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New solution bounds for unified algebraic Lyapunov equation and robust stability in delta operator system

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Abstract. In this paper, we utilize matrix transformations and inequalities to derive a novel upper bound and two lower bounds to solve the unified algebraic Lyapunov matrix equation (UALE). We then review existing bounds for the UALE and compare them with our new bounds, highlighting that our upper bound is the least restrictive among current results. The restrictions of our newly established lower bound are either weaker than the existing lower bounds or consistent with them. Our upper and lower bounds demonstrate increased accuracy over existing results through some numerical examples. As an application to linear systems, we illustrate how our upper bound can be employed to analyze the robust stability of the unified system based on the delta operator. Finally, we validate the effectiveness and superiority of our results through a series of numerical examples.

Keywords: unified algebraic Lyapunov matrix equation; matrix solution bounds; delta operator system; robust stability.

1. INTRODUCTION

The delta operator theory has developed rapidly in the control field in recent years. Its ability to handle high-speed sampling scenarios effectively, mitigating the numerical instability and performance degradation that often plague traditional shift operators in fast-sampling systems [1], has garnered widespread attention from researchers worldwide. Consequently, delta operator approach has been extensively adopted and deeply integrated into diverse cutting-edge domains, including but not limited to digital control systems, real-time signal processing, automatic control [2–6]. Moreover, the delta operator has established a unified mathematical framework for both continuous and discrete systems, providing a convenient approach for studying the commonalities and characteristics of these two types of systems.

The following symbol conventions are used in this paper. $R^{m\times n}(C^{m\times n})$ denotes the set of $m\times n$ real(complex) matrices. Let $\mathbb{S}_n$ be the set of $n\times n$ real symmetric matrices. $\mathbb{S}_n^{++}$ ($\mathbb{S}_n^{+}$) means that the set of $n\times n$ real symmetric positive definite (semidefinite) matrices. $M\succ N$ ($\succeq N$) means $M-N\in\mathbb{S}_n^{++}(\mathbb{S}_n^{+})$. $\mathbb{ST}_n=\{L : Re\lambda_i(L)<0, i=1,2,\dots,n\}$ denotes the set of $n\times n$ Hurwitz matrices. The identity matrix with appropriate dimensions is represented by I. We assume that the eigenvalues of $A\in\mathbb{S}_n$ are arranged so that $\lambda_1(A)\geq\lambda_2(A)\geq\dots\geq\lambda_n(A)$, $\rho(A)$ represents the spectral radius of $A\in R^{n\times n}$. $\theta\geq 0$ is the sampling period, δ is called the delta operator, defined by $\delta=d/dt$ $(\theta=0)$; $\delta=(q-1)/\theta$ $(\theta>0)$. q is the traditional shift operator.

Considering the following continuous-time linear system:

$$\dot{x}(t)=A_c x(t), \tag{1}$$

where $x(t)\in\mathbb{R}^{n\times 1}$ is the state variable; $A_c\in\mathbb{R}^{n\times n}$ is known constant matrices. In the design of high sampling rate digital control systems, such as unmanned aircraft flight control [7], autonomous car [8], we often need to discretize the above continuous systems for analysis. Taking into account the numerical superiority of the delta operator in the case of high-speed sampling, the system (1) should be discretized using the delta operator approach, resulting in the following delta operator system [9]:

$$\delta x(t_k)=Ax(t_k), \tag{2}$$

where $A=(e^{\theta A_c}-I)/\theta$, which implies $\theta A+I$ is non-singular.

In [10], Middleton *et al.* pointed out that there are two equivalent conditions for the system (2) to be asymptotically stable. One is that the system matrix A satisfies $\frac{\theta}{2}|\lambda_i(A)|+Re\lambda_i(A)<0$, the other is that the following so-called unified algebraic Lyapunov matrix equation (UALE) has a unique positive definite solution:

$$A^TP+PA+\theta A^TPA+Q=0, \tag{3}$$

which is considered in this paper, where $Q\in\mathbb{S}_n^{++}(\mathbb{S}_n^{+})$, $P\in\mathbb{S}_n^{++}(\mathbb{S}_n^{+})$ is the solution of the UALE (3) above. Apart from the stability of the system (2), robust stability analysis [11, 12], performance index calculation [8] and other properties of the delta operator system [9] can also be translated into a discussion of the UALE (3) above. When $\theta=0$, the UALE (3) degenerates into continuous algebraic Lyapunov matrix equation (CALE)

$$A^TP+PA=-Q. \tag{4}$$

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When $\theta = 1$, substituting A_1 for $A + I$, (3) becomes the discrete algebraic Lyapunov equation (DALE)

$$A_1^T P A_1 - P = -Q. \tag{5}$$

Therefore, the UALE (3) holds significant research value. It is not only crucial in the analysis and design of digital control and sampling period systems, but also its research outcomes can be smoothly transformed into the results of the CALE (4) and the DALE (5) by adjusting the sampling period. However, calculating the exact solution of the UALE can become computationally expensive when the dimension of the system matrix is high. In practical applications, a tight bound is usually sufficient.

Hence, many scholars have studied UALE's bounds and obtained some results over the past 30 years. Mrabti *et al.* [13] gave some lower bounds for simultaneous eigenvalues of UALE (3), Mori *et al.* [14] proposed a method to obtain new UALE's bounds by using the bounds of CALE and DALE, and the new lower bound for the determinant of UALE obtained improved some of the results in [13]. Suchomski *et al.* [15] derived an estimate for the condition number of the solution for the UALE. The matrix bounds are the most general findings [16–20], which can directly derive the corresponding eigenvalue bounds such as bounds of the extreme eigenvalues, the summation of eigenvalues, and so on. It's worth noting that the upper bounds in [16, 18–20] are derived under the following hypothesis

$$A + A^T + \theta A A^T \prec 0 \tag{6}$$

or stronger. Zhang and Liu [17] used a special similarity transformation to eliminate the hypothesis (6) of the upper bound. However, the solution range does not include the case of $\theta = 0$, which prevents us from directly linking the results of UALE and CALE studies. Furthermore, existing findings often exhibit significant errors in certain scenarios. For these reasons, we investigate the new estimate of matrix bounds for the UALE (3). The new upper bound contains the solvable range of the existing result, and the solvable range of the new lower bounds is consistent with or wider than the existing result. Furthermore, some experimental examples show that the new upper and lower bounds are tighter than the existing ones most of the time.

System parametric uncertainties, arising from environmental disturbances and measurement errors are inherent in practical engineering systems. Unmodeled uncertainties may induce performance degradation or instability in control strategies developed under idealized assumptions, particularly in critical domains such as intelligent manufacturing, renewable energy systems, and autonomous unmanned systems [21, 22]. Strong time-varying operational conditions and asynchronous information exchange intensify the coupling between uncertainties and time-delay effects, challenging conventional deterministic system frameworks. During recent decades, stability analysis of uncertain systems has remained a persistent research focus. Zhou *et al.* [23] proposed a robust stability bound for a continuous uncertain system by using matrix singular value inequalities and Lyapunov stability theory. The robust stability bound for discrete systems with time-varying disturbances is presented in [24]. Sipahi *et al.* [25] studied the complete robust stability of third-order LTI multiple time-delay systems by clustering the characteristic roots. Chesi *et al.* [26] established a unified analysis framework based on linear matrix inequalities (LMI) and sum-of-squares of polynomials (SOS) to address the robust stability problem of uncertain systems. This framework is capable of effectively handling various types of uncertainties. Based on Lyapunov stability and time-invariant methods, robust stability bounds for the discrete microgrid system with parameter perturbations was proposed [27]. Using these bounds, a robust controller was designed. However, the obtained robust stability bound requires solving the DALE, which is expensive when the system scale is large. The existing literature indicates that robust stability for unified systems with unstructured perturbations was uniquely addressed in [11], where singular value inequalities and Lyapunov stability theorems established stability bounds based on the upper bound of maximum singular values for the UALE's solution. However, in many practical problems, using the upper bound of the maximum singular value of a positive definite matrix instead of the upper bound of the positive definite matrix itself is extremely conservative. Therefore, this study applies novel upper matrix bounds for UALE solutions (3) to derive enhanced sufficient conditions for robust stability in unified systems with unstructured perturbations.

