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Revisiting In-Sample Bayesian Comparison of Alternative Multiplicative Error Models: Evidence from Trade Durations and Intraday Trading Volumes for the Polish Stock Market

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Abstract

The main aim of the paper is to compare the modelling ability of several alternative multiplicative error models for describing the dynamics of trade durations and intraday trading volumes. To formally compare the relative explanatory power of multiplicative error specifications, the Bayesian rules of comparing statistical models are applied. In the paper, we revisit Bayesian model comparison and compare the Bayes factors obtained using different approximations of the values of the marginal data densities. The Newton and Raftery's harmonic mean estimator, the corrected harmonic mean estimator of Pajor and Osiewalski, the corrected arithmetic mean estimator proposed by Pajor, and a standard Monte Carlo with importance sampling technique are approximations used. We consider twelve multiplicative error models that differ in the structure of the conditional mean equation and the distribution of innovations. The analysis considers models with the Burr and generalised gamma distributions for the error term. The MCMC methods are suitably adopted to obtain samples from the posterior densities of interest. The empirical part of the work includes modelling historical trade durations and more recent intraday trading volumes for selected equities on the Polish stock market.

Keywords: multiplicative error models, trade duration, intraday volume, Bayesian model selection, Bayes factors

JEL Classification: C50, C58, C11, C22

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1 Introduction

Methods and models for high-frequency data analysis are still of growing importance in empirical finance and financial econometrics. The accessibility of information about every individual transaction and its characteristics means that data describing individual financial events, i.e., financial ultra-high-frequency data, have become widely used in empirical analyses. The modelling of high-frequency financial data is a crucial aspect of research on market microstructure theories and liquidity issues. Very important tasks in financial practice include analysing and forecasting intraday volatility, intraday trading volumes, financial durations, bid-ask spreads, market depth, and trading costs to better optimise order placement and execution. Transaction durations, among many various financial durations, play precisely a key role in market microstructure theory (see Glosten and Milgrom, 1985; Easley and O'Hara, 1987), where they are used as a proxy variable indicative of the presence of new information in the market. Trading volumes and their relationship with financial asset returns are also significant for understanding the operational efficiency and dynamics of private and public information in a financial market.

In recent decades, the class of multiplicative error models (MEMs), as named by Engle (2002), has been successfully applied to appropriately capture the dynamics of high-frequency data. Of these, autoregressive conditional duration (ACD) models originally introduced by Engle and Russell (1998) have become a useful econometric framework in modelling the durations between selected events of the transaction process (trade durations, price durations, volume durations, quote durations). In turn, autoregressive conditional volume (ACV) models have been proposed by Manganelli (2005) to model the dynamics of intraday trading volumes. The underlying basic idea of MEMs is that the conditional mean function interacts multiplicatively with positive-valued error terms. In their seminal paper, Engle and Russell (1998) developed a linear ACD model. To provide additional flexibility in the linear specification of the conditional mean equation in the original ACD model, numerous extensions have been proposed in the financial literature. These extensions include the logarithmic ACD model of Bauwens and Giot (2000) (see also Ng, Peiris and Gerlach, 2014), Box-Cox ACD model of Dufour and Engle (2000), threshold ACD model of Zhang, Russell and Tsay (2001), the asymmetric ACD and asymmetric logarithmic ACD models of Bauwens and Giot (2003), the stochastic conditional duration model of Bauwens and Veredas (2004), the stochastic volatility duration model of Ghysels, Gouriéroux and Jasiak (2004), augmented ACD model of Fernandes and Grammig (2006) or the autoregressive conditional marked duration model of Kwok, Li and Yu (2009). Undoubtedly, in the literature, ACD-type processes have already enjoyed a well-established position. Fernandes and Grammig (2006), Pacurar (2008), Hautsch (2012), and Bhogal and Variyam (2019) provide informative, relevant, and interesting reviews on the various ACD models and their applications.

Additionally, it has already been established that the choice of the error distribution for the observed process in ACD/ACV modelling is crucial, as it affects the ACD/ACV model's ability to capture the characteristics of duration or volume data accurately. Engle and Russell (1998) consider the linear ACD model with the standard exponential distribution (i.e., with parameter equal to 1) for innovations, and, as the simplest extension, the Weibull distribution with a shape parameter to be estimated. However, the exponential and Weibull distributions have been too restrictive, and thus other error distributions have been proposed. Examples of distributions allowing for greater flexibility and more complex modelling of duration processes encompass the log-normal distribution (Allen et al., 2008; Xu, 2013), the generalised gamma (GG) distribution (Lunde, 1999), the Burr type XII (Burr) distribution (Grammig and Maurer, 2000), the generalised beta of the second kind (GB2) distribution or the generalised F distribution as a special case of GB2 (Hautsch, 2003; Bień-Barkowska, 2016, 2017), the Pareto distribution (De Luca and Zuccolotto, 2006), the Birnbaum-Saunders (BS) distribution (Bhatti, 2010; Leiva et al., 2014; Cunha et al., 2020; Fernando, Jeremias and Saulo, 2021), the inverse Gaussian distribution (Balakrishna and Rahul, 2014), the extended Weibull distribution (Yatigammana, Choy and Chan, 2016), the Fréchet distribution (Zheng, Li and Li, 2016), the mixture of GB2 distributions (Yatigammana, Chan and Gerlach, 2019), the extended generalised inverse Gaussian (EGIG) distribution (Tan et al., 2022). All these distributions demonstrate different hazard function shapes, encompassing monotonically increasing, monotonically decreasing, and non-monotonic patterns.

In the case of ACD models, inference about the parameters is usually based on the classical approach, i.e., the Quasi-Maximum Likelihood (QML) method or the Maximum Likelihood (ML) method. However, it must be emphasised that the properties of QML estimators hold for the basic linear ACD model, assuming an exponential distribution for the innovation term, and need not necessarily hold for more general specifications of ACD models. Thus, given the problems surrounding the use of ML and QML estimation methods (see Huptas, 2014) and furthermore still the not so-well-known properties of maximum likelihood estimators for ACD/ACV models with conditional distributions other than the exponential distribution, the Bayesian approach relying on the Monte Carlo techniques has appeared a theoretically consistent estimation method, even if it is numerically demanding and time consuming. Only a few publications can be found on the Bayesian approach to duration and intraday volume modelling in the financial markets exclusively by means of ACD/ACV-type models. The studies utilising this approach include Brownlees and Vannucci (2013), Huptas (2013, 2014), Men, Kolkiewicz, and Wirjanto (2015a, 2015b), Gerlach, Peiris, and Lin (2016), Fernando, Jeremias and Saulo (2021), and Tabash et al. (2024). Moreover, regarding a formal Bayesian comparison of ACD models, the only known work is an unpublished manuscript of Huptas (2013).

The main objective of the paper is to compare the modelling ability of alternative multiplicative error models (MEMs) for describing the dynamics of trade durations and intraday trading volumes. In doing so, in this paper, we revisit the study presented in the unpublished manuscript of the Ph.D. dissertation of Huptas (2013) and extend it considerably. In a narrow sense, we simply return to a formal Bayesian comparison of ACD models. To formally compare the relative explanatory power of multiplicative error specifications, Bayes factors are applied and obtained through Markov Chain Monte Carlo (MCMC) techniques. On the other hand, in a broad sense, we extend the study on Bayesian comparison of multiplicative error models by using different approximations of the marginal data density value, which is the key quantity for Bayesian model selection, to verify the robustness of the models' rankings across these approximations. The Newton and Raftery's (1994) harmonic mean estimator, the corrected harmonic mean estimator of Pajor and Osiewalski (2013), the corrected arithmetic mean estimator proposed by Pajor (2017), and a standard Monte Carlo with importance sampling technique are the approximations used. We consider twelve multiplicative error models that differ in the structure of the conditional mean equation and the distribution of innovations. In particular, the linear ACD (or ACV) model, logarithmic ACD (or ACV) model, Box-Cox ACD (or ACV) model, asymmetric logarithmic ACD (or ACV) model, augmented Box-Cox ACD (or ACV) model, and augmented (Hentschel) ACD (or ACV) model are taken into account. The analysis considers models with the Burr and generalised gamma distributions for the innovation term. The MCMC methods, including the Metropolis-Hastings algorithm and a standard Monte Carlo with importance sampling technique, are suitably adopted to obtain samples from the posterior densities of interest and Bayes factors. The empirical part of the work includes modelling of very historical trade durations and more recent intraday trading volumes of selected equities listed on the Polish stock exchange. It should be noted that all models considered in this paper, and partially these datasets, are admittedly from several years ago. However, the use of these models and datasets is intended to facilitate a direct comparison of the results obtained from these methods in this analysis with those models and techniques of Huptas (2013).

This research adds new merits to the existing literature on modelling high-frequency data and ACD/ACV models. First, to the best of our knowledge, a formal Bayesian comparison of various ACD/ACV models through their Bayes factors – as an alternative method for selection of competing models in relation to ACD/ACV models – has not appeared in the econometric literature so far, except for the unpublished study of Huptas (2013). In the context of the financial and microstructure literature, this application of the proposed approach therefore offers a fresh perspective on the issue of the selection of ACD and ACV models. Second, in this study, we assess the stability of Bayes factors and MEMs' ranks with respect to the four various approximations of the marginal data density values. Third, this research contributes to the line of research on the microstructure of the Polish stock market, which remains one of the most important and rapidly growing markets in Central and Eastern Europe.

The remainder of the paper is structured as follows. In Section 2, the Bayesian statistical methodology, including Bayes factors for model selection, and some computational aspects, are briefly presented. In Section 3, the model framework is introduced, and the specifications of multiplicative error models are described in detail. Section 4 presents the data, ranks the MEM-type models using Bayes factors obtained by means of different approximations of the marginal data densities, and discusses the empirical results. Section 5 of the paper contains concluding remarks.

