \end{aligned}\right\}. \tag{8}$$

The substitution of the relations (8) into Eq. (3) for $x(\tau)$ and into higher derivatives $x^{(2)}(\tau), x^{(3)}(\tau), \ldots, x^{(n-1)}(\tau)$

$$\left.\begin{aligned} x^{(2)}(\tau) &= \sum_{k=1}^{n} s_k^2 A_k e^{s_k\tau} \\ x^{(3)}(\tau) &= \sum_{k=1}^{n} s_k^3 A_k e^{s_k\tau} \\ \ldots &= \ldots\ldots\ldots\ldots \\ x^{(n-1)}(\tau) &= \sum_{k=1}^{n} s_k^{n-1} A_k e^{s_k\tau} \end{aligned}\right\}$$

gives

$$\left.\begin{aligned} & x^{(2)}(\tau) = \\ & = \frac{a_n(c_1c_3 - c_2^2)x(\tau)}{a_nc_1^2 + a_{n-1}c_1c_2 + a_{n-2}c_2^2 + a_{n-3}c_2c_3 + a_{n-4}c_2c_4 + \ldots + c_2c_n} \\ & x^{(3)}(\tau) = \\ & = \frac{a_n(c_1c_4 - c_2c_3)x(\tau)}{a_nc_1^2 + a_{n-1}c_1c_2 + a_{n-2}c_2^2 + a_{n-3}c_2c_3 + a_{n-4}c_2c_4 + \ldots + c_2c_n} \\ & x^{(4)}(\tau) = \\ & = \frac{a_n(c_1c_5 - c_2c_4)x(\tau)}{a_nc_1^2 + a_{n-1}c_1c_2 + a_{n-2}c_2^2 + a_{n-3}c_2c_3 + a_{n-4}c_2c_4 + \ldots + c_2c_n} \\ & \ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots \\ & x^{(n-1)}(\tau) = \\ & = \frac{a_n(c_1c_n - c_2c_{n-1})x(\tau)}{a_nc_1^2 + a_{n-1}c_1c_2 + a_{n-2}c_2^2 + a_{n-3}c_2c_3 + a_{n-4}c_2c_4 + \ldots + c_2c_n} \end{aligned}\right\} \tag{9}$$

where in the denominator we have all possible products of the initial conditions $c_1, \ldots, c_n$ and with the coefficients $a_n, \ldots, a_1$ whose weight together is equal to $n+2$.

In the paper [2] the general relation was proved between $x(\tau), x^{(2)}(\tau)$, …, $x^{(n-1)}(\tau)$ and $c_1, c_2, \ldots, c_n$, $a_1, a_2, \ldots, a_n$.

$$\begin{aligned} & \prod_{k=1}^{n} \sum_{\substack{j=1,\\ j\neq 2}}^{n} (-1)^j \varphi_{n-j}^{(k)}\, x^{(j-1)}(\tau) \\ & = e^{-a_1\tau} \prod_{k=1}^{n} \sum_{j=1}^{n} (-1)^j \varphi_{n-j}^{(k)} c_j, \end{aligned} \tag{10}$$

where $\varphi_r^{(j)}$ is the fundamental symmetric function of the r-th order of $(n-1)$ variables $s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n$, $r = 0, 1, \ldots, n-1$

$$\left.\begin{aligned} \varphi_0^{(j)} &= 1, & a_0 &= 1 \\ \varphi_r^{(j)} &= \sum_{i=0}^{r} (-1)^r a_{r-i} s_j^i, & j &= 1, 2, \ldots, n-1 \end{aligned}\right\}. \tag{11}$$

Both sides of Eq. (10) are composed of the symmetric polynomials of variables $s_1, \ldots, s_n$. Due to this it is possible to present these terms as the polynomials of the coefficients $a_1, \ldots, a_n$. Using Viete's relations it is possible to replace the roots s_k by the coefficients a_k and to avoid calculation of the roots by the solution of algebraic Eqs. (2). Using the substitution of the relations (9) into Eq. (10) we obtain the general formulae for calculation of $x(\tau)$:

$$\begin{aligned} & x^n(\tau)e^{a_1\tau} = \\ & = \frac{(a_nc_1^2 + a_{n-1}c_1c_2 + a_{n-2}c_2^2 + a_{n-3}c_2c_3 + \ldots + c_2c_n)^n}{a_n^{n-1}(a_nc_1^n + a_{n-1}c_1^{n-1}c_2 + a_{n-2}c_1^{n-2}c_2^2 + \ldots + a_1c_1c_2^{n-1} + c_2^n)}. \end{aligned} \tag{12}$$

The weight of each term in numerator is equal to $(n+2)$ and the number of terms is maximally $(n+1)$, when all the initial conditions $c_1, \ldots, c_n$ are different from zero. In the brackets of denominator we have only two initial conditions c_1, c_2 and n coefficients $a_1, \ldots, a_n$. The weight of each term in the brackets is equal to $2n$. The maximal number of terms is equal to $2n+1$, when both initial conditions c_1, c_2 are different from zero. For maintaining the asymptotic stability conditions it is required that all the coefficients $a_1, \ldots, a_n$ must be positive. The discussion of the particular cases illustrates this method [4].

4. Particular cases

- $n = 2$.