In this paper, we aim to propose tighter bounds for the unique positive definite solution of UALE, without the requirement that $A + A^T + \theta A A^T \prec 0$ or $\theta > 0$. Subsequently, using the obtained bounds, we design a sufficient condition to determine the asymptotic stability of the delta-operator perturbation system, thereby avoiding the excessive use of singular value inequalities that would strictly limit the judgment conditions. The main contributions of this paper are as follows.

1. We have proposed some new solution bounds for UALE, eliminating the restrictions of $A + A^T + \theta A A^T \prec 0$ or $\theta > 0$. Therefore, the boundaries we proposed have a wider range of applications.
2. Although the different theoretical methods prevent us from proving that the obtained bounds are tighter than the existing results, some numerical experiments can demonstrate that our bounds are tighter than the existing ones in most cases.
3. Based on the upper bound obtained for the positive definite solution of the UALE, we establish a sufficient condition for determining the asymptotic stability of a class of delta operator uncertain systems. This result generalizes some existing findings, and experiments demonstrate that our condition is weaker than previously established criteria.

The paper consists of the following sections. Section 2 provides some new estimations of the matrix bounds for the UALE (3). In Section 3, the previous solution bounds of UALE (3) are summarized and we continue some theoretical analysis and numerical comparison one by one. Section 4 gives a new robust stability bound for a unified system with unstructured disturbances. The conclusion is provided in Section 5.

2. NEW SOLUTION BOUNDS OF THE UNIFIED ALGEBRAIC LYAPUNOV EQUATION

The solution bounds of UALE (3) are obtained under different conditions in [16–20], the conditions of the upper bound in Zhang and Liu [17] are weaker than other results, and the conditions of the lower bound are better than or consistent with other existing results.

Theorem A. [17] If there exist a nonsingular matrix U such that $\lambda_1\left[(\theta\tilde{A}+I)(\theta\tilde{A}+I)^{\mathrm{T}}\right]<1$, then the solution P of UALE (3) has following upper and lower bounds:

$$P \succeq \frac{\theta\lambda_n(\tilde{Q})(\theta A+I)^{\mathrm{T}}\left(UU^{\mathrm{T}}\right)^{-1}(\theta A+I)}{1-\lambda_n\left[(\theta\tilde{A}+I)^{\mathrm{T}}(\theta\tilde{A}+I)\right]}+\theta Q=P_{us4},$$

$$P \preceq \frac{\theta\lambda_1(\tilde{Q})(\theta A+I)^{\mathrm{T}}\left(UU^{\mathrm{T}}\right)^{-1}(\theta A+I)}{1-\lambda_1\left[(\theta\tilde{A}+I)^{\mathrm{T}}(\theta\tilde{A}+I)\right]}+\theta Q=P_{ux5},$$

where $\tilde{A}=UAU^{-1}$, $\tilde{Q}=UQU^T$, U is nonsingular. However, $\lambda_1\left[(\theta\tilde{A}+I)(\theta\tilde{A}+I)^{\mathrm{T}}\right]<1$ the extended solution range still does not include $\theta=0$. We will give new upper and lower bounds that have the widest solvable range, improving Theorem A.

To get the main results, the following useful lemmas should be reviewed.

Lemma 1. For $A\in C^{n\times n}$, $B=(qI-A)^{-1}(qI+A)$. q is a constant that makes $qI-A$ nonsingular. Then

$$\lambda_i(B)=\frac{q+\lambda_i(A)}{q-\lambda_i(A)},\quad i=1,2,\cdots,n. \tag{7}$$

Proof. Suppose $C=qI-A$, so $\lambda_i(C)=q-\lambda_i(A)$. Then $\lambda_i(C^{-1})=\dfrac{1}{q-\lambda_i(A)}$. $B=C^{-1}(C+2A)=I+2C^{-1}(ql-C)=-(I-2qC^{-1})$. Hence, $\lambda_i(B)=-(1-2q\lambda_i(C^{-1}))=\dfrac{q+\lambda_i(A)}{q-\lambda_i(A)}$. □

Lemma 2. [28] For $A\in R^{n\times n}$, $\rho(A)<1$ if and only if there exists a nonsingular matrix D so that

$$\sigma_1\left(DAD^{-1}\right)<1.$$

In 2011, the algorithm for selecting the similarity matrix D was also given.

The following upper and lower matrix bounds of the UALE (3) can be obtained using some matrix transformations and inequalities.

Theorem 1. If there exists a nonsingular matrix U such that

$$\theta\tilde{A}^T\tilde{A}+\tilde{A}+\tilde{A}^T\prec 0, \tag{8}$$

for any constant $q>0$, the solution P of the UALE (3) has the following upper and lower bounds:

$$P\preceq (AU)^{-T}(\hat{A}^T(AU)^TP_{s1}(AU)\hat{A}+\overline{Q})(AU)^{-1}\equiv P_{s2}, \tag{9}$$

$$P\succeq (AU)^{-T}(\hat{A}^T(AU)^TP_{x1}(AU)\hat{A}+\overline{Q})(AU)^{-1}\equiv P_{x2}, \tag{10}$$

$$P\succeq (AU)^{-T}(\hat{A}^T(AU)^TP_{u1}(AU)\hat{A}+\overline{Q})(AU)^{-1}\equiv P_{ux3}, \tag{11}$$

where $\tilde{A}$, $\overline{A}$, $\hat{A}$, $\overline{Q}$, P_{x1}, P_{s1}, P_{u1}, respectively, are defined as

$$\begin{aligned}
\tilde{A}&=U^{-1}AU,\\
\tilde{Q}&=U^{\mathrm{T}}QU,\\
\overline{A}&=\tilde{A}^{-1}\left(I+\frac{\theta\tilde{A}}{2}\right),\\
\hat{A}&=\left(qI+\overline{A}\right)\left(qI-\overline{A}\right)^{-1},\\
\overline{Q}&=2q\left(qI-\overline{A}\right)^{-T}\tilde{Q}\left(qI-\overline{A}\right)^{-1},\\
P_{x1}&=(AU)^{-T}(\hat{A}^T\overline{Q}\hat{A}+\overline{Q})(AU)^{-1},\\
P_{s1}&=(AU)^{-T}\left(\frac{\lambda_1(\overline{Q})\hat{A}^T\hat{A}}{1-\lambda_1(\hat{A}^T\hat{A})}+\overline{Q}\right)(AU)^{-1},\\
P_{u1}&=(AU)^{-T}\left(\frac{\lambda_n(\overline{Q})\hat{A}^T\hat{A}}{1-\lambda_n(\hat{A}^T\hat{A})}+\overline{Q}\right)(AU)^{-1}.
\end{aligned}$$