2 Bayesian methodology and Bayes factors for model selection

Let $\mathbf{x} = (x_1, \ldots, x_T) \in X \subset \mathbb{R}^T$ denote the vector of observations (trade durations or intraday trading volumes) x_i (in this study and approach it is a vector of deseasonalized observations). The vector of unknown parameters in the model M_j is marked $\theta_j \in \Theta_j$, where $j = 1, \ldots, k$. The j-th Bayesian model M_j describing the dynamics of x_i is uniquely determined by the density of the joint distribution of the vector of observations and the vector of parameters:

$$p\left(\mathbf{x}, \theta_j \middle| x_{(0)}, M_j\right) = p\left(\mathbf{x} \middle| \theta_j, x_{(0)}, M_j\right) p\left(\theta_j | M_j\right), \tag{1}$$

where $x_{(0)}$ denotes initial conditions, $p\left(\mathbf{x} \middle| \theta_j, x_{(0)}, M_j\right)$ is the density of the conditional distribution of $\mathbf{x}$ when θ_j is given, i.e. sampling distribution, and $p\left(\theta_j | M_j\right)$ stands for prior density function. Estimation of the vector of model parameters involves determining the conditional distribution for this vector of parameters given the vector of observations $\mathbf{x}$ i.e. a posterior distribution with a density determined as follows:

$$p\left(\theta_j \middle| \mathbf{x}, x_{(0)}, M_j\right) = \frac{p\left(\mathbf{x} \middle| \theta_j, x_{(0)}, M_j\right) p\left(\theta_j | M_j\right)}{p\left(\mathbf{x} \middle| x_{(0)}, M_j\right)}, \tag{2}$$

where $p\left(\mathbf{x} \middle| x_{(0)}, M_j\right)$ denotes marginal data density in the model M_j expressed by the following formula:

$$p\left(\mathbf{x} \middle| x_{(0)}, M_j\right) = \int_{\Theta_j} p\left(\mathbf{x} \middle| \theta_j, x_{(0)}, M_j\right) p\left(\theta_j | M_j\right) d\theta_j. \tag{3}$$

The statistical inference is based on the posterior distributions given by formula (2), while the marginal densities $p\left(\mathbf{x} \middle| x_{(0)}, M_j\right)$ are the crucial components in model comparison. Comparison of competing models (we assume that models are mutually non-nested and jointly exhaustive models) is based on posterior probabilities calculated for the models considered in the study (see Zellner, 1971; Osiewalski, 2001) using Bayes' theorem:

$$P\left(M_j \middle| \mathbf{x}, x_{(0)}\right) = \frac{P\left(M_j\right) p\left(\mathbf{x} \middle| x_{(0)}, M_j\right)}{\sum_{i=1}^{k} P\left(M_i\right) p\left(\mathbf{x} \middle| x_{(0)}, M_i\right)} \qquad j = 1, \ldots, k, \tag{4}$$

where $P(M_j)$ denotes prior probability of model M_j and $p\left(\mathbf{x}|x_{(0)}, M_j\right)$ is the marginal data density in the model M_j expressed by the formula (3).
The models' posterior probabilities determined on the basis of formula (4) depend on prior probabilities $P(M_j)$ used by the researcher. The simplest solution encountered in pertinent literature involves the use of the same prior probabilities $P(M_j)$, that is, use of a discrete uniform distribution such that $P(M_j) = 1/k$ for $j = 1, \ldots, k$. Another approach favours choosing prior probabilities $P(M_j)$ in accordance with the so-called Ockham's razor principle, i.e., admitting simpler models as the more likely ones (Osiewalski, 2001). Of course, this situation raises the problem of determining which models are simpler. One approach is to adhere to the principle that simpler models are those that have fewer parameters, i.e., *a priori* prefer more sparsely parameterized models. Then prior probabilities $P(M_j)$ can be calculated based on the condition:

$$P(M_j) \propto 2^{-l_j}, \tag{5}$$

where l_j is a dimension of the specific parameter space of a j-model.
Within the Bayesian inference when pairs of models are compared it also can be used the so-called posterior odds ratio $P\left(M_i|\mathbf{x}, x_{(0)}\right)/P\left(M_j|\mathbf{x}, x_{(0)}\right)$. In this case, the effect of prior probabilities is reduced. The posterior odds ratio is a product of the prior odds ratio and the ratio of the marginal data densities:

$$P_{ij} = \frac{P\left(M_i|\mathbf{x}, x_{(0)}\right)}{P\left(M_j|\mathbf{x}, x_{(0)}\right)} = \frac{P(M_i)}{P(M_j)} \cdot \frac{p\left(\mathbf{x}|x_{(0)}, M_i\right)}{p\left(\mathbf{x}|x_{(0)}, M_j\right)}, \tag{6}$$

where the ratio of the marginal data densities is called the Bayes factor:

$$B_{ij} = \frac{p\left(\mathbf{x}|x_{(0)}, M_i\right)}{p\left(\mathbf{x}|x_{(0)}, M_j\right)}. \tag{7}$$

If models' prior probabilities equal $P(M_i) = P(M_j)$, then the posterior odds ratio equals the Bayes factor:

$$P_{ij} = \frac{P\left(M_i|\mathbf{x}, x_{(0)}\right)}{P\left(M_j|\mathbf{x}, x_{(0)}\right)} = \frac{P(M_i)}{P(M_j)} \cdot \frac{p\left(\mathbf{x}|x_{(0)}, M_i\right)}{p\left(\mathbf{x}|x_{(0)}, M_j\right)} = B_{ij}. \tag{8}$$

The Bayes factor gives information about the valuation of the model M_i relative to the model M_j, which arises exclusively from the observed data and assumptions underlying the Bayesian models.
It is worth noting that posterior probability and Bayes factors in the models strongly depend on the marginal data density determined in formula (3). Analytical computation of the marginal data density value is extremely complex, or indeed impossible, so literature recommends different numerical methods to estimate it directly or to compute Bayes factors (Newton and Raftery, 1994; Chib, 1995; Kass and Raftery, 1995; for a review see, e.g., Ardia et al., 2012; Friel and Wyse, 2012).

A widely used method of approximating the marginal likelihood is the Laplace-Metropolis approximation (see Kass and Raftery, 1995; Raftery, 1996). Among the numerical methods used to estimate the marginal likelihood, a very popular approach is the approach of Chib (1995). The method of Chib (1995) aims at the estimation of the marginal likelihood from Gibbs sampling output. The idea of Chib (1995) was subsequently extended by Chib and Jeliazkov (2001). Very commonly used approaches for computing Bayes factors or marginal posterior probabilities are based on the reversible jump algorithm defined by Green (1995), on bridge sampling methods proposed by Meng and Wong (1996) or on an extension of the bridge sampling identity of Meng and Wong (1996) and combined with the reversible jump algorithm suggested by Bartolucci, Scaccia and Mira (2006).
One of the most popular ways of estimating $p\left(\mathbf{x}\middle|x_{(0)}, M_j\right)$, due to its simplicity, assuming a proper prior distribution involves the use of Newton and Raftery's method (Newton and Raftery, 1994). When we have a pseudo-random sample ${\theta_j}^{(1)}, \dots, {\theta_j}^{(N)}$ from a joint posterior distribution of unknown parameters, we can approximate the value of the marginal data density by means of a harmonic mean of the likelihood values. The Harmonic Mean Estimator (HME) proposed by Newton and Raftery (1994) is given by the following formulae:

$$\widehat{p}_{HME}\left(\mathbf{x}\middle|x_{(0)}, M_j\right) = \left(\frac{1}{N}\sum_{q=1}^{N}\frac{1}{p\left(\mathbf{x}\middle|{\theta_j}^{(q)}, x_{(0)}, M_j\right)}\right)^{-1}, \tag{9}$$

where ${\theta_j}^{(1)}, \dots, {\theta_j}^{(N)}$ are drawn from the posterior distribution of parameters by means of a Markov Chain Monte Carlo method. Even though HME is consistent, very easy to calculate, a universal technique to approximate the marginal data density and, it seems by far the most commonly used in the literature, it has some serious theoretical weaknesses. It can be unstable, is without finite asymptotic variance, and it overestimates the marginal data density.
For this reason a number of alternatives to HME have been proposed (see, e.g., Gelfand and Dey, 1994; Chib and Jeliazkov, 2001) as well as methods to improve on it (see, e.g., Raftery et al., 2007; Lenk, 2009; Osiewalski and Osiewalski, 2013; Pajor and Osiewalski, 2013; Pajor, 2017). The adjusted Harmonic Mean Estimator proposed by Lenk (2009) and correcting the so-called "pseudo-bias" of HME, is given by

$$\widehat{p}_{AHME}\left(\mathbf{x}\middle|x_{(0)}, M_j\right) = \widehat{P}(A)\left(\frac{1}{N}\sum_{q=1}^{N}\frac{1}{p\left(\mathbf{x}\middle|{\theta_j}^{(q)}, x_{(0)}, M_j\right)}\right)^{-1}, \tag{10}$$

where $\widehat{P}(A)$ is an assessment of the prior probability of subset $A \subseteq \Theta$, of which the posterior probability is greater than $1-\epsilon$ for small $\epsilon > 0$. Pajor and Osiewalski (2013) have shown that Lenk's correction can be used regardless of the posterior probability accumulated in the chosen subset of the parameters' space, so it is possible to select a subset A and improve numerical properties of the adjusted Harmonic Mean estimator. They have proposed the following Corrected Harmonic Mean estimator (CHME):

$$\widehat{p}_{CHME}\left(\mathbf{x}\middle|x_{(0)},M_j\right)=\widehat{P}(A)\left(\frac{1}{N}\sum_{q=1}^{N}\frac{I_A\left(\theta_j{}^{(q)}\right)}{p\left(\mathbf{x}\middle|\theta_j{}^{(q)},x_{(0)},M_j\right)}\right)^{-1}, \tag{11}$$

or equivalently,

$$\widehat{p}_{A,CHME}\left(\mathbf{x}\middle|x_{(0)},M_j\right)=\frac{\widehat{P}(A)}{\widehat{P}\left(A|\mathbf{x}\right)}\left(\frac{1}{N}\sum_{q=1}^{N}\frac{1}{p\left(\mathbf{x}\middle|\theta_{j;A}^{(q)},x_{(0)},M_j\right)}\right)^{-1}, \tag{12}$$

where $\theta_j{}^{(1)},\ldots,\theta_j{}^{(N)}$ and $\theta_{j;A}^{(1)},\ldots,\theta_{j;A}^{(N)}$ are drawn from $p\left(\theta_j\left|\mathbf{x},x_{(0)},M_j\right.\right)$ and $p\left(\theta_j\middle|\mathbf{x},x_{(0)},M_j;A\right)$, respectively.
In turn, motivated by the problems with HME and its corrections, Pajor (2017) proposed an estimator of the marginal data density value based on simulation from the prior distribution restricted to any (but reasonable) subset of the space of parameters. The Corrected Arithmetic Mean estimator (CAME) proposed by Pajor (2017) is expressed as

$$\widehat{p}_{CAME}\left(\mathbf{x}\middle|x_{(0)},M_j\right)=\frac{\widehat{P}(A)}{\widehat{P}\left(A|\mathbf{x}\right)}\cdot\frac{1}{N}\sum_{q=1}^{N}p\left(\mathbf{x}\middle|\theta_j{}^{(q)},x_{(0)},M_j\right)I_A\left(\theta_j{}^{(q)}\right), \tag{13}$$

where $\theta_j{}^{(1)},\ldots,\theta_j{}^{(N)}$ are drawn from the prior distribution $p\left(\theta_j|M_j\right)$, or equivalently,

$$\widehat{p}_{A,CAME}\left(\mathbf{x}\middle|x_{(0)},M_j\right)=\frac{\widehat{P}(A)}{\widehat{P}\left(A|\mathbf{x}\right)}\cdot\frac{1}{N}\sum_{q=1}^{N}p\left(\mathbf{x}\middle|\theta_{j;A}^{(q)},x_{(0)},M_j\right), \tag{14}$$

where $\theta_{j;A}^{(1)},\ldots,\theta_{j;A}^{(N)}$ are drawn from the prior distribution, restricted to the subset A, $p\left(\theta_j\middle|\mathbf{x},x_{(0)},M_j;A\right)$. In order to assess the posterior probability of the subset A, $\widehat{P}\left(A|\mathbf{x}\right)$, sampling from the posterior distribution is also required.
Finally, it is worth noting that the marginal data density can also be approximated using the pure Monte Carlo–Importance Sampling (MC-IS) method of Geweke (1989). The pure MC-IS estimator of the marginal data density is a sample arithmetic mean of values of the weighting function and is given by the following formulae:

$$\widehat{p}_{MCIS}\left(\mathbf{x}\middle|x_{(0)},M_j\right)=\frac{1}{N}\sum_{q=1}^{N}w\left(\theta_j{}^{(q)}\right)=\frac{1}{N}\sum_{q=1}^{N}\frac{p\left(\mathbf{x}\middle|\theta_j{}^{(q)},x_{(0)},M_j\right)p\left(\theta_j{}^{(q)}\middle|M_j\right)}{s\left(\theta_j{}^{(q)}\right)}, \tag{15}$$

where $w(\cdot)$ denotes the so-called weighting function, $s(\cdot)$ is called the importance function (defined on the positive support) and ${\theta_j}^{(1)}, \ldots, {\theta_j}^{(N)}$ are drawn independently from the distribution with density function $s(\cdot)$.
Since this paper builds on previous empirical research of the unpublished manuscript of Huptas (2013) and we return to the Bayesian comparison of MEM-type models (in particular to ACD models), to verify stability of model comparisons, competing models in this study are compared pairwise through the Bayes factors obtained using four different approximations of the marginal data densities such as the Harmonic Mean Estimator, the Corrected Harmonic Mean Estimator, the Corrected Arithmetic Mean Estimator, and the standard MC-IS estimator. It must also be noted that the issue of stability of Bayes factors and model ranks with respect to the method used to approximate marginal data density values has already been discussed in the literature by Osiewalski and Pipień (2004). In the context of Bayesian comparison of bivariate ARCH-type models, authors have used and compared three alternative approximations of the marginal data density values, namely the Harmonic Mean Estimator, the Chib and Jeliazkov's (2001) estimator, and the Laplace approximation.