We have a differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + a_1\frac{\mathrm{d}x}{\mathrm{d}t} + a_2x = 0, \tag{13}$$

with initial conditions

$$\left.\begin{aligned} x(0) &= c_1 \\ x^{(1)}(0) &= c_2 \end{aligned}\right\}.$$

Solution of Eq. (13) is

$$x(t) = \frac{c_1}{s_2 - s_1}\left(s_2e^{s_1t} - s_1e^{s_2t}\right) - \frac{c_2}{s_2 - s_1}\left(e^{s_1t} + e^{s_2t}\right),$$

where s_1, s_2 are real different roots of the characteristic equation

$$s^2 + a_1s + a_2 = 0.$$

The derivative

$$x^{(1)}(t)=c_1\frac{s_1s_2}{s_2-s_1}\left(e^{s_1t}-e^{s_2t}\right)-\frac{c_2}{s_2-s_1}\left(s_1e^{s_1t}+s_2e^{s_2t}\right).$$

From the necessary condition $x^{(1)}(\tau)=0$ we obtain

$$\tau=\frac{1}{s_1-s_2}\ln\frac{s_2(c_1s_1-c_2)}{s_1(c_1s_2-c_2)}$$

or using Viete's formulae

$$\tau=\frac{1}{\sqrt{a_1^2-4a_2}}\ln\frac{2a_2c_1+(a_1+\sqrt{a_1^2-4a_2})c_2}{2a_2c_1+(a_1-\sqrt{a_1^2-4a_2})c_2},$$

$$a_1^2\geq 4a_2.$$

It is to find using equation $x^{(1)}(\tau)=0$ that

$$x(\tau)=\frac{c_1s_1-c_2}{s_1}\,e^{s_2\tau},\tag{14}$$

$$x^{(2)}(\tau)=-(c_1s_1-c_2)s_2e^{s_2\tau}.\tag{15}$$

Elimination of $e^{s_2\tau}$ from Eqs. (14) and (15) gives

$$x^{(2)}(\tau)=-s_1s_2x(\tau)=-a_2x(\tau).$$

We have also from (10)

$$x^2(\tau)e^{a_1\tau}=c_1^2+\frac{a_1}{a_2}\,c_1c_2+\frac{1}{a_2}\,c_2^2.$$

• $n=3$.

The differential equation has the form

$$\frac{d^3x(t)}{dt^3}+a_1\frac{d^2x(t)}{dt^2}+a_2\frac{dx(t)}{dt}+a_3x(t)=0,\tag{16}$$

with the initial conditions

$$\left.\begin{aligned}x(0)&=c_1\\x^{(1)}(0)&=c_2\\x^{(2)}(0)&=c_3\end{aligned}\right\}.\tag{17}$$

Solution of Eq. (16) with (17) is

$$\begin{aligned}x(t)&=\\=c_1&\left(\frac{s_2s_3e^{s_1t}}{(s_1-s_2)(s_1-s_3)}+\frac{s_3s_1e^{s_2t}}{(s_2-s_3)(s_2-s_1)}+\frac{s_1s_2e^{s_3t}}{(s_3-s_1)(s_3-s_2)}\right)\\-c_2&\left(\frac{(s_2+s_3)e^{s_1t}}{(s_1-s_2)(s_1-s_3)}+\frac{(s_3+s_1)e^{s_2t}}{(s_2-s_3)(s_2-s_1)}+\frac{(s_1+s_2)e^{s_3t}}{(s_3-s_1)(s_3-s_2)}\right)\\+c_3&\left(\frac{e^{s_1t}}{(s_1-s_2)(s_1-s_3)}+\frac{e^{s_2t}}{(s_2-s_3)(s_2-s_1)}+\frac{e^{s_3t}}{(s_3-s_1)(s_3-s_2)}\right).\end{aligned}$$

From the equation $x^{(1)}(t)=0$ we have

$$\begin{aligned}&x^{(1)}(t)=c_1s_1s_2s_3\\&\left(\frac{e^{s_1t}}{(s_1-s_2)(s_1-s_3)}+\frac{e^{s_2t}}{(s_2-s_3)(s_2-s_1)}+\frac{e^{s_3t}}{(s_3-s_1)(s_3-s_2)}\right)\\&-c_2\left(\frac{s_1(s_2+s_3)e^{s_1t}}{(s_1-s_2)(s_1-s_3)}+\frac{s_2(s_3+s_1)e^{s_2t}}{(s_2-s_3)(s_2-s_1)}+\frac{s_3(s_1+s_2)e^{s_3t}}{(s_3-s_1)(s_3-s_2)}\right)\\&+c_3\left(\frac{s_1e^{s_1t}}{(s_1-s_2)(s_1-s_3)}+\frac{s_2e^{s_2t}}{(s_2-s_3)(s_2-s_1)}+\frac{s_3e^{s_3t}}{(s_3-s_1)(s_3-s_2)}\right)=0.\end{aligned}\tag{18}$$