Proof. Applying the following congruent transformation to UALE (3)

$$\tilde{P}\tilde{A}+\tilde{A}^T\tilde{P}+\theta\tilde{A}^T\tilde{P}\tilde{A}+\tilde{Q}=0, \tag{12}$$

where $\tilde{P}=U^{\mathrm{T}}PU$, $\tilde{Q}=U^{\mathrm{T}}QU$, $\tilde{A}=U^{-1}AU$. Then (12) can be rewritten as

$$\tilde{A}^T\tilde{P}\left(I+\frac{\theta\tilde{A}}{2}\right)+\left(I+\frac{\theta\tilde{A}}{2}\right)^T\tilde{P}\tilde{A}=-\tilde{Q}, \tag{13}$$

make an identical transformation on the left-hand side of (13)

$$\tilde{A}^T\tilde{P}\tilde{A}\left[\tilde{A}^{-1}\left(I+\frac{\theta\tilde{A}}{2}\right)\right]+\left[\tilde{A}^{-1}\left(I+\frac{\theta\tilde{A}}{2}\right)\right]^T\tilde{A}^T\tilde{P}\tilde{A}=-\tilde{Q}. \tag{14}$$

Then we get the following the CALE

$$\overline{P}\,\overline{A}+\overline{A}^T\overline{P}=-\tilde{Q}, \tag{15}$$

where $\overline{P}=\tilde{A}^T\tilde{P}\tilde{A}$. For any constant $q>0$, equation (15) can be transformed into the following form

$$\left(qI-\overline{A}\right)^T\overline{P}\left(qI-\overline{A}\right)-\left(qI+\overline{A}\right)^T\overline{P}\left(qI+\overline{A}\right)=2q\tilde{Q}. \tag{16}$$

Condition (8) implies $Re\lambda_i(\overline{A})<0$ $i=1,\cdots,n$. It means that $(qI-\overline{A})$ is invertible, then we can obtain the following DALE

$$\overline{P}=\left(qI-\overline{A}\right)^{-T}\left(qI+\overline{A}\right)^T\overline{P}\left(qI+\overline{A}\right)\left(qI-\overline{A}\right)^{-1}+\overline{Q}, \tag{17}$$

where $\overline{Q}=2q(qI-\hat{A})^{-T}\tilde{Q}(qI-\hat{A})^{-1}$. From equation (17) and the fact $\overline{P}\leq\lambda_1(\overline{P})I$, we have

$$\overline{P}\preceq\lambda_1(\overline{P})(qI-\overline{A})^{-T}(qI+\overline{A})^T(qI+\overline{A})(qI-\overline{A})^{-1}+\overline{Q}, \tag{18}$$

which implies

$$\lambda_1(\overline{P}) \le \lambda_1(\lambda_1(\overline{P})(qI-\overline{A})^{-T}(qI+\overline{A})^T(qI+\overline{A})(qI-\overline{A})^{-1}+\overline{Q}) \le \lambda_1(\overline{P})\lambda_1[(qI-\overline{A})^{-T}(qI+\overline{A})^T(qI+\overline{A})(qI-\overline{A})^{-1}] + \lambda_1(\overline{Q}). \tag{19}$$

Note that condition (8) is equivalent to

$$\tilde{A}^{-T}(\theta\tilde{A}^T\tilde{A}+\tilde{A}+\tilde{A}^T)\tilde{A}^{-1} \prec 0,$$

and then

$$\tilde{A}^{-1}+\frac{\theta}{2}I+\tilde{A}^{-T}+\frac{\theta}{2}I \prec 0 \quad i.e. \quad \overline{A}+\overline{A}^T \prec 0.$$

It is the same thing as

$$2\overline{A}^T\overline{A}+(qI-\overline{A})^T\overline{A}+\overline{A}^T(qI-\overline{A}) \prec 0. \tag{20}$$

Multiply the left-hand side of inequality (20) by $(qI-\overline{A})^{-T}$, multiply the right-hand side of inequality (20) by $(qI-\overline{A})^{-1}$, then

$$4(qI-\overline{A})^{-T}\overline{A}^T\overline{A}(qI-\overline{A})^{-1}+2\overline{A}(qI-\overline{A})^{-1}+2(qI-\overline{A})^{-T}\overline{A}^T \prec 0,$$

which implies

$$(I+2\overline{A}(qI-\overline{A})^{-1})^T(I+2\overline{A}(qI-\overline{A})^{-1}) \prec I, \tag{21}$$

notice that $I=(qI-\overline{A})(qI-\overline{A})^{-1}$, then

$$\lambda_1((qI-\overline{A})^{-T}(qI+\overline{A})^T(qI+\overline{A})(qI-\overline{A})^{-1}) < 1, \tag{22}$$

so inequalities (8) and (22) are equivalent, we get

$$\lambda_1(\overline{P}) \le \frac{\lambda_1(\overline{Q})}{1-\lambda_1((qI-\overline{A})^{-T}(qI+\overline{A})^T(qI+\overline{A})(qI-\overline{A})^{-1})}. \tag{23}$$

Substituting (23) into (17) gives

$$P \preceq (AU)^{-T}\left(\frac{\lambda_1(\overline{Q})\hat{A}^T\hat{A}}{1-\lambda_1(\hat{A}^T\hat{A})}+\overline{Q}\right)(AU)^{-1} \equiv P_{s1}. \tag{24}$$

Then substituting (24) into Eq.(17), the upper bound (9) is derived. Since equation (17) implies

$$\overline{P} \succeq \overline{Q}, \tag{25}$$

substituting inequality (25) into (17) and leads to the lower bound (10). Meanwhile, according to (17), we can easily obtain the following inequality

$$\overline{P} \succeq \lambda_n(\overline{P})(qI-\overline{A})^{-T}(qI+\overline{A})^T(qI+\overline{A})(qI-\overline{A})^{-1}+\overline{Q}, \tag{26}$$

and then

$$\lambda_n(\overline{P}) \ge \lambda_n\Big(\lambda_n(\overline{P})(qI-\overline{A})^{-T}(qI+\overline{A})^T(qI+\overline{A})(qI-\overline{A})^{-1}+\overline{Q}\Big) \ge \lambda_n(\overline{P})\lambda_n(\hat{A}^T\hat{A})+\lambda_n(\overline{Q}). \tag{27}$$

Note that condition (8) contains $\lambda_n(\hat{A}^T\hat{A})<1$, thus

$$\lambda_n(\overline{P}) \ge \frac{\lambda_n(\overline{Q})}{1-\lambda_n(\hat{A}^T\hat{A})}. \tag{28}$$

The lower bound (11) can be obtained by substituting the inequality (28) into the inequality (26) and performing an iteration similar to P_{s2}. □

Remark 1. When $\theta=0$, condition (8) is equivalent to $A \in \mathbb{ST}$ by Lemma 2.

Remark 2. Theorem A is derived on the premise that $\theta>0$, and Theorem 1 does not require this condition. Moreover, select $\theta>0$, $q=\frac{\theta}{2}$, then the lower bound P_{ux3} can degenerate into P_{ux5}.

Remark 3. Through the derivation of Theorem 1, it is easy to find that P_{s1} is also the upper bound of the UALE's solution, P_{x1} and P_{u1} are the lower bounds of the UALE's solution. So we have the following deduction.