3 Basic model framework and competing MEM-type specifications

Let x_i denote an observed time series of transaction durations or intraday trading volumes for $i = 1, 2, ..., T$ with T standing for the total number of observations. The transaction duration is the time interval between successive transactions that occur at the moments t_{i-1} and t_i. The general ACD/ACV model for the observations x_i is defined multiplicatively as:

$$x_i = \Psi_i \cdot \varepsilon_i, \tag{16}$$

$$\Psi_i = E\left(x_i | \theta_j; \mathcal{F}_{i-1}\right), \tag{17}$$

where $\mathcal{F}_{i-1}$ denotes the set of information available at time t_{i-1}, θ_j is the vector of unknown parameters of model M_j, Ψ_i represents the conditional expected duration or trading volume, and ε_i denotes an error term, which is a sequence of positive, identically and independently distributed random variables with density function $f_\varepsilon(\varepsilon_i)$, mean value $E\left(\varepsilon_i\right) = 1$ and a finite variance. It can be noted that equations (16)-(17) formulate a very general setup that allows for a variety of specific models.
The basic specification of the conditional mean, Ψ_i, which remains in line with Engle and Russell (1998) (for ACD model) and Manganelli (2005) (for ACV model), is usually assumed to have the following autoregressive linear form and is called a linear ACD/ACV model – ACD(1,1) (or ACV(1,1)):

$$\Psi_i = \omega + \alpha \cdot x_{i-1} + \beta \cdot \Psi_{i-1}, \tag{18}$$

where $\omega > 0,\ \alpha \geq 0,\ \beta \geq 0,\ \alpha + \beta < 1$. These inequality restrictions are imposed in order to ensure positive conditional durations/volumes for all possible realizations of random variables x_i, existence of the unconditional mean of duration/trading volume, and stationarity of the model.

With respect to specifications underlying the description of the functional form of the conditional expected duration/trading volume, Ψ_i, below we present only selected augmented ACD/ACV specifications which are used in the empirical study described in the following part of the article. These specifications include:

i) logarithmic ACD (or ACV) model – LACD(1,1) (or LACV(1,1)) model (Bauwens and Giot, 2000; Lunde, 1999):

$$\ln \Psi_i = \omega + \alpha \cdot \ln \varepsilon_{i-1} + \beta \cdot \ln \Psi_{i-1} = \omega + \alpha \cdot \ln x_{i-1} + (\beta - \alpha) \cdot \ln \Psi_{i-1}, \quad (19)$$

where $|\beta| < 1$ (the condition of non-explosiveness and strict stationarity of the process),

ii) Box-Cox ACD (or ACV) model – BCACD(1,1) (or BCACV(1,1)) model (Hautsch, 2002):

$$\Psi_i^{\delta_1} = \omega + \alpha \cdot \varepsilon_{i-1}^{\delta_2} + \beta \cdot \Psi_{i-1}^{\delta_1}, \quad (20)$$

where $\omega > 0,\ \alpha > 0,\ 0 < \beta < 1$ (the condition of non-explosiveness and strict stationarity of the process), $\delta_1 > 0$, $\delta_2 > 0$,

iii) asymmetric logarithmic ACD (or ACV) model – AsLACD(1,1) (or AsLACV(1,1)) model (Fernandes and Grammig, 2006):

$$\ln \Psi_i = \omega + \alpha \cdot [|\varepsilon_{i-1} - b| + c \cdot (\varepsilon_{i-1} - b)] + \beta \cdot \ln \Psi_{i-1}, \quad (21)$$

where $b > 0$ and $|\beta| < 1$ (the condition of non-explosiveness and strict stationarity of the process),

iv) augmented Box-Cox ACD (or ACV) model – ABCACD(1,1) (or ABCACV(1,1)) model (Hautsch, 2004, 2012):

$$\Psi_i^{\delta_1} = \omega + \alpha \cdot [|\varepsilon_{i-1} - b| + c \cdot (\varepsilon_{i-1} - b)]^{\delta_2} + \beta \cdot \Psi_{i-1}^{\delta_1}, \quad (22)$$

where $\omega > 0,\ \alpha > 0,\ 0 < \beta < 1$ (the condition of non-explosiveness and strict stationarity of the process), $\delta_1 > 0$, $\delta_2 > 0$, $b > 0$, $|c| \leq 1$ (this restriction has to be imposed in order to circumvent complex values whenever $\delta_2 \neq 1$),

v) augmented ACD (or ACV) model – AACD(1,1) (or AACV(1,1)) model (Fernandes and Grammig, 2006):

$$\Psi_i^{\delta_1} = \omega + \alpha \cdot \Psi_{i-1}^{\delta_1} \cdot [|\varepsilon_{i-1} - b| + c \cdot (\varepsilon_{i-1} - b)]^{\delta_2} + \beta \cdot \Psi_{i-1}^{\delta_1}, \quad (23)$$

where $\omega > 0,\ \alpha > 0,\ 0 < \beta < 1$ (the condition of non-explosiveness of the process), $\delta_1 > 0$, $\delta_2 > 0$, $b > 0$, $|c| \leq 1$ (this restriction has to be imposed in order to circumvent complex values whenever $\delta_2 \neq 1$).

The functional forms described above for the conditional mean Ψ_i are considered in line with Huptas (2013) and do not represent an exhaustive set of all possible directions of generalisations of the basic ACD/ACV process. The presentation of these generalisations draws primarily on Fernandes and Grammig (2006) as well as Hautsch (2004, 2012).

As far as the distribution of innovations ε_i is concerned, the following probability distributions defined on the set of positive real numbers can be considered: the exponential distribution, Weibull distribution, the log-normal distribution, the gamma distribution, the generalized gamma distribution and Burr distribution, among others (see Engle and Russell 1998; Lunde, 1999; Grammig and Maurer, 2000; Bauwens and Giot, 2001, 2003; Hautsch, 2002; Bauwens et al., 2004; De Luca and Gallo, 2008; Fernandes and Grammig, 2006; De Luca and Zuccolotto, 2006; Allen et al., 2008). In this study in ACD/ACV models (in line with Huptas (2013)) we assume that ε_i follows a Burr type XII distribution $\{\varepsilon_i\} \sim Burr(\lambda,\ \kappa,\ \eta)$ with parameter $\lambda = \eta^{1+\frac{1}{\kappa}}\Gamma\left(1+\frac{1}{\eta}\right) / \left[\Gamma\left(1+\frac{1}{\kappa}\right)\Gamma\left(\frac{1}{\eta}-\frac{1}{\kappa}\right)\right]$ or a generalized gamma distribution $\{\varepsilon_i\} \sim GG(\lambda,\ \gamma,\ \nu)$ with parameter $\lambda = \left(\Gamma\left(\frac{\nu}{\gamma}\right)/\Gamma\left(\frac{1+\nu}{\gamma}\right)\right)^{\gamma}$.

Thus, in order to model transaction durations or intraday trading volumes in the empirical part of the paper, twelve (Bayesian) ACD/ACV models, with one of six specifications of the conditional mean, Ψ_i, given by (18), (19), (20), (21), (22) or (23), are specified: six models with the Burr distribution for the random term and six models with the generalised gamma distribution for innovations. These include: Burr-ACD/ACV, Burr-LACD/LACV, Burr-BCACD/BCACV, Burr-AsLACD/AsLACV, Burr-ABCACD/ABCACV, Burr-AACD/AACV, GGam-ACD/ACV, GGam-LACD/LACV, GGam-BCACD/ABACV, GGam-AsLACD/AsLACV, GGam-ABCACD/ABCACV and GGam-AACD/AACV models.

The alternative ACD/ACV models proposed are estimated and compared in the Bayesian framework, see, e.g. Zellner (1971), Osiewalski (2001), Geweke (2005).

The sample density is the product of appropriate conditional densities of transaction durations (in the case of ACD models) or intraday trading volumes (in the case of ACV models):

$$p\left(\mathbf{x}|\theta_j, x_{(0)}, M_j\right) = \prod_{i=1}^{T} f\left(x_i|\theta_j, x_{(0)}, M_j; \mathcal{F}_{i-1}\right) \tag{24}$$

where the conditional duration/intraday trading volume density $f\left(x_i|\theta_j, x_{(0)}, M_j; \mathcal{F}_{i-1}\right)$ is expressed by the following formulas:

1. for an ACD/ACV model with a Burr distribution:

$$f\left(x_i|\theta_j, x_{(0)}, M_j; \mathcal{F}_{i-1}\right) = \\ = \frac{\kappa}{x_i}\left(\frac{x_i}{\Psi_i}\cdot\frac{\Gamma\left(1+\frac{1}{\kappa}\right)\Gamma\left(\frac{1}{\eta}-\frac{1}{\kappa}\right)}{\eta^{1+\frac{1}{\kappa}}\Gamma\left(1+\frac{1}{\eta}\right)}\right)^{\kappa}\cdot \\ \cdot \left[1+\eta\left(\frac{x_i}{\Psi_i}\cdot\frac{\Gamma\left(1+\frac{1}{\kappa}\right)\Gamma\left(\frac{1}{\eta}-\frac{1}{\kappa}\right)}{\eta^{1+\frac{1}{\kappa}}\Gamma\left(1+\frac{1}{\eta}\right)}\right)^{\kappa}\right]^{-1-\frac{1}{\eta}}$$

where $\kappa > \eta > 0$,

2. for an ACD/ACV model with a generalized gamma distribution:

$$f\left(x_i|\theta_j, x_{(0)}, M_j; \mathcal{F}_{i-1}\right) = \\ = \frac{\gamma}{x_i\Gamma\left(\frac{\nu}{\gamma}\right)}\left(\frac{x_i}{\Psi_i}\cdot\frac{\Gamma\left(\frac{1+\nu}{\gamma}\right)}{\Gamma\left(\frac{\nu}{\gamma}\right)}\right)^{\nu}\exp\left[-\left(\frac{x_i}{\Psi_i}\cdot\frac{\Gamma\left(\frac{1+\nu}{\gamma}\right)}{\Gamma\left(\frac{\nu}{\gamma}\right)}\right)^{\gamma}\right]$$

where $\gamma > 0$ and $\nu > 0$.
In both above cases Ψ_i is given by one of these specifications: (18), (19), (20), (21), (22) or (23).
In order to fully specified a Bayesian model, prior distributions of parameters $p(\theta_j|M_j)$ must be determined. It should be emphasised that these models are too complex to analytically determine the reference priors in the vein of Jeffreys and Bernardo. Therefore, prior distributions are assumed that reflect subjectively weak preliminary knowledge about the parameters. The vector θ_j of all parameters in the model M_j can be decomposed into two prior independent random subvectors: the subvector of parameters characteristic for a given specification Ψ_i, and the subvector of parameters characteristic for the error term ε_i.
In ACD/ACV models with the Burr distribution (these are Bayesian models M_j for $j = 1, \dots 6$) the vector of parameters of the distribution of ε_i is (κ, η), where $\kappa > \eta > 0$. In models with this distribution, it was assumed that:

$$p(\kappa, \eta) \propto f_N\left(\kappa|\mu_\kappa, \sigma_\kappa^2\right)\cdot f_N\left(\eta|\mu_\eta, \sigma_\eta^2\right)\cdot I_{(0,\infty)}(\kappa)\cdot I_{(0,\infty)}(\eta)\cdot I_{(\kappa>\eta)}(\kappa, \eta),$$

where $\mu_\kappa = 0$, $\sigma_\kappa = 5$, $\mu_\eta = 0$, $\sigma_\eta = 5$ and $f_N\left(\cdot|\mu_0, \sigma_0^2\right)$ denotes the density of a normal distribution with an expected value of μ_0 and variance of σ_0^2. Thus, the initial knowledge of the vector of the parameters (κ, η) is reflected in truncated normal distributions.
In turn, in the ACD/ACV models with a generalized gamma distribution (Bayesian models M_j for $j = 7, \dots 12$) the vector of parameters of ε_i is (γ, ν), where $\gamma > 0$ and

$\nu > 0$. Then the prior density is:

$$p(\gamma,\nu) = p(\gamma)p(\nu) \propto f_N\left(\gamma|\mu_\gamma,\sigma_\gamma^2\right) \cdot I_{(0,\infty)}(\gamma) \cdot f_N\left(\nu|\mu_\nu,\sigma_\nu^2\right) \cdot I_{(0,\infty)}(\nu),$$

where $\mu_\gamma = 0$, $\sigma_\gamma = 5$, $\mu_\nu = 0$, $\sigma_\nu = 30$. The initial knowledge of the vector of parameters (γ,ν) is embodied in the above appropriate truncated normal distributions, similarly as in the Burr distribution.
The specification of the prior density $p\left(\theta_j|M_j\right)$ in the model M_j must be supplemented with the prior density of the vector of parameters characteristic for a given specification of the conditional expected duration/volume Ψ_i. We assume prior independence of parameters. Then the joint prior density of the vector of these parameters is a product of the prior densities of its coordinates. The exception here is the linear ACD model with additional restrictions imposed on selected parameters. We assume that prior distributions for the parameters with values that span the entire set of real numbers are normal with zero mean and standard deviation of five. For the remaining parameters of the models considered we assume normal distributions with zero mean and standard deviation of five, adequately truncated, due to the restrictions imposed on the parameters by the individual models. The exception is the parameter b in the augmented Box-Cox ACD model and the augmented ACD model for which the prior distribution is an inverted gamma distribution with density:

$$p(b) = f_{IG}\left(b|r,s\right) = \frac{s^r}{\Gamma(r)}\left(\frac{1}{b}\right)^{r+1}\exp\left(-\frac{s}{b}\right)I_{(0,\infty)}(b),$$

where $r = 1$ and $s = 0.3$. All above prior distributions reflect the subjectively weak preliminary knowledge of the parameters.

4 Data sets and empirical study

Since in this paper we revisit and return to the previous research of the unpublished manuscript of Huptas (2013), we use exactly the same data (regarding transaction durations) as in Huptas (2013). It must be emphasised that the data is very historical, but, in fact, we simply return to a formal Bayesian comparison of ACD models. However, we try to verify the stability of the comparisons of those models using three extra different approximations of the marginal data densities values to calculate the Bayes factors. The sample covers transaction durations between 5 May 2009 and 19 June 2009. Although the transaction duration data is relatively old, it still reveals very similar features as can also be found for more recent data (though the trading frequency arguably has increased since that time). Moreover, in order to check the robustness of the approach applied, we also use in this paper more recent data of intraday trading volumes.

4.1 Data sets – transaction durations and intraday trading volumes

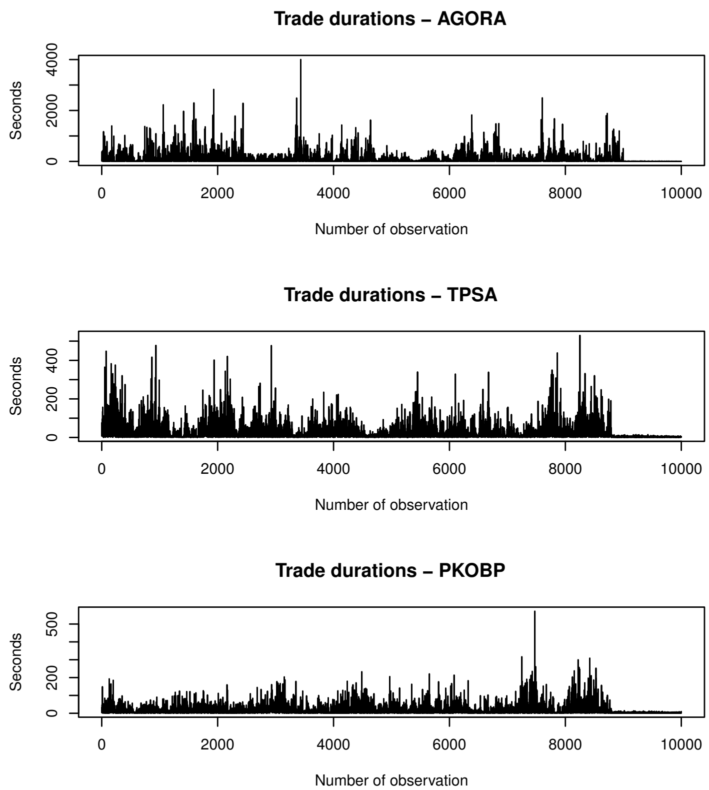
The empirical study was carried out based on data from the Warsaw Stock Exchange (WSE) in Poland, which is the stock market with the highest capitalisation in Eastern and Central Europe. The number of works focusing on ACD or ACV models and their empirical applications to Polish financial data is still limited; see e.g., Doman (2005), Bień (2006, 2006a), Doman (2008), Doman and Doman (2010), Doman (2011), Bień-Barkowska (2011, 2013, 2014, 2015, 2016, 2017, 2020), Huptas (2013, 2014, 2016, 2017, 2018, 2019), Gurgul and Syrek (2016).
In order to empirically verify the approaches presented above, calculate Bayes factors, and compare competing ACD models, we use transaction durations calculated for three companies listed in the Warsaw Stock Exchange WIG20 index, namely the Polish Telecom company (TPSA), the media company Agora SA (AGORA), and the PKOBP SA bank (PKOBP). The empirical study is based on relatively old tick-by-tick quotations between 9 May 2009 and 19 June 2009 (at that time the data was derived from the Stooq.pl website). It must be noted that at that time, the stocks were selected to reveal differences in liquidity (trading intensity) during the study period. The analysis covers only transactions carried out in the continuous trading phase, i.e., at that time in the case of the Warsaw Stock Exchange between 10:00 and 16:10 of local time. When several transactions were recorded in the same second, data were partially aggregated, and such transactions were deemed to constitute a single transaction with the price being a volume-weighted average. In addition, the transaction durations between the end of the session and the beginning of the next day's session were removed. In this case, we followed the most common convention proposed in the financial literature. Our original time series of transaction durations under analysis consists of 10,000 observations. It is worth nothing that, due to different liquidity of considered stocks, ACD models were used to model the dynamics of transaction durations covering the following periods: from 5 May 2009 to 19 June 2009 for the company AGORA, from 6 June 2009 to 19 June 2009 for the company TPSA and from 8 June 2009 to 19 June 2009 for the company PKOBP.
The basic descriptive statistics for the transaction durations of the stocks surveyed are shown in Table 1. We are dealing with three companies that exhibit divergent patterns of trading activity. The majority of the transactions involved PKOBP equities, for which the average duration between transactions is approximately 16 seconds. The company belongs to a group of the most liquid WSE entities. The fewest transactions were reported for AGORA, for which the average transaction duration is about 83 seconds. With an average duration between transactions of approximately 23 seconds, TPSA represents an average liquid company. In addition, the medians of the series are markedly smaller than the average values, which means that the duration distributions are characterised by a strong right-sided asymmetry. Analysis of the descriptive statistics of empirical duration distributions reveals their striking overdispersion.

Table 1: Descriptive statistics of transaction durations, autocorrelation coefficients and the Ljung-Box statistics for the analysed stocks

	plain durations			adjusted (deseasonalised) durations		
	AGORA	TPSA	PKOBP	AGORA	TPSA	PKOBP
Number of observations	10000	10000	10000	10000	10000	10000
Mean	83.065	22.825	16.442	1.029	0.927	1.230
Standard deviation (SD)	179.548	38.011	26.246	2.097	1.449	1.791
Dispersion index (=SD/Mean)	2.162	1.665	1.596	2.038	1.563	1.456
Median	19	9	7	0.267	0.420	0.564
First quartile	5	3	3	0.769	0.164	0.229
Third quartile	80	26	19	1.035	1.047	1.421
Kurtosis	62.930	31.384	42.003	49.714	29.456	21.626
Skewness	5.866	4.303	4.485	5.257	4.092	3.510
Minimum	1	1	1	0.006	0.023	0.043
Maximum	4003	530	572	44.065	22.281	24.439
ACF(1)	0.234	0.223	0.185	2.038	0.202	0.157
Q(5)	1642.9	1551.6	1427.7	1522.8	1176.8	904.1
Q(20)	3798.9	4200.7	3952.3	2933.5	2711.1	2194.5

Note: ACF(k) – the value of the k-th order autocorrelation coefficient; Q(k) – the value of the Ljung-Box Q-statistic of k-th order.