We need an additional equation for the determination of τ. We take

$$\frac{dx^{(1)}}{dc_3}=0$$

and obtain that

$$s_1(s_2-s_3)e^{s_1\tau}+s_2(s_3-s_1)e^{s_2\tau}+s_3(s_1-s_2)e^{s_3\tau}=0\tag{19}$$

and in (18) remains the equation

$$\begin{aligned}&s_1s_2s_3c_1\left[(s_2-s_3)e^{s_1\tau}+(s_3-s_1)e^{s_2\tau}+(s_1-s_2)e^{s_3\tau}\right]\\&\quad-c_2\Big[s_1(s_2^2-s_3^2)e^{s_1\tau}+s_2(s_3^2-s_1^2)e^{s_2\tau}\\&\quad+s_3(s_1^2-s_2^2)e^{s_3\tau}\Big]=0.\end{aligned}\tag{20}$$

From Eqs. (19) and (20) we have, see (8)

$$e^{s_1\tau_1}=\frac{s_3(s_1c_1-c_2)}{s_1(s_3c_1-c_2)}\,e^{s_3\tau_1},\tag{21}$$

$$e^{s_2\tau_2}=\frac{s_3(s_2c_1-c_2)}{s_2(s_3c_1-c_2)}\,e^{s_3\tau_2}.\tag{22}$$

Equations (21) and (22) determine two values (if they exist) of τ_1 and τ_2. We look for a common $\tau_1=\tau_2$ for these two equations and obtain from (21) and (22) that

$$\tau=\tau_1=\tau_2=\frac{1}{s_1-s_2}\,\ln\frac{s_2(s_1c_1-c_2)}{s_1(s_2c_1-c_2)}.$$

The substitution of (21) and (22) for common τ to $x(\tau)$ and $x^{(2)}(\tau)$, after elimination of $e^{s_3\tau}$ lead to a relation between $x^{(2)}(\tau)$ and $x(\tau)$, see (9)

$$x^{(2)}(\tau)=\frac{a_3(c_1c_3-c_2^2)}{a_3c_1^2+a_2c_1c_2+a_1c_2^2+c_2c_3}\,x(\tau).\tag{23}$$

We assume that $c_1c_3-c_2^2\neq 0$ in order to avoid an inflection point.
We have also from the relation (10) see [2] that

$$\begin{aligned}&e^{a_1\tau}\{a_3^2x^3(\tau)+a_1a_3x^{(2)}(\tau)x^2(\tau)+a_2\left[x^{(2)}(\tau)\right]^2x(\tau)\\&+\left[x^{(2)}(\tau)\right]^3\}=a_3^2c_1^3+2a_2a_3c_2c_1^2+(a_1a_3+a_2^2)\,c_2^2c_1\\&+(a_1a_2-a_3)\,c_2^3+(a_1a_2+3a_3)c_1c_2c_3+a_1a_3c_1^2c_3\\&+a_2c_1c_3^2+(a_1^2+a_2)c_2^2c_3\\&+2a_1c_2c_3^2+c_3^3.\end{aligned}\tag{24}$$

The substitution of (23) to (24) gives finally

$$x^3(\tau)e^{a_1\tau}=\frac{(a_3c_1^2+a_2c_1c_2+a_1c_2^2+c_2c_3)^3}{a_3^2(a_3c_1^3+a_2c_1^2c_2+a_1c_1c_2^2+c_2^3)},\tag{25}$$

which is the final result for this case, compare with (12).

• $n=4$.

Following the same way as for $n=3$ we have to use two additional equations. These equations are obtained from two additional conditions

$$\left.\begin{aligned} \frac{\mathrm{d}x^{(1)}}{\mathrm{d}c_4} &= 0 \\ \frac{\mathrm{d}x^{(2)}}{\mathrm{d}c_3} &= 0 \end{aligned}\right\}. \tag{26}$$

Equations (26) with the basic equation $x^{(1)}(t) = 0$ give the solution of the problem in this case:

$$\left.\begin{aligned} e^{s_1\tau_1} &= \frac{s_4(s_1c_1 - c_2)}{s_1(s_4c_1 - c_2)}\, e^{s_4\tau_1} \\ e^{s_2\tau_2} &= \frac{s_4(s_2c_1 - c_2)}{s_2(s_4c_1 - c_2)}\, e^{s_4\tau_2} \\ e^{s_3\tau_3} &= \frac{s_4(s_3c_1 - c_2)}{s_3(s_4c_1 - c_2)}\, e^{s_4\tau_3} \end{aligned}\right\}.$$

Substituting these equations to $x(\tau)$, $x^{(2)}(\tau)$ and $x^{(3)}(\tau)$ gives the following result

$$\frac{x^{(2)}(\tau)}{x(\tau)} = \frac{a_4(c_1c_3 - c_2^2)}{a_4c_1^2 + a_3c_1c_2 + a_2c_2^2 + a_1c_2c_3 + c_2c_4},$$

$$\frac{x^{(3)}(\tau)}{x(\tau)} = \frac{a_4(c_1c_4 - c_2c_3)}{a_4c_1^2 + a_3c_1c_2 + a_2c_2^2 + a_1c_2c_3 + c_2c_4},$$

which substituted to the equation similar to (24)