Corollary 1. If condition (8) holds, then

$$P_{s2} \preceq P_{s1},\ P_{x2} \succeq P_{x1},\ P_{ux3} \succeq P_{u1}.$$

Proof.

$$\begin{aligned} P_{s2}-P_{s1} &= \widetilde{A}^T(AU)^TP_{s1}(AU)\widetilde{A}-\frac{\lambda_1(\overline{Q})}{1-\lambda_1(\hat{A}^T\hat{A})^{-1})}\widetilde{A}^T\widetilde{A} \\ &= \widetilde{A}^T(\frac{\lambda_1(\overline{Q})(\hat{A}^T\hat{A}-I)}{1-\lambda_1(\hat{A}^T\hat{A})}+\overline{Q})\widetilde{A} \\ &\preceq \widetilde{A}^T(\frac{\lambda_1(\overline{Q})(\lambda_1(\hat{A}^T\hat{A})-1)}{1-\lambda_1(\hat{A}^T\hat{A})}+\lambda_1(\overline{Q})I)\widetilde{A} \\ &= \widetilde{A}^T(-\lambda_1(\overline{Q})+\lambda_1(\overline{Q}))\widetilde{A} \\ &= 0, \end{aligned} \tag{29}$$

where $\widetilde{A}=(qI+\overline{A})(qI-\overline{A})^{-1}(AU)^{-1}$. The proofs of the $P_{x2} \succeq P_{x1}$ and $P_{ux3} \succeq P_{u1}$ are similar to the above. □

Remark 4. Compare the lower bound P_{x2} with P_{ux3}, when

$$\lambda_n((qI-\overline{A})^{-T}(qI+\overline{A})^T(qI+\overline{A})(qI-\overline{A})^{-1}) \ge 1-\frac{\lambda_n(\overline{Q})}{\lambda_1(\overline{Q})},$$

we have $P_{ux3} \succeq P_{x2}$, noticed

$$1 \ge \lambda_n((qI-\overline{A})^{-T}(qI+\overline{A})^T(qI+\overline{A})(qI-\overline{A})^{-1}) \ge 0,$$

then

$$\frac{\lambda_n(\overline{Q})}{1-\lambda_n((qI-\overline{Q})^{-T}(qI+\overline{Q})^T(qI+\overline{Q})(qI-\overline{Q})^{-1})} \ge \lambda_n(\overline{Q}).$$

So we know that when $Q \in \mathbb{S}_n^+$, there is $P_{ux3} \preceq P_{x2}$.

3. COMPARISON OF THE NEW UPPER AND LOWER BOUNDS WITH THE EXISTING RESULTS

We summarize the existing results on estimating the UALE bounds, and then compare the required conditions. The condition (8) of Theorem 1 is either weaker than or consistent with the existing results. Because of the different methods, the forms of the existing bounds vary greatly, so it is not easy to make a theoretical comparison. We illustrated that our upper and lower bounds are more accurate than the existing results in most cases by some numerical examples.

3.1. Comparison of the new upper bounds with the existing results

Surveying the literature, existing upper solution bounds of the UALE (3) are summarized in Table 1.

Table 1
Upper solution bounds of the UALE (3)

(U1) if $\lambda_1\left(A+A^T+\theta A^T A\right)<0$, M be chosen such that $Q \succeq M \succ 0$,	
$P \preceq \left[Q+(A+I)^T P_{us3}(A+I)-(1-\theta)A^T P_{ux6} A\right] \equiv P_{us5}$	[16]
(U2) if $\lambda_1(D)<1$,	
$P \preceq \bar{A}^T P_{u0}\bar{A}+2\theta(A-I)^{-T}A^T P_{u0}A(A-I)^{-1}+\bar{Q} \equiv P_{us6}$	[20]
$P \preceq (A+I)^T P_{us6}(A+I)+(\theta-1)A^T P_{ux7}A+Q \equiv P_{us7}$	[20]
$P \preceq (A+I)^T P_{us6}(A+I)+(\theta-1)A^T P_{ux8}A+Q \equiv P_{us8}$	[20]
(U3) if $\lambda_1(\bar{D})<1$,	
$P \preceq (A+I)^T \bar{P}_{u0}(A+I)+\theta A^T \bar{P}_{u0}A+Q-A^T P_{ux7}A \equiv P_{us9}$	[20]
$P \preceq (A+I)^T \tilde{P}_{u0}(A+I)+\theta A^T \tilde{P}_{u0}A+Q-A^T P_{ux8}A \equiv P_{us10}$	[20]
(U4) if $\sigma_1(F)<1$, $\sigma_1(F)\lambda_1(\theta Q)<1$, $\sigma_1(\Gamma^{\frac{1}{2}})\sigma_1(F)\sigma_1((\theta Q)^{\frac{1}{2}})<1$,	
$P \preceq \left[\theta Q^{\frac{1}{2}}F^T\left[\Gamma^{-1}-F(\theta Q)^{-1}F^T\right]^{-1}FQ^{\frac{1}{2}}+\frac{1}{4}\theta^2 Q^2\right]^{\frac{1}{2}} +\frac{1}{2}\theta Q \equiv P_{us11}$	[18]

The symbols in Table 1 are described as follows:

$$D \equiv \bar{A}^T\bar{A}+2\theta(A-I)^{-T}A^T A(A-I)^{-1},$$
$$\bar{D} \equiv (A+I)^T(A+I)+\theta A^T A,$$
$$\bar{Q} \equiv 2(A-I)^{-T}Q(A-I)^{-1},$$
$$\bar{A} \equiv (A+I)(A-I)^{-1},$$
$$P_{u0} \equiv \varphi\left[\bar{A}^T\bar{A}+2\theta(A-I)^{-T}A^T A(A-I)^{-1}\right]+\bar{Q},$$
$$\bar{P}_{u0} \equiv \bar{\varphi}\left[(A+I)^T(A+I)+\theta A^T A\right]+Q-A^T P_{ux7}A,$$
$$\tilde{P}_{u0} \equiv \tilde{\varphi}\left[(A+I)^T(A+I)+\theta A^T A\right]+Q-A^T P_{ux8}A,$$
$$\alpha \equiv \frac{\lambda_n(\bar{Q})}{1-\lambda_n(D)},$$
$$\bar{\varphi} \equiv \frac{\lambda_1\left(Q-A^T P_{ux5}A\right)}{1-\lambda_1(\bar{D})},$$
$$\tilde{\varphi} \equiv \frac{\lambda_1\left(Q-A^T P_{ux6}A\right)}{1-\lambda_1(\bar{D})},$$
$$\varphi \equiv \frac{\lambda_1(\bar{Q})}{1-\lambda_1(D)},$$
$$P_{ux6} \equiv S^{-1}\left(S\left(Q-M+\theta\eta A^T A\right)S\right)^{1/2}S^{-1},$$
$$P_{ux7} \equiv \bar{A}^T P_{l0}\bar{A}+\bar{Q}+2\theta(A-I)^{-T}A^T P_{l0}A(A-I)^{-1},$$
$$P_{ux8} \equiv \bar{A}^T \bar{P}_{l0}\bar{A}+\bar{Q}+2\theta(A-I)^{-T}A^T \bar{P}_{l0}A(A-I)^{-1},$$
$$S \equiv \left(AM^{-1}A^T\right)^{1/2},$$
$$\eta \equiv \frac{\theta\sigma_n^2(AS)+\sqrt{\theta^2\sigma_n^4(AS)+4\lambda_1^2\left(AM^{-1}A^T\right)\lambda_n[S(Q-M)S]}}{2\lambda_1^2\left(AM^{-1}A^T\right)},$$
$$\Gamma = \zeta(\theta A+I)^T(\theta A+I)+\theta Q,$$
$$\zeta = \frac{\lambda_1(\theta Q)}{1-\sigma_1^2(\theta A+I)},$$
$$F = \theta A+I.$$

Remark 5. Theorem A in [17] improves the result in [19], but the extended solution range of Theorem A still does not include $\theta=0$, which has been improved by Theorem 1. When $\theta>0$, the conditions (8) for Theorem 1 are the same as Theorem A.

Remark 6. Condition (U2) is equivalent to condition (6) from (U1) by simple calculation. Condition (8) of P_{s2} is weaker than conditions (U1) and (U2) by Lemma 2.

Remark 7. Condition (U3) and condition (U4) are both stronger than conditions (U1) and (U2). Hence, they are also stronger than condition (8). It is worth noting that the relationship of A, Q and θ in (U4) is coupled, and (U4) is also related to the result in [19], so it is more computationally intensive.