Figure 1: Time series plots of plain transaction durations for AGORA, TPSA, and PKOBP companies



The dispersion indices (variation coefficients) are generally very high, which may indicate the high dynamics of the surveyed series. The values of the variation coefficients range from 1.6 to 1.7. With its dispersion index ascertained at a high 2.16, the company AGORA is an exception. It is worth noting that the higher the transaction intensity, the lower the dispersion index. The dynamics of the transaction durations can be seen in Figure 1, showing graphs for the time series considered. In addition, the graphs clearly indicate clustering of short and long transaction durations. This suggests the presence of a strong autocorrelation in the tested series. Polish durations are characterised by properties analogous to data recorded at mature foreign markets, i.e., a strong right-sided asymmetry, overdispersion or clustering of shorter and longer durations (see Hautsch, 2004; Hautsch, 2012; Bauwens and Giot, 2001, and numerous other works). The autocorrelation of the transaction durations is very strong. The values of the Ljung-Box Q statistics in Table 1 allow formally (but in a non-Bayesian approach) to reject easily, at the significance level of 0.05, the null hypothesis whereby there is no autocorrelation of durations respectively from the first to the twentieth order delay for all three companies (the critical values of χ^2 distribution for the significance level of 0.05 are $\chi^2(5) = 11.070$ and $\chi^2(20) = 31.410$). A very strong autocorrelation of transaction durations may be due to the presence of intraday seasonality patterns of trading activity. It is well documented in the financial literature that financial durations exhibit a well-pronounced intraday periodic pattern. In a way similar to Bauwens and Veredas (2004) (see also Veredas, Rodriguez-Poo and Espasa, 2001; Huptas, 2014, 2016, 2018, 2019), the intraday periodic pattern for transaction durations was estimated using the Nadaraya-Watson kernel estimator of regression of the duration on the time of the day (the *quartic* kernel with an optimal bandwidth of $(2.78 \cdot s \cdot n^{\frac{-1}{5}})$ was used). It is worth noting that the kernel method applied is a non-Bayesian one. To further analyse the series surveyed on the basis of Bayesian ACD models, the seasonality effect from the series was eliminated, which helped to reduce the detected autocorrelation. Following Engle and Russell (1998), the time-of-day-adjusted transaction durations were computed by dividing the plain durations by the estimated periodic component.

The descriptive statistics of the durations after the elimination of the seasonal effect are also shown in Table 1. The elimination of the seasonal factor from the data resulted in a partial reduction in transaction duration autocorrelation. The values of the Ljung-Box test statistics declined for the companies surveyed, but still remain very high. The rather high values of low-order autocorrelation coefficients suggest that the rather strong clustering of short and long durations will continue. The null hypothesis of the absence of autocorrelation is still rejected at any reasonable level of significance. Of course, this indicates that the dynamics of transaction durations are influenced by factors other than a purely deterministic seasonality effect arising from the structure of the stock market.

Furthermore, we also consider equity trading volumes. We analyse the number of shares of the KGHM SA enterprise (KGHM), a company listed on the WIG20 Index of the Warsaw Stock Exchange (WSE), which is very actively traded during the continuous trading phase of market opening hours (9:00 -16:50 in Warsaw local time). The analysis covers the period from March 23, 2022, until June 15, 2022, which includes 56 trading days. The frequency of observations is 10 minutes and complies with the financial literature. The dataset has been taken from the Refinitiv Eikon Database (formerly Thomson Reuters Eikon Database). The plain 10-minute trading volumes for KGHM are plotted in Figure 2. In turn, the basic descriptive statistics of 10-minute trading volumes for the stock surveyed are reported in Table 2. Similar to transaction durations, the empirical trading volume distribution also shows overdispersion. The coefficient of variation is also very high, which may indicate the high dynamics of the surveyed series. The value of the variation coefficient climbed to 1.11. The high skewness value indicates that the trading volume distribution is characterised by a strong right-sided asymmetry, similar to distributions of transaction durations. The very high value of skewness, overdispersion, and modal value on the far side of zero indicate that the empirical distribution in this case is very far from an exponential distribution. The existence of intraday periodicities in trading volumes is also well documented in the financial literature. The intraday periodic pattern for 10-minute trading volumes was estimated in a manner similar to transaction durations, using the Nadaraya-Watson kernel estimator to regress the trading volume variable on time of day. To reduce high autocorrelation, following Engle and Russell (1998), intraday trading volumes were deseasonalized by dividing plain data by the diurnal factor to obtain diurnally adjusted volumes. Finally, all ACV models were estimated using seasonally adjusted trading volumes.

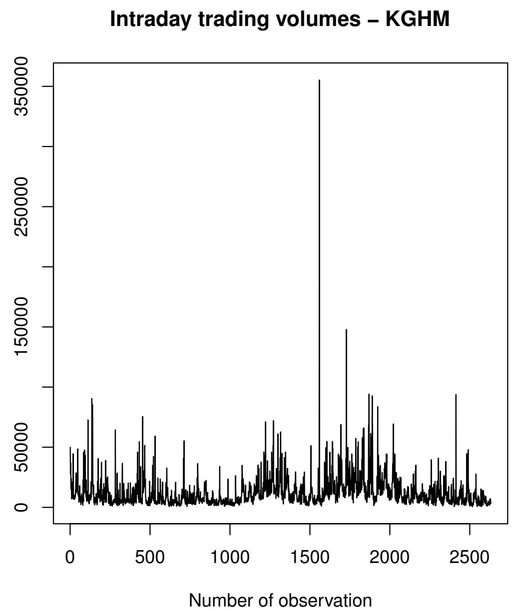
Table 2: Descriptive statistics of 10-minute trading volumes, autocorrelation coefficients, and the Ljung-Box statistics for KGHM

KGHM	plain intraday trading volumes	adjusted (deseasonalised) intraday trading volumes
Number of observations	2632	2632
Mean	12096.930	1.011
Standard deviation (SD)	13436.970	1.067
Dispersion index (=SD/Mean)	1.111	1.055
Median	8257	0.745
First quartile	4770	0.452
Third quartile	14938	1.244
Kurtosis	172.624	202.243
Skewness	8.338	9.356
Minimum	570	0.047
Maximum	355282	29.288
ACF(1)	0.436	0.361
Q(5)	1426.5	895.1
Q(20)	2215.8	2082.4

4.2 Bayes factors for competing models and stability of model comparisons

This section presents the results of a formal Bayesian comparison of the explanatory power of multiplicative error models conducted for a series of transaction durations and intraday trading volumes derived from data on the Polish stock market. It should be emphasised once again here that the question of a formal Bayesian comparison

Figure 2: Time series plot of plain 10-minute trading volumes for KGHM



(based on Bayes factors) of different ACD/ACV models has not been considered so far in the literature.

In order to calculate the ultimate characteristics of the posterior distribution of parameters, in each model the Monte Carlo methods based on Markov chains (MCMC) are used. The Metropolis-Hastings algorithm is used with a symmetric proposal density (see e.g., Hastings, 1970; O'Hagan, 1994). For a candidate generating distribution, we use the multivariate Student's t distribution with three degrees of freedom, for which the expected value is equal to the previous state of the Markov chain, and the covariance matrix is obtained based on a numerical strategy using the Monte Carlo-Importance Sampling method. The length of the generated Markov chain and the number of burnt-in states depend on the algorithm's convergence speed within the model framework. For the more general models, e.g., the augmented Box-Cox ACD/ACV model and the augmented ACD/ACV model, we made 6,000,000 draws, including 1,000,000 burn-in states. On the other hand, in simpler, less parameterized models, we made 2,200,000 draws, including 200,000 burnt-in states. To assess the convergence of the Metropolis and Hastings algorithms, we used CUMSUM plots (Yu and Mykland, 1998).

It must also be mentioned that for the CHME and CAME estimators, we assume that the subset A is an intersection of the parameter space, where the conditional density function $p\left(\mathbf{x} \left|\theta_j, x_{(0)}, M_j\right.\right)$ exceeds the smallest value of $p\left(\mathbf{x} \left|\theta_j, x_{(0)}, M_j\right.\right)$ evaluated at pseudo-random sample $\left\{\theta_j{}^{(q)}\right\}_{q=1}^{N}$ from the posterior distribution, and the hypercuboid limited by the range of the sampler output,

$$\begin{aligned} A \;=\; & \underset{i}{\otimes}\left[\min\left\{\theta_{i;j}{}^{(q)}\right\}, max\left\{\theta_{i;j}{}^{(q)}\right\}\right] \cap \\ & \cap\left\{\theta_j : p\left(\mathbf{x}\middle|\theta_j, x_{(0)}, M_j\right) \geq min\left\{p\left(\mathbf{x}\middle|\theta_j{}^{(q)}, x_{(0)}, M_j\right),\ q=1,\ldots,N\right\}\right\}, \end{aligned}$$

where $\left\{\theta_{i;j}{}^{(q)}\right\}$ is the i-th component of the vector $\left\{\theta_j{}^{(q)}\right\}$ in the model M_j. Consequently, $\widehat{P}\left(A|\mathbf{x}\right) = 1$, and the prior simulation support is in regions where the likelihood is significant. To estimate $p\left(\mathbf{x}\middle|x_{(0)}, M_j\right)$ in the model M_j using (13), we applied the MC-IS method with the importance function being the multivariate Student's t distribution with three degrees of freedom and with the mean and covariance matrix of the posterior distribution (estimated from the MCMC draws). Additionally, to approximate P(A), similar to Lenk (2009) and Pajor (2017), we also used the MC-IS with the multivariate Student's t distribution with three degrees of freedom.

All empirical results presented were obtained using the author's own codes implemented in the GAUSS econometric environment. The initial observations were used as the initial conditions $x_{(0)}$.

The first issue to be examined involves the comparison of the explanatory power of Bayesian ACD models for trade durations. We also discuss the stability of the Bayes factors and the ranks of the models with respect to the method used to approximate the values of the marginal data density. Of course, all calculations and comparisons are based on the particular prior distributions proposed for the model parameters.

Tables 3, 5, and 7 show the decimal logarithms of the marginal data density values using all four approximations, along with the model rankings for the three companies surveyed. On the other hand, the decimal logarithms of the Bayes factors against the simplest Burr-ACD(1,1) model and the model rankings for these three stocks are reported in Tables 4, 6, and 8.

First, let us focus on general findings. The overall qualitative conclusion (based on all four methods used to approximate the marginal data density values) is that the results reveal the absolute inadequacy of the ACD models with the Burr conditional distribution. The ACD models with a conditional generalised gamma distribution have substantially higher marginal data density values, consequently, more posterior probability, than ACD models with a Burr conditional distribution and group virtually all the posterior probability mass under any reasonable prior model probabilities. It thus appears that the conditional generalised gamma distribution is much more adequate than the conditional Burr distribution, even if these two depend on two free parameters.

Moreover, in line with previous literature and research on approximations of the marginal data density, the HME estimator overestimates the marginal data density. In turn, in all cases, the estimates of the marginal data density provided by the other three alternative methods are almost the same. Consequently, the Bayes factors obtained by HME differ from the corresponding values obtained by the other three estimators, but the ranking of the models for HME is similar to the rankings for CHME, MC-IS, and CAME. Fortunately, while the Bayes factors are different, their values and the resulting model ranks are similar enough to indicate the leading position of the GGam-BCACD(1,1) specification for CHME, MC-IS, and CAME, and the GGam-BCACD and GGam-ABCACD models for HME in the case of all stocks considered.