$$\begin{aligned}
& e^{a_1t_e}\Big[a_4^3x_e^4 + 2a_2a_4^2x_e^3x_e^{(2)} + a_1a_4^2x_e^3x_e^{(3)} \\
& +\left(a_2^2 + a_1a_3 + 2a_4\right)a_4x_e^2\left(x_4^{(2)}\right)^2 \\
& +\left(a_1a_2 + 3a_3\right)a_4x_e^2x_e^{(2)}x_e^{(3)} + a_2a_4x_e^2\left(x_e^{(3)}\right)^2 \\
& +\left(a_1a_2a_3 + a_1^2a_4 - a_3^2 + 2a_2a_4\right)x_e\left(x_e^{(2)}\right)^3 \\
& +\left(a_1^3a_3 + a_2a_3 + 5a_1a_4\right)x_e\left(x_e^{(2)}\right)^2x_e^{(3)} \\
& +2\left(a_1a_3 + 2a_4\right)x_ex_e^{(2)}\left(x_e^{(3)}\right)^2 \\
& +a_3x_e\left(x_e^{(3)}\right)^3 + \left(a_1^2a_2 - a_1a_3 + a_4\right)\left(x^{(2)}\right)^4 \\
& +\left(a_1^3 + 2a_1a_2 - a_3\right)\left(x_2^{(2)}\right)^3x_e^{(3)} \\
& +\left(3a_1^2 + a_2\right)\left(x_e^{(2)}\right)^2\left(x_e^{(3)}\right)^2 + 3a_1x_e^{(2)}\left(x_e^{(3)}\right)^3 + \left(x_e^{(3)}\right)^4\Big] \\
& = a_4^3c_1^4 + 3a_3a_4^2c_1^3c_2 + 2a_2a_4^2c_1^3c_3 \\
& +a_1a_4^2c_1^3c_4 + \left(3a_3^2 + a_2a_4\right)a_4c_1^2c_2^2 \\
& +\left(4a_2a_3 + 3a_1a_4\right)a_4c_1^2c_2c_3 + 2\left(a_1a_3 + 2a_4\right)a_4c_1^2c_2c_4 \\
& +\left(a_2^2 + a_1a_3 + 2a_4\right)a_4c_1^2c_3^2 + \left(a_1a_2 + 3a_3\right)a_4c_1^2c_3c_4 \\
& +a_2a_4c_1^2c_4^2 + \left(a_3^3 + 2a_2a_3a_4 - a_1a_4^2\right)c_1c_2^3 \\
& +2\left(a_2a_3^2 + a_2^2a_4 + 2a_1a_3a_4 - 2a_4^2\right)c_1c_2^2c_3 \\
& +\left(a_1a_3^2 + a_1a_2a_4 + 5a_3a_4\right)c_1c_2^2c_4 \\
& +\left(a_2^2a_3 + a_1a_3^2 + 5a_1a_2a_4 - a_3a_4\right)c_1c_2c_3^2 \\
& +\left(a_1a_2a_3 + 3a_1^2a_4 + 3a_3^2 + 4a_2a_4\right)c_1c_2c_3c_4 \\
& +\left(a_2a_3 + 3a_1a_4\right)c_1c_2c_4^2 \\
& +\left(a_1a_2a_3 + a_1^2a_4 - a_3^2 + 2a_2a_4\right)c_1c_3^3 \\
& +\left(a_1^2a_3 + a_2a_3 + 5a_1a_4\right)c_1c_3^2c_4 \\
& +2\left(a_1a_3 + 2a_4\right)c_1c_3c_4^2 + a_3c_1c_4^3 \\
& +\left(a_2a_3^2 - a_1a_3a_4 + a_4^2\right)c_2^4 \\
& +\left(2a_2^2a_3 + a_1a_3^2 - a_1a_2a_4 - a_3a_4\right)c_2^3c_3 \\
& +\left(a_1a_2a_3 - a_1^2a_4 + a_3^2 + 2a_2a_4\right)c_2^3c_4 \\
& +a_2\left(a_2^2 + 3a_1a_3 - 3a_4\right)c_2^2c_3^2 + \\
& +\left(a_1a_2^2 + a_1^2a_3 + 5a_2a_3 - a_1a_4\right)c_2^2c_3c_4 \\
& +\left(a_2^2 + a_1a_3 + 2a_4\right)c_2^2c_4^2 \\
& +\left(2a_1a_2^2 + a_1^2a_3 - a_2a_3 - a_1a_4\right)c_2c_3^3 \\
& +2\left(a_1^2a_2 + a_2^2 + 2a_1a_3 - 2a_4\right)c_2c_3^2c_4 \\
& +\left(4a_1a_2 + 3a_3\right)c_2c_3c_4^2 + 2a_2c_2c_4^3 \\
& +\left(a_1^2a_2 - a_1a_3 + a_4\right)c_3^4 \\
& +\left(a_1^3 + 2a_1a_2 - a_3\right)c_3^3c_4 \\
& +\left(3a_1^2 + a_2\right)c_3^2c_4^2 + 3a_1c_3c_4^3 + c_4^4
\end{aligned} \tag{27}$$

gives finally, compare with (12)

$$x^4(\tau)e^{a_1\tau} = \frac{(a_4c_1^2 + a_3c_1c_2 + a_2c_2^2 + a_1c_2c_3 + c_2c_4)^4}{a_4^3(a_4c_1^4 + a_3c_1^3c_2 + a_2c_1^2c_2^2 + a_1c_1c_2^3 + c_2^4)}.$$