Remark 8. Based on the above comparison, condition (8) is weaker than all the conditions of available results. Because of the different theoretical methods, it is difficult for us to prove theoretically that P_{s2} is tighter than the existing results. We will use following numerical examples to demonstrate the superiority and validity of P_{s2}.

Example 1 will show that condition (8) is weaker than all known results and that the upper bound P_{s2} is superior.

Example 1. [17] Consider the UALE (3) with

$$A=\begin{bmatrix}-18.1 & 5.2 & 2.3 & 1.2\\ 0 & -1.8 & 3.3 & -5.5\\ 7.1 & 0 & -5.8 & -4.3\\ -3.2 & 1.1 & 6.4 & -10\end{bmatrix},\quad Q=\mathrm{diag}(1,1,5,1),\ \ \theta=0.1.$$

Compute $\lambda_1\left(A+A^T+\theta A^T A\right)=7.3452>0$, $\lambda_1(D)=1.212>1$, $\lambda_1(\bar{D})\approx 450>1$, $\rho(\theta A+I)=0.7848<1$. So no upper bound can be used except P_{s2}, P_{us4}, by using the Algorithm to find similarity matrix U derived from [28], there exists similarity matrix

$$U=\begin{bmatrix}0.6947 & 0.4795 & -0.3616 & -0.3414\\ 0.2998 & 0.4359 & -0.4507 & -0.8301\\ -0.2362 & -0.0227 & -0.4772 & -0.3526\\ 0.6097 & 0.7613 & -0.6621 & -0.2646\end{bmatrix},$$

which satisfies $\lambda_1((\theta\tilde{A}+I)(\theta\tilde{A}+I)^T) = 0.6159 < 1$. Choosing $q = 0.5$, we have

$$P_{s2} = \begin{bmatrix} 10.5808 & -4.8549 & 5.8866 & -5.6980 \\ -4.8549 & 3.1987 & -3.3999 & 2.1073 \\ 5.8866 & -3.3999 & 5.2163 & -2.8105 \\ -5.6980 & 2.1073 & -2.8105 & 3.6902 \end{bmatrix},$$

$$P_{us4} = \begin{bmatrix} 38.1775 & -14.1710 & 10.9259 & -16.1065 \\ -14.1710 & 21.3979 & -8.9296 & -2.5062 \\ 10.9259 & -8.9296 & 9.8765 & -2.1963 \\ -16.1065 & -2.5062 & -2.1963 & 13.2805 \end{bmatrix}.$$

Obviously, $P_{s2} \prec P_{us4}$. Consider the UALE (3) with

$$A = \begin{bmatrix} -21 & 1 & 1 & 3 \\ 4 & -12 & 4 & 0 \\ 1 & 2 & -3 & 1 \\ 3 & 3 & 2 & -10 \end{bmatrix}, \quad Q = \mathrm{diag}(1,1,1,2), \quad \theta = 0.$$

Then UALE(3) degenerates into CALE, there are no upper bound can be used except P_{s2}, P_{us6}, P_{us7}, P_{us8}. Selecting $q = \rho(\overline{A}) = 0.6482$ and by calculation we have

$$P_{s2} = \begin{bmatrix} 0.0439 & 0.0217 & 0.0354 & 0.027 \\ 0.0217 & 0.0894 & 0.0691 & 0.0292 \\ 0.0354 & 0.0691 & 0.342 & 0.0504 \\ 0.027 & 0.0292 & 0.0504 & 0.137 \end{bmatrix},$$

$$P_{us6} = \begin{bmatrix} 1.5356 & -0.1127 & -0.0365 & -0.1613 \\ -0.1127 & 1.1327 & -0.2786 & -0.1489 \\ -0.0365 & -0.2786 & 0.6156 & -0.2155 \\ -0.1613 & -0.1489 & -0.2155 & 1.0124 \end{bmatrix},$$

$$P_{us7} = \begin{bmatrix} 670.3064 & -82.5461 & 4.475 & -143.9162 \\ -82.5461 & 168.3795 & -50.3016 & -25.7203 \\ 4.475 & -50.3016 & 27.9436 & -14.1589 \\ -143.9162 & -25.7203 & -14.1582 & 104.2188 \end{bmatrix},$$

$$P_{us8} = \begin{bmatrix} 670.2642 & -82.4839 & 4.4426 & -143.8971 \\ -82.4839 & 168.8937 & -50.4364 & -25.9386 \\ 4.4426 & -50.4364 & 28.1722 & -14.55 \\ -143.8971 & -25.9386 & -14.55 & 105.366 \end{bmatrix},$$

$P_{s2} \prec \min\{P_{us6}, P_{us7}, P_{us8}\}$ by calculation, hence we can conclude that only P_{s2} applies and without losing superiority in two cases above.

Next example will compare P_{s2} with P_{us4}, P_{us5}, P_{us6}, P_{us7}, P_{us8} under the condition that $\lambda_1\left(A + A^T + \theta A^T A\right) < 0$.

Example 2. We randomly construct 100 UALEs to compare bound P_{s2} with P_{us4}, P_{us5}, P_{us6}, P_{us7}, P_{us8} under the condition that $\lambda_1\left(A_i + A_i^T + \theta A_i^T A_i\right) < 0$, $i = 1, 2, \ldots, 100$. Let

$$A_i = \begin{bmatrix} -5|a_{11}| & a_{12} & a_{13} & a_{14} \\ a_{21} & -5|a_{22}| & a_{23} & a_{24} \\ a_{31} & a_{32} & -5|a_{33}| & a_{34} \\ a_{41} & a_{42} & a_{43} & -5|a_{44}| \end{bmatrix},$$

$Q = I$, $\theta = 0.1$, where a_{jl} ($j,l = 1$ to 4) follow a uniform distribution on the interval $[0,1]$. And then we go through a loop to find 100 matrices A satisfies $\lambda_1\left(A_i + A_i^T + \theta A_i^T A_i\right) < 0$. Choose $q = \rho(\overline{A})$, for each i, we record the value of $\lambda_1(P_{s2} - P_{usk})$ ($k = 4,5,6,7,8$) and $\lambda_1(P_{usk} - P_{s2})$ ($k = 4,5,6,7,8$). The results are presented in Fig. 1 and Fig. 2, respectively.

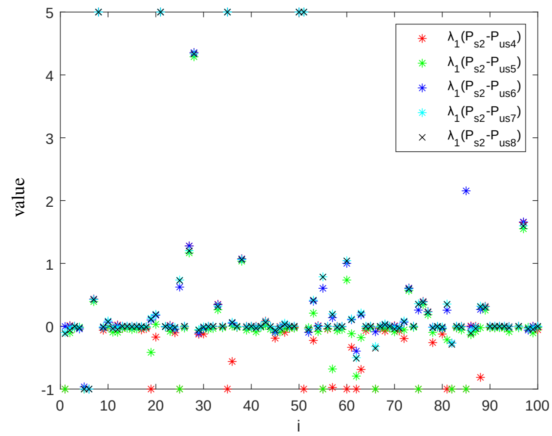

Fig. 1. The value of $\lambda_1(P_{s2} - P_{usk})$ ($k = 4,5,6,7,8$)

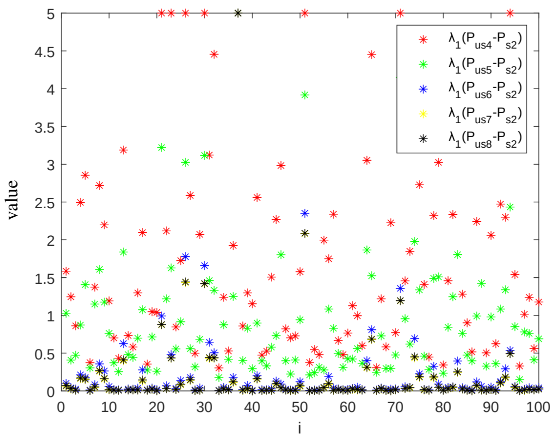

Fig. 2. The value of $\lambda_1(P_{usk} - P_{s2})$ ($k = 4,5,6,7,8$)

We can see that in the 100 examples in Fig. 1, P_{s2} outperformed P_{us4} in 79 examples, P_{us5} in 78 examples, P_{us6} in 61 examples, P_{us7} in 65 examples, and P_{us8} in 64 examples. In Fig. 2, all the points are above 0, This means that there is not a single instance in which the bound is better than P_{s2} in these 100 cases. From these numerical examples, we show that the upper bound P_{s2} is tighter than $P_{us4}, P_{us5}, P_{us6}, P_{us7}, P_{us8}$ in most instances.