Now, let us analyse the results in more detail. In the case of TPSA, the greatest explanatory power (according to the ranking based on the CAME estimator) was revealed for the model M_9, that is, the BCACD model with a generalized gamma distribution for the error term. The decimal logarithm of the Bayes factor of M_9 against the simplest Burr-ACD(1,1) model M_1 (calculated using the CAME estimator), $\log(BF_{9\ 1}) = 62.71$, indicates that – under equal prior model probabilities – M_9 is about 63 orders of magnitude more probable a posteriori than M_1. Additionally, M_9 is about 73 orders of magnitude more probable a posteriori than M_2, the LACD model with Burr innovations, which occupies the last position in the ranking. Moreover, it also seems that the GGam-BCACV model would receive almost all posterior probability mass, assuming equal prior model probabilities. Indeed, the second and third places in the ranking (based on the CAME estimator) are occupied by the GGam-AsLACD and GGam-ABCACD models, but they are about 2 and 2.6 orders of magnitude worse than the GGam-BCACV model, respectively. The data also decisively disqualify the two simplest ACD specifications with a conditional generalised gamma distribution, i.e., the GGam-ACD model and GGam-LACD model. It should therefore be emphasised that high Bayes factors coupled with a rather sparse parameterisation, confirm the high explanatory power of the GGam-BCACD model. Furthermore, the explanatory power of ACD models with a conditional Burr distribution is very poor, i.e., between 50 and 73 orders of magnitude smaller than the explanatory power of M_9, which can be attributed to the fact that the Burr error distribution plays little role in explaining the observed dynamics of TPSA transaction durations.

It must also be noted that applying the CAME estimator does not change much in terms of overall model ranking compared to the well-known HME estimator, which overestimates the value of marginal data density. According to the ranking based on the HME estimator, the BCACV model with a generalised gamma distribution for the error term is also the best model among all 12 models under consideration. However, its marginal data density value after using the CAME estimator dropped by about 12.7 orders of magnitude compared to the HME approximation. It seems that the more complex the functional forms of conditional duration, the bigger the

Table 3: Bayesian model comparison for TPSA – the decimal logarithms of marginal data density values using various estimators and the models' rankings

TPSA	HME		CHME		MC-IS		CAME	
Model	log HME	Rank	log CHME	Rank	log MC-IS	Rank	log CAME	Rank
M_1 Burr-ACD(1,1)	-3179.13	11	-3187.18	11	-3187.11	11	-3187.11	11
M_2 Burr-LACD(1,1)	-3187.31	12	-3197.32	12	-3197.14	12	-3197.14	12
M_3 Burr-BCACD(1,1)	-3162.07	8	-3174.96	7	-3174.54	7	-3174.54	7
M_4 Burr-As-LACD(1,1)	-3163.97	10	-3175.63	8	-3175.37	8	-3175.32	8
M_5 Burr-ABCACD(1,1)	-3161.58	7	-3177.08	9	-3177.27	9	-3177.23	9
M_6 Burr-AACD(1,1)	-3163.57	9	-3179.61	10	-3179.68	10	-3179.67	10
M_7 GGam-ACD(1,1)	-3131.85	5	-3139.89	5	-3139.71	5	-3139.71	5
M_8 GGam-LACD(1,1)	-3132.64	6	-3142.64	6	-3142.53	6	-3142.52	6
M_9 GGam-BCACD(1,1)	-3111.69	1	-3124.54	1	-3124.39	1	-3124.40	1
M_{10} GGam-As-LACD(1,1)	-3114.85	4	-3126.59	2	-3126.44	2	-3126.41	2
M_{11} GGam-ABCACD(1,1)	-3111.79	2	-3127.17	3	-3127.09	3	-3127.05	3
M_{12} GGam-AACD(1,1)	-3114.62	3	-3130.53	4	-3130.70	4	-3130.65	4

Note: 'log' stands for decimal logarithm, 'HME' – the Harmonic Mean estimator, 'CHME' – the Corrected Harmonic Mean estimator, 'MC-IS' – the standard Monte Carlo with Importance Sampling-based estimator, 'CAME' – the Corrected Arithmetic Mean estimator.

Table 4: Bayesian model comparisons for TPSA – the decimal logarithms of Bayes factors against the Burr-ACD (1,1) model and the models' rankings

TPSA	HME		CHME		MC-IS		CAME	
Model	$\log(\mathrm{BF}_{i\ 1})$	Rank	$\log(\mathrm{BF}_{i\ 1})$	Rank	$\log(\mathrm{BF}_{i\ 1})$	Rank	$\log(\mathrm{BF}_{i\ 1})$	Rank
M_1 Burr- ACD(1,1)	0.00	11	0.00	11	0.00	11	0.00	11
M_2 Burr- LACD(1,1)	-8.18	12	-10.14	12	-10.03	12	-10.03	12
M_3 Burr- BCACD(1,1)	17.06	8	12.22	7	12.57	7	12.57	7
M_4 Burr- As-LACD(1,1)	15.16	10	11.55	8	11.74	8	11.79	8
M_5 Burr-ABCACD(1,1)	17.55	7	10.10	9	9.84	9	9.88	9
M_6 Burr- AACD(1,1)	15.56	9	7.57	10	7.43	10	7.44	10
M_7 GGam- ACD(1,1)	47.28	5	47.29	5	47.40	5	47.40	5
M_8 GGam- LACD(1,1)	46.49	6	44.54	6	44.58	6	44.59	6
M_9 GGam-BCACD(1,1)	67.44	1	62.64	1	62.72	1	62.71	1
M_{10} GGam- As-LACD(1,1)	64.28	4	60.59	2	60.67	2	60.70	2
M_{11} GGam-ABCACD(1,1)	67.34	2	60.01	3	60.02	3	60.06	3
M_{12} GGam- AACD(1,1)	64.51	3	56.65	4	56.41	4	56.46	4

Note: 'log' stands for decimal logarithm, 'HME' – the Harmonic Mean estimator, 'CHME' – the Corrected Harmonic Mean estimator, 'MC-IS' – the standard Monte Carlo with Importance Sampling-based estimator, 'CAME' – the Corrected Arithmetic Mean estimator, and BF denotes the Bayes factor.

correction of the marginal data density. At the extremes, the correction of the marginal data density is roughly between 15.2 and 16.1 orders of magnitude for the ABCACD and AACD models. In turn, the smallest correction for HME pseudo-bias after the CAME approximation can be observed in linear ACD models, i.e., the Burr-ACD(1,1) and GGam-ACD(1,1) specifications, where their marginal data density dropped by nearly 8 orders of magnitude.

A look at AGORA reveals that the greatest explanatory power (according to the ranking based on the CAME estimator) was ascertained for the BCACD and ABCACD models with generalised gamma innovations. With a discrete uniform distribution of prior model probabilities, the GGam-BCACD model receives almost all posterior probability mass, being about 5.86 orders of magnitude more probable a posteriori than the second-ranked GGam-ABCACD model (the decimal logarithm of the Bayes factor of M_9 against model M_{11}, i.e., $\log(BF_{9\ 11})$ – calculated using the CAME estimator – is about 5.86) and 7.66 orders of magnitude better than the GGam-LACD model, which occupied the third place in the ranking. It seems that the GGam-BCACD, GGam-ABCACD and GGam-LACD models reveal almost 100% posterior probability. Other specifications are improbable in the light of the modelled data. Note the far weaker, than in the case of TPSA, ranking of the GGam-AsLACD model. This time it ranks only the fifth place. As in the case of TPSA, the results show that ACD models with a conditional generalized gamma distribution are a posteriori more probable than ACD models with a conditional Burr distribution. An exception is made for the GGam-ACD model, which, for the AGORA stock, is ranked second lowest, ahead only of the linear Burr-ACD model. However, while comparing these two worst specifications, the decimal logarithm of the Bayes factor in favour of the GGam-ACD model against the Burr-ACD model (M_1), i.e., $\log(BF_{7\ 1})$ – calculated using the CAME estimator – is close to 38.1, thus the modelled dataset of trading durations of AGORA stock supports decisively superiority of the conditional generalized gamma distribution rather than the Burr distribution. This situation also shows that in the case of a less liquid company, a switch from the conditional Burr distribution to the conditional generalized gamma distribution in the simplest linear ACD model is less favourable than changing the conditional mean equation and retaining the Burr distribution. Thus, we can say that in the case of AGORA, the generalized gamma distribution has less explanatory power than it does for TPSA. We must admit, however, that ACD models with a conditional Burr distribution generally describe the dynamics of the transaction durations far worse than the ACD models with a conditional generalized gamma distribution. The explanatory power of the ACD models with a conditional Burr distribution is between 35.5 and 110.2 orders of magnitude smaller than the explanatory power of M_9. It is clear that all specifications built on the basis of the conditional Burr distribution receive little data support.

We also examine the sensitivity of the inference to the method used to approximate the marginal data density values. Applying the CAME estimator instead of the

Table 5: Bayesian model comparison for AGORA – the decimal logarithms of marginal data density values using various estimators and the models' rankings

AGORA	HME		CHME		MC-IS		CAME	
Model	log HME	Rank	log CHME	Rank	log MC-IS	Rank	log CAME	Rank
M_1 Burr- ACD(1,1)	-2502.10	12	-2510.03	12	-2508.75	12	-2508.75	12
M_2 Burr- LACD(1,1)	-2435.12	9	-2444.53	8	-2444.38	8	-2444.38	8
M_3 Burr- BCACD(1,1)	-2419.89	7	-2433.59	6	-2433.02	6	-2433.02	6
M_4 Burr- As-LACD(1,1)	-2457.10	10	-2468.01	10	-2469.64	10	-2469.64	10
M_5 Burr-ABCACD(1,1)	-2419.05	6	-2440.09	7	-2440.04	7	-2439.98	7
M_6 Burr- AACD(1,1)	-2423.95	8	-2444.70	9	-2444.93	9	-2444.86	9
M_7 GGam- ACD(1,1)	-2463.84	11	-2471.34	11	-2470.65	11	-2470.65	11
M_8 GGam- LACD(1,1)	-2397.05	4	-2406.33	3	-2406.62	3	-2406.28	3
M_9 GGam-BCACD(1,1)	-2384.58	1	-2398.62	1	-2398.22	1	-2398.62	1
M_{10} GGam- As-LACD(1,1)	-2417.16	5	-2429.48	5	-2429.41	5	-2429.41	5
M_{11} GGam-ABCACD(1,1)	-2384.62	2	-2404.61	2	-2404.02	2	-2404.48	2
M_{12} GGam- AACD(1,1)	-2390.49	3	-2409.88	4	-2409.75	4	-2409.68	4

Table 6: Bayesian model comparisons for AGORA – the decimal logarithms of Bayes factors against the Burr-ACD (1,1) model and the models' rankings

AGORA	HME		CHME		MC-IS		CAME	
Model	log $(BF_{i\ 1})$	Rank	log $(BF_{i\ 1})$	Rank	log $(BF_{i\ 1})$	Rank	log $(BF_{i\ 1})$	Rank
M_1 Burr- ACD(1,1)	0.00	12	0.00	12	0.00	12	0.00	12
M_2 Burr- LACD(1,1)	66.98	9	65.50	8	64.37	8	64.37	8
M_3 Burr- BCACD(1,1)	82.21	7	76.44	6	75.73	6	75.73	6
M_4 Burr- As-LACD(1,1)	45.00	10	42.02	10	39.11	10	39.11	10
M_5 Burr-ABCACD(1,1)	83.05	6	69.94	7	68.71	7	68.77	7
M_6 Burr- AACD(1,1)	78.15	8	65.33	9	63.82	9	63.89	9
M_7 GGam- ACD(1,1)	38.26	11	38.69	11	38.10	11	38.10	11
M_8 GGam- LACD(1,1)	105.05	4	103.70	3	102.13	3	102.47	3
M_9 GGam-BCACD(1,1)	117.52	1	111.41	1	110.53	1	110.13	1
M_{10} GGam- As-LACD(1,1)	84.94	5	80.55	5	79.34	5	79.34	5
M_{11} GGam-ABCACD(1,1)	117.48	2	105.42	2	104.73	2	104.27	2
M_{12} GGam- AACD(1,1)	111.61	3	100.15	4	99.00	4	99.07	4

HME estimator, which overestimates the value of the marginal data density, does not significantly change the ranking of the models. Fortunately, according to the ranking based on the HME estimator, the BCACD model with generalised gamma innovations is also the best model among all models. However, it must be noted that its marginal data density value after using the CAME estimator dropped by about 14 orders of magnitude compared to the HME approximation. The GGam-ABCACD model took second place in the HME ranking, while the GGam-AACD ranked third. Although there are some minor differences in the HME ranking of the models with Burr innovations compared to the ranking based on the CAME estimator, the last position in the ranking is also occupied by the Burr-ACD model. As in the case of TPSA, it seems that the more complex the functional forms of conditional duration, the larger the correction to the marginal data density can be reported. The smallest correction for HME pseudo-bias after the CAME approximation is observed in linear ACD models, i.e., the Burr-ACD(1,1) and GGam-ACD(1,1) specifications, where the marginal data density dropped by nearly 7 orders of magnitude. In turn, at the extremes, the correction of the marginal data density is roughly between 19.2 and 20.9 orders of magnitude for the ABCACD and AACD specifications.