• $n = 5$.

$$\left.\begin{aligned} e^{s_1\tau_1} &= \frac{s_5(s_1c_1 - c_2)}{s_1(s_5c_1 - c_2)}\, e^{s_5\tau_1} \\ e^{s_2\tau_2} &= \frac{s_5(s_2c_1 - c_2)}{s_2(s_5c_1 - c_2)}\, e^{s_5\tau_2} \\ e^{s_3\tau_3} &= \frac{s_5(s_3c_1 - c_2)}{s_3(s_5c_1 - c_2)}\, e^{s_5\tau_3} \\ e^{s_4\tau_4} &= \frac{s_5(s_4c_1 - c_2)}{s_4(s_5c_1 - c_2)}\, e^{s_5\tau_4} \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{x^{(2)}(\tau)}{x(\tau)} &= \frac{a_5(c_1c_3 - c_2^2)}{a_5c_1^2 + a_4c_1c_2 + a_3c_2^2 + a_1c_2c_4 + c_2c_5} \\ \frac{x^{(3)}(\tau)}{x(\tau)} &= \frac{a_5(c_1c_4 - c_2c_3)}{a_5c_1^2 + a_4c_1c_2 + a_3c_2^2 + a_1c_2c_4 + c_2c_5} \\ \frac{x^{(4)}(\tau)}{x(\tau)} &= \frac{a_5(c_1c_5 - c_2c_4)}{a_5c_1^2 + a_4c_1c_2 + a_3c_2^2 + a_1c_2c_4 + c_2c_5} \end{aligned}\right\}$$

and finally

$$\begin{aligned} & x^{(5)}(\tau)e^{a_i\tau} \\ &= \frac{\left(a_5c_1 + a_4c_1c_2 + a_3c_2^2 + a_2c_2c_3 + a_1c_2c_4 + c_2c_5\right)^5}{a_5^4\left(a_5c_1^5 + a_4c_1^4c_2 + a_3c_1^3c_2^2 + a_2c_1^2c_2^3 + a_1c_1c_2^4 + c_2^5\right)} \end{aligned} \tag{28}$$

which agrees with the general formulae (12).

5. Numerical results

• $n = 3$.

We assume the values of the roots

$$s_1 = -1, \quad s_2 = -2, \quad s_3 = -3$$

from Eqs. (21) and (22) we have

$$e^{-\tau} = \frac{(-3)(-c_1 - c_2)}{(-1)(-3c_1 - c_2)} e^{-3\tau},$$

$$e^{-2\tau} = \frac{(-3)(-2c_1 - c_2)}{(-2)(-3c_1 - c_2)} e^{-3\tau}.$$

The common τ must fulfill the equation

$$\left(\frac{3}{2} \cdot \frac{c_2 + 2c_1}{c_2 + 3c_1}\right)^2 = \frac{3}{1} \cdot \frac{c_2 + c_1}{c_2 + 3c_1}$$

from which we have an equation

$$c_2^2 + 4c_1 c_2 = 0.$$

There are two possibilities

1^o. $c_2 = 0$ which gives $\tau_1 = 0$ or
2^o. $c_2 = -4c_1$ which gives $\tau_2 = \ln 3$.

From Eq. (25) we obtain that

$$x(\tau_2) = x^3(\tau_1) \cdot (3)^6 = -\frac{1}{216} \cdot (58c_1 - 4c_3)^3.$$

If we assume $c_1 = 1$ and $x(\tau) = 1$ we have that $c_3 = 28$ and from (23) we obtain $x^{(2)}(\tau_1) = \frac{18}{11}$.

Remark 1. For $n \geq 4$ the proposed method may not lead to success. It is caused by the fact that for example $n = 4$ two additional equations having the same root τ are required. It may be not fulfilled for the given roots s_1, s_2, s_3, s_4.

For that reason another, more general method is proposed. In the proposed method we take for consideration only one additional equation from the set of Eqs. (6). If this Eq. (6) has $(n-1)$ zeroes then with the Eq. (4) we can determine $(n-2)$ ratios $\frac{c_1}{c_n}, \frac{c_2}{c_n}, \ldots, \frac{c_{n-2}}{c_n}$.

In the case when Eq. (6) has less than $(n-1)$ zeroes we obtain a better possibility for the choice of the ratios of the initial conditions. In the case when none from Eqs. (6) has zeroes it is not possible to determine the initial conditions.

In conclusion, if any of Eqs. (6) is fulfilled for $\tau > 0$ then there are sufficient conditions for solutions of Eqs. (4) and (6) together.

6. The second general method for solution of transcendental equations

For the sake of simplicity we illustrate the proposed method on example equations with three exponential functions and with four exponential functions.