Next we will compare P_{s2} with P_{us9}, P_{us10} under the condition that $\lambda_1((A+I)^T(A+I)+\theta A^T A) < 1$.

Example 3. We compare bounds P_{s2} with P_{us9}, P_{us10} in the similar way to Example 2. The difference is that we set

$$A_i = \begin{bmatrix} -5|a_{11}| & 0 & 0 & a_{14} \\ 0 & -5|a_{22}| & a_{23} & a_{24} \\ 0 & a_{32} & -5|a_{33}| & a_{34} \\ a_{41} & a_{42} & a_{43} & -5|a_{44}| \end{bmatrix}. \tag{30}$$

This is easily to satisfy $\lambda_1((A+I)^T(A+I)+\theta A^T A) < 1$. For each i, we select $q = 0.5 * \rho(\overline{A})$ and record the value of $\lambda_1(P_{s2} - P_{usk})$ $(k = 9, 10)$ and $\lambda_1(P_{usk} - P_{s2})$ $(k = 9, 10)$. The results are shown in Fig. 3 and Fig. 4, respectively.

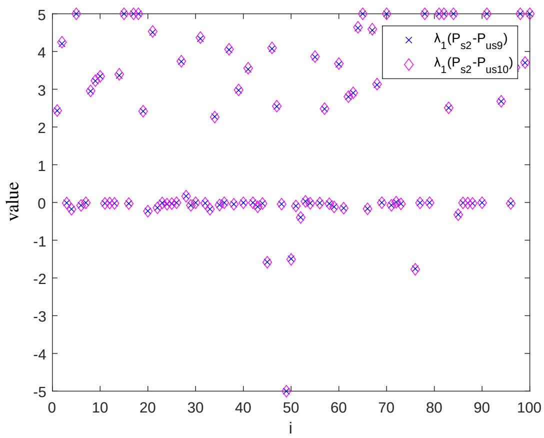

Fig. 3. The value of $\lambda_1(P_{s2} - P_{usk})$ $(k = 9, 10)$

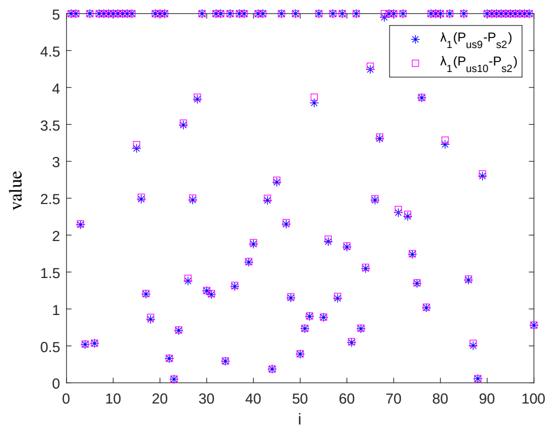

Fig. 4. The value of $\lambda_1(P_{usk} - P_{s2})$ $(k = 9, 10)$

We can see that in the 100 examples in Fig. 3, $P_{s2} \preceq P_{us9}$ in 49 examples, $P_{s2} \preceq P_{us10}$ in 49 examples. In Fig. 4, all the points are above 0. From these numerical examples, we determine that the upper bound P_{s2} is more tight than P_{us9}, P_{us10} in most instances.

3.2. Comparison of the new lower bounds with the existing results

Surveying the literature, we summarize existing matrix lower solution bounds of the UALE (3) in Table 2. The symbols in Table 2 are described as follows:

$$\begin{aligned}
&\tilde{Q} = U^T Q U, \quad \tilde{A} = U^{-1} A U,\\
&P_{l0} \equiv \bar{A}^T \bar{Q} \bar{A} + \bar{Q} + 2\theta(A-I)^{-T} A^T \bar{Q} A (A-I)^{-1},\\
&\bar{P}_{l0} \equiv \alpha\left[\bar{A}^T \bar{A} + 2\theta(A-I)^{-T} A^T A (A-I)^{-1}\right] + \bar{Q},\\
&\Gamma = \zeta(\theta A+I)^T(\theta A+I) + \theta Q,\\
&\zeta = \frac{\lambda_1(\theta Q)}{1-\sigma_1^2(\theta A+I)},\\
&\bar{Q} \equiv 2(A-I)^{-T} Q (A-I)^{-1},\\
&\bar{A} \equiv (A+I)(A-I)^{-1},\\
&\alpha \equiv \frac{\lambda_n(\bar{Q})}{1-\lambda_n(D)},\\
&F = \theta A + I.
\end{aligned}$$

Table 2
Lower solution bounds of the UALE (3)

(L1) The positive constant matrix M be chosen such that $Q \succeq M$,	
$P \succeq S^{-1}\left(S\left(Q - M + \theta\eta A^T A\right)S\right)^{1/2} S^{-1} \equiv P_{ux6}$	[16]
(L2) $P \succeq \bar{A}^T P_{l0} \bar{A} + \bar{Q} + 2\theta(A-I)^{-T} A^T P_{l0} A (A-I)^{-1} \equiv P_{ux7}$	[20]
$P \succeq \bar{A}^T \bar{P}_{l0} \bar{A} + \bar{Q} + 2\theta(A-I)^{-T} A^T \bar{P}_{l0} A (A-I)^{-1} \equiv P_{ux8}$	[20]
(L3) $\sigma_1(F) < 1$, $\sigma_1(F)\lambda_1(\theta Q) < 1$, $\sigma_1(\Gamma^{\frac{1}{2}})\sigma_1(F)\sigma_1(\theta Q)^{\frac{1}{2}} < 1$,	
$P \succeq \left[\theta Q^{\frac{1}{2}} F^T[(\theta Q)^{-1} - F\Gamma^{-1}F^T]^{-1} F Q^{\frac{1}{2}} + \frac{1}{4}\theta^2 Q^2\right]^{\frac{1}{2}} + \frac{1}{2}\theta Q \equiv P_{ux9}$	[18]

Remark 9. Using Lemma (2), the conditions (8), (L1), and (L2) are identical if there exists a unique positive definite solution for the UALE (3).

Remark 10. The condition of P_{ux5} in Theorem A cannot be applied when $\theta = 0$. Compared with P_{ux6}, choosing a positive definite matrix M will lead to more computation. Since the part one of condition (L3) contains condition (8), (L1) and (L2), condition (L3) of P_{ux9} is the strictest conditions.

Due to different theoretical methods, it is difficult to directly compare the precision of lower bound P_{ux3} and the existing results. Three numerical examples will illustrate the advantages and effectiveness of the P_{ux3}. Example 4 will compare P_{ux3} with P_{ux5}, P_{ux6}, P_{ux7}, P_{ux8} under condition $\rho(\theta A+I) < 1$.