In the case of PKOBP, the ACD models with a conditional generalised gamma distribution also have, in general, greater explanatory power and hence capture virtually more posterior probability than ACD models with a conditional Burr distribution. As it can be seen from Table 8 and according to the ranking based on the CAME estimator, the GGam-BCACD model (M_9), this time also turns out to be the best by far. This is similar to the case of previous companies. The GGam-BCACD model fits the data better than other models. Moreover, the GGam-AsLACD model (M_{10}) ranks second, as in the case of TPSA, while the GGam-ABCACD model (M_{11}) is third. In terms of the Bayes factors, assuming equal prior model probabilities, the GGam-BCACD model is only about 0.68 orders of magnitude better than the second-ranked GGam-AsLACD model (the decimal logarithm of the Bayes factor in favour of M_9 against model M_{10}, i.e., $\log(BF_{9\ 10})$ – calculated using CAME estimator – is about 0.68) and 2.30 orders of magnitude more probable a posterior than the GGam-ABCACD model. However, the differences in explanatory power between the leading specifications in the ranking are smaller than for previous stocks. It must be admitted that the logarithm of the Bayes factor in favour of the GGam-BCACD specification over the GGam-AsLACD one is less than one. Thus, according to the scale presented by Kass and Raftery (1995), we can observe only positive, but not strong, evidence in favour of the BCACD model with generalised gamma innovations. It means that with regard to PKOBP, although the GGam-AsLACD and GGam-ABCACD models do not surpass the GGam-BCACD model, they have smaller Bayes factors in favour of the best model and consequently produced virtually much higher posterior probabilities than was in the case for TPSA and AGORA. Yet, one can assume that for very liquid companies, the inclusion of the asymmetry effect in the function of reaction to disturbance is not without its significance, and in the case of favouring simpler models, the GGam-AsLACD model is more plausible in the light of the data. Of the $M_7, \ldots, M_{12}$ models, data also disqualifies the two simplest specifications, i.e., the GGam-ACD model (M_7) and the GGam-LACD model (M_8), both displaying quite high Bayes factors in favour of the GGam-BCACD model (M_9).

Moreover, the ACD structures with Burr innovations are strongly rejected by the data, as the decimal logarithms of marginal data density values are much smaller than the decimal logarithms of marginal data density values of the models with a conditional generalised gamma distribution. Specifically, the best specification M_3 in the ranking among models built on the basis of the conditional Burr distribution, receives negligible data support, as it is over 78.7 orders of magnitude worse than the first-ranked GGam-BCACD model. Furthermore, it is worth noting that, compared with the results for the above-mentioned companies, one can also say that the generalised gamma distribution carries much greater explanatory power for PKOBP than it does for TPSA and AGORA. Under a given specification for conditional expected duration, models with the conditional generalised gamma distribution are about 75-80 orders of magnitude better than the ACD models with a conditional Burr distribution. In the case of TPSA and AGORA, the results were, respectively, about 47-58 and 35-40 orders of magnitude better.

It must also be noted that in the case of PKOBP, applying the well-known HME estimator it changes a little model ranking in terms of the best specifications compared to the ranking based on the CAME approximation. The GGam-ABCACD model this time turns out to be the best. The GGam-BCACD model ranked second best, revealing the explanatory power of only 0.24 orders of magnitude worse than the GGam-ABCACD model. The third place in this ranking is occupied by the GGam-AACD specification, whereas the GGam-AsLACD model now takes the fourth place. The GGam-AACD and GGam-AsLACD models are, respectively, about 1.36 and 1.54 orders of magnitude less probable a posteriori than the GGam-BCACD model. We therefore have a situation that is reversed compared to the CAME ranking. It still seems that the more complex the functional forms of conditional duration, the bigger the correction of the marginal data density. At the extremes, the correction to the marginal data density is roughly between 15 and 16 orders of magnitude for the ABCACD and AACD models. In turn, the smallest correction for HME pseudo-bias after the CAME approximation can be observed in linear ACD specifications, i.e., the Burr-ACD(1,1) and GGam-ACD(1,1) specifications, where their marginal data density dropped nearly by about 7.5-8.5 orders of magnitude.

Let us now proceed with the analysis of time series of intraday trading volumes for the KGHM company and with a formal Bayesian comparison of ACV models, which are counterparts of ACD models for trading volumes. In Tables 9 and 10, we present the results of a Bayesian comparison of all 12 competing specifications. Table 9 displays the decimal logarithms of the marginal data density values using all four approximations and the models' rankings for the KGHM company, while the decimal

Table 7: Bayesian model comparison for PKOBP – the decimal logarithms of marginal data density values using various estimators and the models' rankings

PKOBP	HME		CHME		MC-IS		CAME	
Model	log HME	Rank	log CHME	Rank	log MC-IS	Rank	log CAME	Rank
M_1 Burr- ACD(1,1)	-4588.32	12	-4596.19	12	-4595.88	12	-4595.88	12
M_2 Burr- LACD(1,1)	-4581.60	11	-4591.85	11	-4591.86	11	-4591.86	11
M_3 Burr- BCACD(1,1)	-4570.29	8	-4583.64	7	-4583.68	8	-4583.68	8
M_4 Burr- As-LACD(1,1)	-4571.80	9	-4583.78	8	-4583.56	7	-4583.56	7
M_5 Burr-ABCACD(1,1)	-4568.55	7	-4584.54	9	-4584.36	9	-4584.32	9
M_6 Burr- AACD(1,1)	-4574.91	10	-4584.93	10	-4584.69	10	-4584.67	10
M_7 GGam- ACD(1,1)	-4511.66	6	-4519.95	6	-4519.89	6	-4519.87	6
M_8 GGam- LACD(1,1)	-4499.22	5	-4510.35	5	-4510.06	5	-4510.92	5
M_9 GGam-BCACD(1,1)	-4491.33	2	-4504.54	1	-4504.88	1	-4504.82	1
M_{10} GGam- As-LACD(1,1)	-4492.63	4	-4505.28	2	-4506.49	2	-4505.50	2
M_{11} GGam-ABCACD(1,1)	-4491.09	1	-4507.13	3	-4506.50	3	-4507.12	3
M_{12} GGam- AACD(1,1)	-4492.45	3	-4507.57	4	-4507.89	4	-4507.88	4

Table 8: Bayesian model comparisons for PKOBP – the decimal logarithms of Bayes factors against the Burr-ACD (1,1) model and the models' rankings

PKOBP	HME		CHME		MC-IS		CAME	
Model	$\log(BF_{i\ 1})$	Rank	$\log(BF_{i\ 1})$	Rank	$\log(BF_{i\ 1})$	Rank	$\log(BF_{i\ 1})$	Rank
M_1 Burr- ACD(1,1)	0.00	12	0.00	12	0.00	12	0.00	12
M_2 Burr- LACD(1,1)	6.72	11	4.34	11	4.02	11	4.02	11
M_3 Burr- BCACD(1,1)	18.03	8	12.55	7	12.20	8	12.20	8
M_4 Burr- As-LACD(1,1)	16.52	9	12.41	8	12.32	7	12.32	7
M_5 Burr-ABCACD(1,1)	19.77	7	11.65	9	11.52	9	11.56	9
M_6 Burr- AACD(1,1)	13.41	10	11.26	10	11.19	10	11.21	10
M_7 GGam- ACD(1,1)	76.66	6	76.24	6	75.99	6	76.01	6
M_8 GGam- LACD(1,1)	89.10	5	85.84	5	85.82	5	84.96	5
M_9 GGam-BCACD(1,1)	96.99	2	91.65	1	91.00	1	91.06	1
M_{10} GGam- As-LACD(1,1)	95.69	4	90.91	2	89.39	2	90.38	2
M_{11} GGam-ABCACD(1,1)	97.23	1	89.06	3	89.38	3	88.76	3
M_{12} GGam- AACD(1,1)	95.87	3	88.62	4	87.99	4	88.00	4

Table 9: Bayesian model comparison for intraday trading volumes for KGHM – the decimal logarithms of marginal data density values using various estimators and the models' rankings

KGHM	HME		CHME		MC-IS		CAME	
Model	log HME	Rank	log CHME	Rank	log MC-IS	Rank	log CAME	Rank
M_1 Burr- ACV(1,1)	-649.39	7	-655.43	6	-655.38	6	-655.38	6
M_2 Burr- LACV(1,1)	-617.62	3	-625.66	2	-625.42	2	-625.42	2
M_3 Burr- BCACV(1,1)	-619.84	5	-633.86	5	-633.57	5	-633.54	5
M_4 Burr- As-LACV(1,1)	-617.82	4	-627.43	3	-627.43	3	-627.43	3
M_5 Burr-ABCACV(1,1)	-616.68	2	-631.81	4	-631.42	4	-631.68	4
M_6 Burr- AACV(1,1)	-614.65	1	-625.26	1	-625.26	1	-625.21	1
M_7 GGam- ACV(1,1)	-681.34	12	-688.47	12	-686.13	12	-688.26	12
M_8 GGam- LACV(1,1)	-652.09	11	-661.6	8	-661.35	7	-661.34	7
M_9 GGam-BCACV(1,1)	-650.94	8	-665.99	10	-665.98	10	-665.95	11
M_{10} GGam- As-LACV(1,1)	-652.00	9	-664.13	9	-664.08	9	-664.08	9
M_{11} GGam-ABCACV(1,1)	-652.07	10	-667.39	11	-668.08	11	-665.01	10
M_{12} GGam- AACV(1,1)	-647.52	6	-660.43	7	-662.18	8	-661.70	8

Table 10: Bayesian model comparisons for intraday trading volumes for KGHM – the decimal logarithms of Bayes factors against the Burr-ACV (1,1) model and the models' rankings

KGHM	HME		CHME		MC-IS		CAME	
Model	$\log(BF_{i\ 1})$	Rank	$\log(BF_{i\ 1})$	Rank	$\log(BF_{i\ 1})$	Rank	$\log(BF_{i\ 1})$	Rank
M_1 Burr- ACV(1,1)	0.00	7	0.00	6	0.00	6	0.00	6
M_2 Burr- LACV(1,1)	31.77	3	29.77	2	29.96	2	29.96	2
M_3 Burr- BCACV(1,1)	29.55	5	21.57	5	21.81	5	21.84	5
M_4 Burr- As-LACV(1,1)	31.57	4	28.00	3	27.95	3	27.95	3
M_5 Burr-ABCACV(1,1)	32.71	2	23.62	4	23.96	4	23.70	4
M_6 Burr- AACV(1,1)	34.74	1	30.17	1	30.12	1	30.17	1
M_7 GGam- ACV(1,1)	-31.95	12	-33.04	12	-30.75	12	-32.88	12
M_8 GGam- LACV(1,1)	-2.70	11	-6.17	8	-5.97	7	-5.96	7
M_9 GGam-BCACV(1,1)	-1.55	8	-10.56	10	-10.60	10	-10.57	11
M_{10} GGam- As-LACV(1,1)	-2.61	9	-8.70	9	-8.70	9	-8.70	9
M_{11} GGam-ABCACV(1,1)	-2.68	10	-11.96	11	-12.70	11	-9.63	10
M_{12} GGam- AACV(1,1)	1.87	6	-5.00	7	-6.80	8	-6.32	8

logarithms of the Bayes factors against the Burr-ACV(1,1) model, along with the models' rankings for this asset, are presented in Table 10.