We start with Eq. (18). We must consider three possibilities:

1^o. If we take an additional equation $\frac{dx^{(1)}}{dc_3} = 0$ we obtain the following equation in the explicit form

$$s_1(s_2 - s_3)\, e^{s_1\tau} + s_2(s_3 - s_1)\, e^{s_2\tau} + s_3(s_1 - s_2)\, e^{s_3\tau} = 0. \quad (29)$$

2^o. Similarly for $\frac{dx^{(1)}}{dc_2} = 0$ we have

$$s_1(s_2^2 - s_3^2)\, e^{s_1\tau} + s_2(s_3^2 - s_1^2)\, e^{s_2\tau} + s_3(s_1^2 - s_2^2)\, e^{s_3\tau} = 0. \quad (30)$$

3^o. In the last possibility $\frac{dx^{(1)}}{dc_1} = 0$ we obtain

$$s_1 s_2 s_3 \big[(s_2 - s_3)\, e^{s_1\tau} + (s_3 - s_1) e^{s_2\tau} + (s_1 - s_2)\, e^{s_3\tau}\big] = 0. \quad (31)$$

Equations (29), (30) and (31) have the form independent of the initial conditions. Zeroes of these equations play the most important role in this method.

It is well known that solutions of these equations are very sensitive with respect to the exponents.

The proposed method avoids this sensitivity.

We will find the differential equations from which these equations as results must be obtained, then we solve them by application MATLAB programs.

We assume that Eq. (29) represents the function $y(\tau)$ and look for this function.

For these purposes we twice differentiate the function $y(\tau)$ with respect to τ and obtain the differential equation:

$$\frac{d^3 y(\tau)}{d\tau^3} + b_1 \frac{d^2 y(\tau)}{d\tau^2} + b_2 \frac{dy(\tau)}{d\tau} + b_3 y(\tau) = 0, \quad (32)$$

where

$$y(\tau) = s_1(s_2 - s_3)e^{s_1\tau} + s_2(s_3 - s_1)e^{s_2\tau} + s_3(s_1 - s_2)e^{s_3\tau},$$

$$\frac{dy(\tau)}{d\tau} = s_1^2(s_2 - s_3)e^{s_1\tau} + s_2^2(s_3 - s_1)e^{s_2\tau} + s_3^2(s_1 - s_2)e^{s_3\tau},$$

$$\frac{d^2 y(\tau)}{d\tau^2} = s_1^3(s_2 - s_3)e^{s_1\tau} + s_2^3(s_3 - s_1)e^{s_2\tau} + s_3^3(s_1 - s_2)e^{s_3\tau}.$$

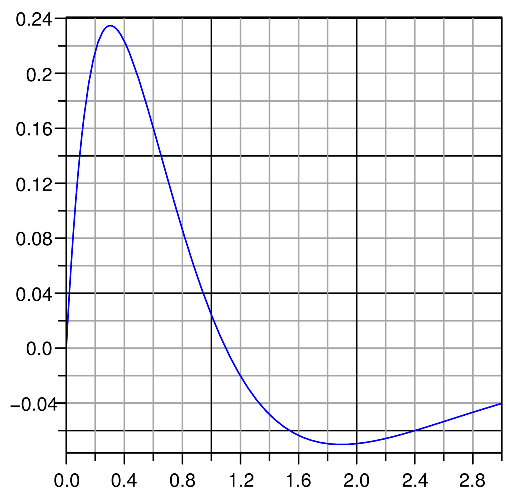

Fig. 1. Solution of Eq. (32) for: $b_1 = 6$, $b_2 = 11$, $b_3 = 6$ and $c_1^* = 0$, $c_2^* = 2$, $c_3^* = -12$

We find the initial conditions assuming $\tau = 0$

$$y(0)=c_1^*=s_1(s_2-s_3)+s_2(s_3-s_1)+s_3(s_1-s_2)=0,$$
$$y^{(1)}(0)=c_2^*=s_1^2(s_2-s_3)+s_2^2(s_3-s_1)+s_3^2(s_1-s_2)\neq 0,$$
$$y^{(2)}(0)=c_3^*=s_1^3(s_2-s_3)+s_2^3(s_3-s_1)+s_3^3(s_1-s_2)\neq 0. \tag{33}$$

Similarly for Eq. (30) we obtain

$$c_1^* = s_1(s_2^2-s_3^2)+s_2(s_3^2-s_1^2)+s_3(s_1^2-s_2^2)\neq 0,$$
$$c_2^* = s_1^2(s_2^2-s_3^2)+s_2^2(s_3^2-s_1^2)+s_3^2(s_1^2-s_2^2)= 0,$$
$$c_3^* = s_1^3(s_2^2-s_3^2)+s_2^3(s_3^2-s_1^2)+s_3^3(s_1^2-s_2^2)\neq 0.$$

and finally for Eq. (31)

$$c_1^* = s_2-s_3+s_3-s_1+s_1-s_2=0,$$
$$c_2^* = s_1(s_2-s_3)+s_2(s_3-s_1)+s_3(s_1-s_2)=0,$$
$$c_3^* = s_1^2(s_2-s_3)+s_2^2(s_3-s_1)+s_3^2(s_1-s_2)\neq 0.$$

Solutions of Eq. (32) for $s_1=-1$, $s_2=-2$, $s_3=-3$ are presented in the corresponding Figs. 1–3.