Example 4. We compare bounds P_{ux3} with P_{ux5}, P_{ux6}, P_{ux7}, P_{ux8} in a similar way to Example 2. We set $\theta = 0.1$ and selected 100 random parameter matrices $A_i \in \mathbb{R}^{n\times n}$ $(i = 1, \ldots, 100)$ which have the same characteristics as A_i in Example 2 to satisfy $\rho(\theta A+I) < 1$. Taking $Q = \mathrm{diag}(1, 5, 10, 20)$, $q = \rho(\overline{A})$. The results are shown in Fig. 5 and Fig. 6.

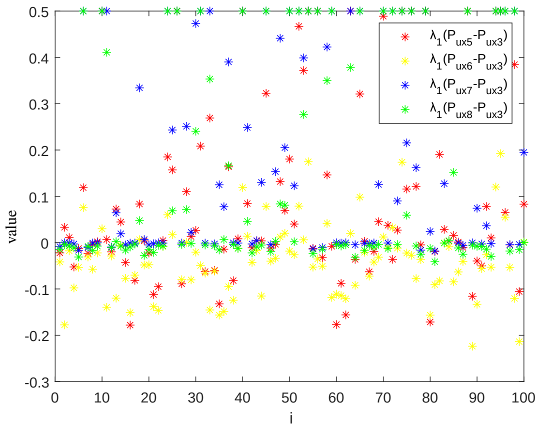

Fig. 5. The value of $\lambda_1(P_{usk} - P_{ux3})$, $k = 5,6,7,8$

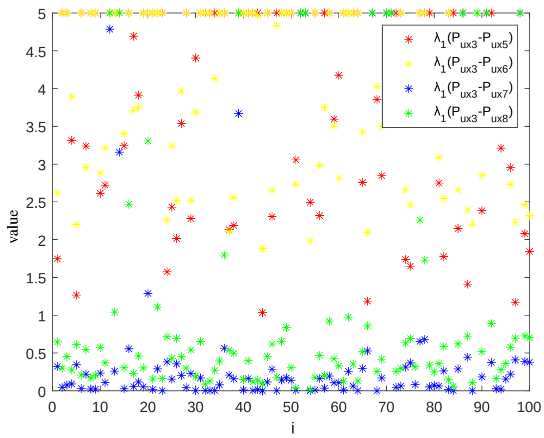

Fig. 6. The value of $\lambda_1(P_{us3} - P_{uxk})$, $k = 5,6,7,8$

We can see that in Fig. 5, P_{ux3} outperformed P_{ux5} in 40 examples, P_{ux6} in 73 examples, P_{ux7} in 42 examples, and P_{ux8} in 55 examples. In Fig. 6, however, all the points are above 0. This means that there is not a single instance in which the bound is tighter than P_{ux3} in these 100 cases. From these numerical examples, we determine that the lower bound P_{ux3} is more precise than P_{ux5}, P_{ux6}, P_{ux7}, P_{ux8} in most instances.

In conjunction with Remarks 7 and 10, the conditions for P_{us11} and P_{ux9} are more complex than most of the available results, the random experiments with a large number of samples are difficult to construct. So we choose the following two examples from [18] that illustrate P_{s2} and P_{ux3} better than P_{us11} and P_{ux9}, respectively in most cases.

Example 5. [18] Consider the UALE (3) with

$$A = \begin{bmatrix} -6.5 & 0.5 & 0.1 \\ -0.3 & -7.5 & 0.3 \\ 0.1 & 0.2 & -12.2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1.69 & 0.2 & -0.05 \\ 0.19 & 1.99 & 1.2 \\ 0.4 & 0.99 & 1.95 \end{bmatrix},$$

$\theta = 0.1$. By computation,

$$P_{s2} = \begin{bmatrix} 0.1932 & 0.0244 & -0.0035 \\ 0.0236 & 0.2148 & 0.1146 \\ 0.0380 & 0.0942 & 0.2044 \end{bmatrix},$$

$$P_{us11} = \begin{bmatrix} 0.1953 & 0.0243 & -0.0033 \\ 0.0237 & 0.2154 & 0.1142 \\ 0.0379 & 0.0940 & 0.2046 \end{bmatrix}.$$

The elements of the set of the spectrum of the matrix $P_{s2} - P_{us11}$ are $\{-0.0021, -0.0007, -0.0001\}$, then $P_{s2} \prec P_{us11}$. P_{ux3} and P_{ux9} also can be used. By computation,

$$P_{ux3} = \begin{bmatrix} 0.798 & 0.381 & -0.018 \\ 0.381 & 0.722 & 0.091 \\ -0.018 & 0.091 & 0.917 \end{bmatrix},$$

$$P_{ux9} = \begin{bmatrix} 0.1893 & 0.0233 & -0.0036 \\ 0.0224 & 0.2132 & 0.1144 \\ 0.0380 & 0.0942 & 0.2035 \end{bmatrix}.$$

The elements of the spectrum of the matrix $P_{ux9} - P_{ux3}$ are $\{-0.0044, -0.0011, -0.0010\}$, we get $P_{ux9} \prec P_{ux3}$.

Example 6. [18] Consider the UALE (3) with

$$A = \begin{bmatrix} -1.62 & 0.1 & -0.01 & 0.13 \\ 0.035 & -1.51 & 0.35 & -0.025 \\ -0.002 & 0.003 & -1.3 & -0.01 \\ 0.0005 & 0.02 & 0.1 & -1.44 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.95 & 0.75 & 0.05 & 0.18 \\ 0.39 & 0.845 & 0.6 & 0.45 \\ 0.15 & 0.5 & 1.8 & 0.3 \\ 0.25 & 0.1 & 0.915 & 4.21 \end{bmatrix}, \quad \theta = 0.5.$$

By computation,

$$P_{s2} = \begin{bmatrix} 0.4953 & 0.4005 & 0.0456 & 0.1018 \\ 0.2119 & 0.4604 & 0.3617 & 0.2534 \\ 0.0922 & 0.3026 & 1.1124 & 0.2153 \\ 0.1384 & 0.0695 & 0.5469 & 2.2884 \end{bmatrix},$$

$$P_{us11} = \begin{bmatrix} 0.5000 & 0.4034 & 0.0432 & 0.1032 \\ 0.2162 & 0.4724 & 0.3821 & 0.2588 \\ 0.0951 & 0.3202 & 1.1538 & 0.2108 \\ 0.1434 & 0.0723 & 0.5425 & 2.3010 \end{bmatrix}.$$

The elements of the set of the spectrum of the matrix $P_{s2} - P_{us11}$ are $\{-0.0507, -0.0164, -0.0027, -0.0008\}$, then $P_{s2} \prec P_{us11}$.

P_{ux3} and P_{ux9} also can be used.

$$P_{ux3} = \begin{bmatrix} 0.4953 & 0.4005 & 0.0456 & 0.1018 \\ 0.2119 & 0.4604 & 0.3617 & 0.2534 \\ 0.0922 & 0.3026 & 1.1124 & 0.2153 \\ 0.1384 & 0.0695 & 0.5469 & 2.2884 \end{bmatrix},$$

$$P_{ux9} = \begin{bmatrix} 0.4953 & 0.3989 & 0.0395 & 0.1003 \\ 0.2116 & 0.4580 & 0.3555 & 0.2539 \\ 0.0897 & 0.2955 & 1.0642 & 0.2003 \\ 0.1396 & 0.0678 & 0.5315 & 2.2777 \end{bmatrix}.$$

The elements of the spectrum of the matrix $P_{ux9} - P_{ux3}$ are $\{-0.0546, -0.0061, -0.0001, -0.0007\}$, we get $P_{ux9} \prec P_{ux3}$.

Combining Examples 1–6, we can show that the lower bound P_{ux3} and the upper bound P_{s2} are tighter than the existing bounds in most cases.