The general conclusion for intraday volumes of KGHM is that the results reveal the total inadequacy of the ACV models with a conditional generalised gamma distribution, irrespective of the approximation used to estimate the marginal data density. Therefore, we face a situation that is completely opposite to the previous results for the time series of transaction durations. In terms of marginal data density values and Bayes factors, the ACV models with Burr innovations fit the volume data much better than the ACV ones with generalised gamma innovations. The ACV models with a conditional Burr distribution have substantially higher marginal data density values than the ACV models with a conditional generalised gamma distribution, and it appears that they group virtually all the posterior probability mass under any reasonable prior model probabilities. Therefore, the conditional Burr distribution is much more adequate than the conditional generalised gamma distribution in the case of modelling the intraday trading volumes of KGHM.

It must be admitted that the completely opposite ranking of MEMs for intraday trading volumes compared to MEMs for transaction durations is extremely interesting and thought-provoking. However, this difference may result from the fact that in both cases we are modelling rather different types of variables. In the case of trading volumes, we model the volumes aggregated into equal 10-minute intervals. In the case of transaction durations, on the other hand, we model the times that have elapsed between successive transactions, which are simply arranged in the order in which they occurred.

As can be seen in Table 10, according to the ranking based on the CAME estimator, the greatest explanatory power was ascertained for the AACV and LACV models with Burr innovations. The Burr-AACV model turns out to be the best in terms of in-sample fit, while the Burr-LACV model takes second place. However, under equal prior model probabilities the Burr-AACV model is only about 0.21 orders of magnitude more probable a posteriori than the second-ranked Burr-LACV model, since the decimal logarithm of the Bayes factor of M_6 against model M_2, i.e., $\log(BF_{6\ 2})$ – calculated using the CAME estimator – is about 0.21, and is less than one. Therefore, it is evident that the modelled dataset of intraday trading volumes of KGHM does not support the superiority of the Burr-AACV specification decisively. Only slightly positive evidence in favour of the AACV model with Burr innovations is observed, although according to the scale presented by Kass and Raftery (1995), such a difference between models is 'not worth more than a bare mention'. It is also important to note that the AACV model nests the LACV model for $\delta_1 \to 0$, $\delta_2 \to 0$, $b = c = 0$. Hence, these two models can have a similar explanatory power, since the reasoning based on the posterior odds ratio favours parsimoniously parameterised models. The third position in the CAME ranking is occupied by the Burr-AsLACV model, which is about 2.22 orders of magnitude less probable than M_6. The high place of this specification shows that the inclusion of the asymmetry effect in the function of reaction to disturbance is not without its significance and is supported by the considered dataset. Consequently, these three models, i.e., M_6, M_2 and M_4, cumulate almost all of the posterior probability mass, making all remaining specifications rather improbable in view of the data. Surprisingly, the Burr-BCACV model ranks only in fifth place. This is the far weaker ranking of this model than in the case of previously modelled transaction durations. The Burr-BCACV fits the data about 8.33 orders of magnitude worse than the best ACV specification. Also, the linear ACV model with Burr innovations leads to very doubtful explanatory power. The model M_1 is strongly rejected by the data, being about 30.2 orders of magnitude less probable a posteriori than the best-ranked Burr-AACV model. As mentioned above, our findings show that the ACV models with a conditional generalised gamma distribution generally describe the dynamics of the intraday trading volumes far worse than the ACV models with a conditional Burr distribution. The explanatory power of the ACV models with a conditional generalised gamma distribution is very poor and is between 36.1 and 63.1 orders of magnitude smaller than the explanatory power of M_6. It is clear that all specifications built on the basis of the conditional generalised gamma distribution receive little data support and fit the data much worse than models with Burr innovations. Therefore, for intraday volumes, switching from the conditional generalised gamma distribution to the conditional Burr distribution is more favourable than changing the conditional mean equation and retaining the generalised gamma distribution.

We continue our analysis by examining the stability of Bayes factors and model comparisons with respect to the method used to approximate the marginal data density values. The application of the HME estimator instead of the CAME estimator, which overestimates the value of marginal data density, slightly changed the model ranking. According to the ranking based on the HME approximation, fortunately, the AACV model with Burr innovations is also the best model among all models. However, it is important to note that its marginal data density value after using the CAME estimator dropped by about 10.6 orders of magnitude compared to the HME approximation. This time, the Burr-ABCACV model took second place in the HME ranking, while the Burr-LACV ranked third. The ACV model with a Box-Cox transformation and Burr innovations also gained fifth place. Although there are some minor differences in the HME ranking of the models with generalised gamma innovations compared to the ranking based on the CAME estimator, in general, these models are worse (in terms of Bayes factors) than the models with Burr distribution, and the last position in the ranking is also occupied by the GGam-ACV specification. However, the most complex AACV model with generalised gamma innovations is slightly ahead of the Burr-ACV(1,1) specification. As in the case of transaction durations, it appears that the more complex the functional form of the conditional duration, the larger the correction to the marginal data density can be observed. The smallest correction for HME pseudo-bias after the CAME approximation is observed in linear ACV models, i.e., the Burr-ACV(1,1) and GGam-ACV(1,1) specifications, where the marginal data density dropped by about 6-7 orders of magnitude. In turn, at the extremes, the correction of the marginal data density is roughly between 13 and 15 orders of magnitude for the ABCACV, BCACV, and GGam-AACV specifications. Turning our attention to two other rankings based on the CHME estimator and the pure MC-IS technique, and comparing them with the CAME ranking, one can see that the decimal logarithms of marginal data density values and the decimal logarithms of Bayes factors obtained using all three approximations differ only slightly. Therefore, their values and the resulting model ranks are sufficiently similar to indicate exactly the same first six positions for the ACV models with Burr innovations.

5 Concluding remarks

In this paper, we have used time series of historical transaction durations and intraday trading volumes for selected companies listed on the Polish stock exchange to compare various univariate multiplicative error models through their Bayes factors. It is worth mentioning again that the Bayesian model comparison through Bayes factors has not been used thus far as an alternative method for the selection of competing models within the whole class of ACD/ACV models discussed in the paper. In the context of the financial and microstructure literature, this application of the proposed approach therefore offers a fresh perspective on the selection of ACD and ACV models. We have investigated the effects of different conditional mean equations and conditional duration/volume distributions on the in-sample model fit.

In particular, the model comparison exercise performed in this study reports the following findings. In general, ACD/ACV models provide an adequate description tool for the dynamics of transaction durations and intraday trading volumes and can be used for modelling. The results indicate that both model parsimony and the flexibility of the conditional sampling distribution are important for modelling the considered time series. The general finding arising from the empirical research is that in the case of transaction durations, the results reveal the utter inadequacy of the ACD models with a Burr conditional distribution. The ACD models with a conditional generalised gamma distribution have substantially higher explanatory power than the ACD models with a Burr conditional distribution. It thus appears that the conditional generalised gamma distribution is much more adequate than the conditional Burr distribution, even if both distributions depend on two free parameters. However, in the case of intraday trading volumes analysis, the situation is completely opposite to the results for the time series of transaction durations. In terms of Bayes factors, the ACV models with Burr innovations fit the volume data much better than the ACV models with generalised gamma innovations. Moreover, the overall qualitative conclusion is also that a formal Bayesian comparison of the explanatory power of the ACD/ACV models through their Bayes factors enables us to confirm the markedly higher adequacy of nonlinear ACD/ACV models than that of the simplest variant of this class, the linear ACD/ACV model.

As regards more specific empirical results, in the case of transaction durations, the BCACD specification always turned out to be the best. However, the observed data also supported the ABCACD specification, which took second or third place in the rankings. It is important to note that the ABCACD specification nests the BCACD one. Thus, in practice, the BCACD model is as flexible as the more parameterised ABCACD model, and, as a result, it wins the competition due to its relative simplicity. On the other hand, in the case of intraday trading volumes, the AACD model had the highest explanatory power, while the LACD model ranked second and was almost as good as the AACD model. Interestingly, the AACD specification encompasses the less parameterised LACD model, and hence a similar explanatory power can be seen in these two specifications. In addition, it should be emphasised that although there is strong evidence that nonlinear specifications of conditional mean help in explaining the data under analysis, the gain from augmenting and generalising the conditional mean equation is much smaller than the gain from replacing conditional innovation distributions.

One of the main objectives of this study was also to assess the stability of Bayes factors and model ranks with respect to the four methods used to approximate the marginal data density values, i.e., the HME, CHME, CAME, and pure MC-IS estimators. It should be noted that, in line with previous literature and research on approximations of the marginal data density, the HME estimator overestimates the marginal data density. Thus, the Bayes factors obtained by HME differ from the corresponding values obtained by the other three estimators, but the ranking of models for HME is quite close to the rankings for the CHME, MC-IS, and CAME estimators. Moreover, the resulting model ranks are similar enough to indicate the same leading specifications. In turn, in all cases, estimates of the marginal data density provided with the CHME, CAME, and pure MC-IS estimators are almost the same, i.e., the decimal logarithms of marginal data density values and the decimal logarithms of Bayes factors obtained using these three approximations are only slightly different. However, it is important to note that the application of an accurate method used to estimate the marginal data density could matter.

To sum up, on the basis of the results one can also state that the Bayes factors provide a universal and convenient tool of in-sample models' comparison. We have restricted our attention to ACD/ACV-type models, but it would be of great interest for future research to make formal in-sample Bayesian comparisons between ACD/ACV and stochastic conditional duration/volume (SCD/SCV) specifications. Additionally, a promising unexplored topic would also be a formal Bayesian comparison of different multiplicative error models with other conditional distributions such as inverse gamma, inverse Gaussian, log-normal, log-normal with time-varying variance, beta prime (or GB2), and Birnbaum-Saunders distributions. This challenge will be faced in future research.

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