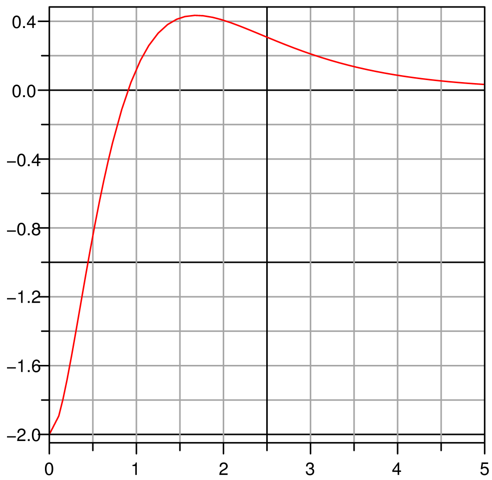

Fig. 2. Solution of Eq. (32) for: $b_1 = 6$, $b_2 = 11$, $b_3 = 6$ and $c_1^* = -2$, $c_2^* = 0$, $c_3^* = 2$

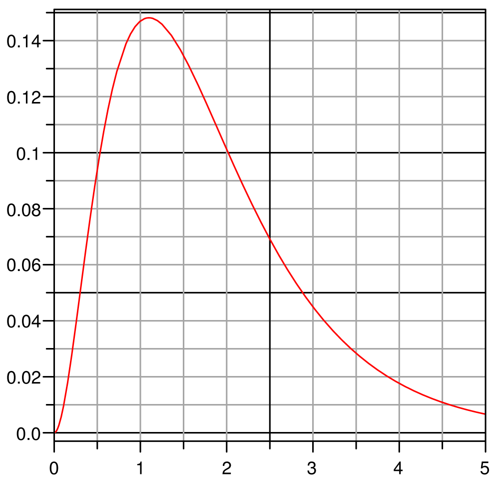

Fig. 3. Solution of Eq. (32) for: $b_1 = 6$, $b_2 = 11$, $b_3 = 6$ and $c_1^* = 0$, $c_2^* = 0$, $c_3^* = 2$

From these we obtain the zeroes of Eqs. (29), (30) and (31). This method can be applied to the equation of n-th-order where $s_1, s_2, \ldots, s_n$ can be real or complex-conjugate. After finding zeroes we return to Eq. (18) and we obtain solutions of this equation for arbitrary initial conditions c_1, c_2, c_3.

7. Numerical results

- $n = 3$.

We assume that

$$s_1 = -1, \quad s_2 = -2, \quad s_3 = -3.$$

From (33) we obtain for equation in point 1^o so that

$$y(0) = c_1^* = 0, \quad y^{(1)}(0) = c_2^* = 2, \quad y^{(2)}(0) = c_3^* = -12.$$

In Fig. 1 the dependence $y(\tau)$ is shown and we see that $y(\tau^*) = 0$ for $\tau_1^* = 0$ and $\tau_2^* = \ln 3 = 1.0986$, there are also coordinates for extremums.

For equation in point 2^o we have

$$y(0) = c_1^* = -2, \quad y^{(1)}(0) = c_2^* = 0, \quad y^{(2)}(0) = c_3^* = 22.$$

In Fig. 2 we find that for $y(\tau^*) = 0$ the value of $\tau^* = 0.905$ and the coordinates of the one extremum are denoted.

For equation in point 3^o we find that for $y(\tau^*) = 0$ we have $\tau^* = 0$ and the coordinates of one extremum are denoted. The initial conditions are

$$y(0) = c_1^* = 0, \quad y^{(1)}(0) = c_2^* = 0, \quad y^{(2)}(0) = c_3^* = 2.$$

- $n = 4$.

Similary as for $n = 3$ we can write the equations for the coefficients c_1, c_2, c_3 and c_4:

$$\begin{aligned}
&s_1e^{s_1\tau}\Big[s_2^3(s_3-s_4)s_3s_4+s_3^3(s_4-s_2)s_2s_4 \\
&\quad +s_4^3(s_2-s_3)s_2s_3\Big] \\
&+s_2e^{s_2\tau}\Big[s_1^3(s_4-s_3)s_3s_4 \\
&\quad +s_3^3(s_1-s_4)s_1s_4+s_4^3(s_3-s_1)s_1s_3\Big] \\
&+s_3e^{s_3\tau}\Big[s_1^3(s_2-s_4)s_2s_4 \\
&\quad +s_2^3(s_4-s_1)s_1s_4+s_4^3(s_1-s_2)s_1s_2\Big] \\
&+s_4e^{s_4\tau}\Big[s_1^3(s_3-s_2)s_2s_3+s_2^3(s_1-s_3)s_1s_3 \\
&\quad +s_3^3(s_2-s_1)s_1s_2\Big]=0,
\end{aligned} \tag{34}$$

$$\begin{aligned}
&s_1e^{s_1\tau}\Big[s_2^3(s_4^2-s_3^2)+s_3^3(s_2^2-s_4^2)+s_4^3(s_3^3-s_2^2)\Big] \\
&+s_2e^{s_2\tau}\Big[s_1^3(s_3^2-s_4^2)+s_3^3(s_4^2-s_1^2)+s_4^3(s_1^2-s_3^2)\Big] \\
&+s_3e^{s_3\tau}\Big[s_1^3(s_4^2-s_2^2)+s_2^3(s_1^2-s_4^2)+s_4^3(s_2^2-s_1^2)\Big] \\
&+s_4e^{s_4\tau}\Big[s_1^3(s_2^2-s_3^2)+s_2^3(s_3^2-s_1^2)+s_3^3(s_1^2-s_2^2)s_1s_2\Big]=0,
\end{aligned}$$

$$s_1 e^{s_1\tau}\Big[s_2^3(s_3-s_4)+s_3^3(s_4-s_2)+s_4^3(s_2-s_3)\Big]$$
$$+s_2 e^{s_2\tau}\Big[s_1^3(s_4-s_3)+s_3^3(s_1-s_4)+s_4^3(s_3-s_1)\Big]$$
$$+s_3 e^{s_3\tau}\Big[s_1^3(s_2-s_4)+s_2^3(s_4-s_1)+s_4^3(s_1-s_2)\Big]$$
$$+s_4 e^{s_4\tau}\Big[s_1^3(s_3-s_2)+s_2^3(s_1-s_3)+s_3^3(s_2-s_1)\Big]=0,$$
$$s_1 e^{s_1\tau}\Big[s_2^2(s_4-s_3)+s_3^2(s_2-s_4)+s_4^2(s_3-s_2)\Big]$$
$$+s_2 e^{s_2\tau}\Big[s_1^2(s_3-s_4)+s_3^2(s_4-s_1)+s_4^2(s_1-s_3)\Big]$$
$$+s_3 e^{s_3\tau}\Big[s_1^2(s_4-s_2)+s_2^2(s_1-s_4)+s_4^2(s_2-s_1)\Big]$$
$$+s_4 e^{s_4\tau}\Big[s_1^2(s_2-s_3)+s_2^2(s_3-s_1)+s_3^2(s_1-s_2)\Big]=0.$$

Similarly to Eq. (32) we have here

$$\frac{\mathrm{d}^4y(\tau)}{\mathrm{d}\tau^4}+10\,\frac{\mathrm{d}^3y(\tau)}{\mathrm{d}\tau^3}+35\,\frac{\mathrm{d}^2y(\tau)}{\mathrm{d}\tau^2}+50\,\frac{\mathrm{d}y(\tau)}{\mathrm{d}\tau}+24y(\tau)=0, \tag{35}$$

where we assume $s_1=-1$, $s_2=-2$, $s_3=-3$, $s_4=-4$. From Eq. (34) we have the for coefficient of c_1 the equation

$$e^{3\tau}-3e^{2\tau}+3e^{\tau}-1=0.$$

The solutions of it are

$$\tau_1=\tau_2=\tau_3=0.$$

Similarly for c_2 we obtain only one solution for equation

$$13e^{3\tau}-57e^{2\tau}+63e^{\tau}-22=0.$$
$$\tau_1=1.07366.$$

For c_3 the equation is

$$3e^{3\tau}-16e^{2\tau}+21e^{\tau}-8=0$$

and the solutions are (see Fig. 4)

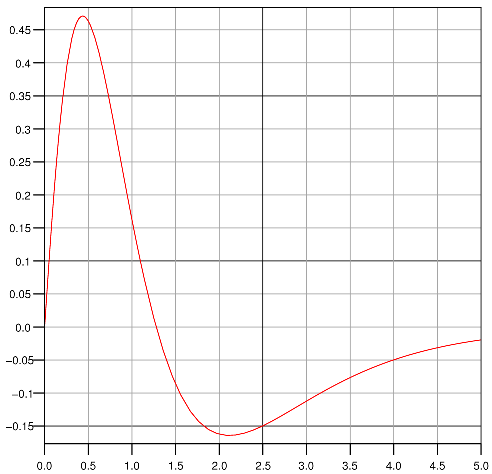

Fig. 4. Solution of Eq. (35) for $c_1^*=0$, $c_2^*=2$, $c_3^*=0$, $c_4^*=-70$

$$\tau_1=0,\qquad \tau_2=\ln 3.591=1.2783.$$

and finally for c_4 the equation

$$e^{3\tau}-6e^{2\tau}+9e^{\tau}-4=0$$

and the solutions (see Fig. 5)

$$\tau_1=\tau_2=0,\qquad \tau_3=\ln 4=1.386.$$

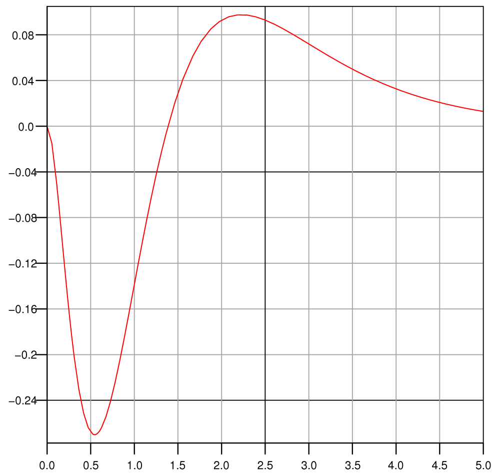

Fig. 5. Solution of Eq. (35) for $c_1^*=0$, $c_2^*=0$, $c_3^*=-6$, $c_4^*=60$

8. Conclusions

The solution of the transcendental equation in an analytical form are presented. The methods are based on the assumption that we look for extremal points τ with respect to the initial conditions c_i.

The existence of such points is connected with the roots of the characteristic equation. These roots may be shifted in the desired location using the well known methods of the poles and zeros locations see [5]. This method opens a new possibility of design of control systems, where the concrete extremal points τ and $x(\tau)$ are required.

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