4. STABILITY ROBUSTNESS BOUND FOR THE DELTA OPERATOR SYSTEM WITH STRUCTURED UNCERTAINTY

In this section, we will apply the upper bound P_{s2} in Theorem 1 to solve the robust stability problem for the systems (2). Let us consider the following delta operator system with structured uncertainty as follows

$$\delta x(t) = (A + E(t))x(t), \tag{31}$$

where $A \in \mathbb{R}_n$ satisfies the condition (8), $E(t) \in \mathbb{R}_n$ is a time-varying uncertainty matrix.

Next, we derive the following stability robustness bound by constructing a positive semi-definite matrix and employing Lyapunov's method in conjunction with some key matrix and singular inequalities.

Theorem 2. The uncertain delta operator system (31) is asymptotically stable, if

$$\sigma_1(E(t)) < \sqrt{\frac{a - \sigma_1\left[Q^{-1/2}(I+\theta A)^T P_{s2}(I+\theta A)Q^{-1/2}\right]}{a(a+\theta)\sigma_1(Q^{-1})\sigma_1(P_{s2})}} \equiv E_u, \tag{32}$$

where $P, Q \in \mathbb{S}_n^{++}$ satisfy the UALE (3), $a > 0$ such that

$$a > \sigma_1\left[Q^{-1/2}(I+\theta A)^T P_{s2}(I+\theta A)Q^{-1/2}\right]. \tag{33}$$

Proof. If (32) holds, then

$$(a+\theta)\sigma_1(P_{s2})\sigma_1^2(E(t)) < \sigma_n(Q) - (1/a)\sigma_n(Q)\sigma_1\left[Q^{-1/2}(I+\theta A)^T P(I+\theta A)Q^{-1/2}\right], \tag{34}$$

which implies that

$$\begin{aligned} 1 - (1/a)\sigma_1&\left[Q^{-1/2}(I+\theta A)^T P(I+\theta A)Q^{-1/2}\right] \\ &> (a+\theta)\sigma_1(P_{s2})\sigma_1^2[E(t)]/\sigma_n(Q) \\ &> (a+\theta)\sigma_1(P_{s2})\sigma_1^2[E(t)]\sigma_1(Q^{-1}) \\ &> (a+\theta)\sigma_1\left[Q^{-1/2}E(t)^T P_{s2}E(t)Q^{-1/2}\right]. \end{aligned} \tag{35}$$

From inequality (35), we can obtain

$$(a+\theta)E(t)^T P_{s2}E(t) \prec Q - (1/a)(I+\theta A)^T P_{s2}(I+\theta A). \tag{36}$$

Since $P_{s2} \succeq P$, we get

$$(a+\theta)E(t)^T PE(t) \prec Q - (1/a)(I+\theta A)^T P(I+\theta A). \tag{37}$$

Notice that

$$\left[-\sqrt{a}E(t)^T P^{\frac{1}{2}} + (1/\sqrt{a})(I+\theta A)^T P^{\frac{1}{2}}\right] \cdot \left[-\sqrt{a}E(t)^T P^{\frac{1}{2}} + (1/\sqrt{a})(I+\theta A)^T P^{\frac{1}{2}}\right]^T \succeq 0. \tag{38}$$

From inequality (38), we have

$$(I+\theta A)^T PE(t) + E(t)^T P(I+\theta A) \preceq aE(t)^T PE(t) + (1/a)(I+\theta A)^T P(I+\theta A). \tag{39}$$

Substituting inequality (39) into inequality (37), we get

$$Q - \theta E(t)^T PE(t) - \theta A^T PE(t) - \theta E(t)^T PA - PE(t) - E(t)^T P \succ 0. \tag{40}$$

Since $Q = -A^T P - PA - \theta A^T PA$, then inequality (40) can be converted to

$$\left[A+E(t)\right]^T P + P\left[A+E(t)\right] + \theta\left[A+E(t)\right]^T P\left[A+E(t)\right] \prec 0, \tag{41}$$

which implies

$$x(t)^T\left(A_E^T P + PA_E + \theta A_E^T PA_E\right)x(t) < 0, \tag{42}$$

where $A_E = A + E(t)$, $x(t)$ is the state vector of system (31). Choosing a Lyapunov function

$$V(t,x) = x^T Px,$$

from inequality (42), we obtain

$$\delta V(t,x) = \delta\left(x^T Px\right) = x^T(A_E^T P + PA_E + \theta A_E^T PA_E)x.$$

Hence $\delta V(t,x) < 0$, according to Lyapunov's second stability method, the system (31) is asymptotically stable. The positive definiteness of (32) is ensured by inequality (33). The proof is complete. □

We will illustrate the effectiveness of the stability robustness bound (32) by the following numerical experiments.

Example 7. [9] Consider a unified system that has a linear perturbations as follows

$$\delta x(t) = \left(\begin{bmatrix} -0.21 & 0.1 & 0 \\ -0.06 & -0.15 & 0 \\ 0 & -0.1 & -0.1 \end{bmatrix} + E \right) x(t), \tag{43}$$

$x(0) = (0.85, 0.85, 0.85)^T$ with $\theta = 0.001$, where $E \in R^{3\times 3}$ is uncertainty matrix.

By choosing $Q = I$, $a = 2\sigma_1[Q^{-1/2}(I+\theta A)^T P_{s2}(I+\theta A)Q^{-1/2}]$ and using Theorems 1 and 2, we obtain $E_u = 0.0718$ $(\theta = 0.001)$. Applying the judgment criteria for such discrete uncertain systems from [11] and [24], we obtain $\sigma_1(E) < 0.0592$, where P is calculated through 'dlyap' in MATLAB. We construct three random matrices E_i $(i = 1,2,3)$ such that $\sigma_1(E) < 0.0718$. Details are as follows:

$$E_1 = \begin{bmatrix} 0.0212 & 0.0145 & 0.0171 \\ 0.0399 & 0.0286 & 0.0053 \\ 0.0141 & 0.0081 & 0.0381 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0.0155 & 0.0060 & 0.0213 \\ 0.0036 & 0.0530 & 0.0126 \\ 0.0082 & 0.0207 & 0.0351 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} 0.0192 & 0.0094 & 0.0445 \\ 0.0210 & 0.0133 & 0.0141 \\ 0.0469 & 0.0213 & 0.0054 \end{bmatrix},$$

where $\sigma_1(E_1) = 0.063$, $\sigma_1(E_2) = 0.066$, $\sigma_1(E_3) = 0.069$. It is easy for us to observe that the conditions given by [11] and [24] cannot determine the stability of the system under the aforementioned three disturbances. However, according to Theorem 2, we can determine that the system is stable under all three disturbances. Figures 7–9 illustrate that the system (31) remains asymptotically stable under the three different perturbations mentioned above.

Example 8. Consider the following joint error open-loop disturbance system of the drone internal loop disturbance observer (DO) and attitude control [29]:

$$\dot{x}(t) = (\Xi + E)x(t), \tag{44}$$

where

$$\Xi = \begin{bmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{bmatrix},$$

$$\xi_{11} = \xi_{13} = \xi_{31} = \xi_{32} = 0^{3\times 3}, \quad \xi_{12} = I^{3\times 3},$$

$$\xi_{21} = \begin{bmatrix} -500 & 0 & 0 \\ 0 & -519.2 & 0 \\ 0 & 0 & -540.5 \end{bmatrix},$$

$$\xi_{22} = \begin{bmatrix} -200 & 0 & 0 \\ 0 & -207.7 & 0 \\ 0 & 0 & -202.7 \end{bmatrix},$$

$$\xi_{23} = \begin{bmatrix} -100 & 0 & 0 \\ 0 & -103.8 & 0 \\ 0 & 0 & -67.57 \end{bmatrix},$$

$$\xi_{33} = \begin{bmatrix} -1000 & 0 & 0 \\ 0 & -1038 & 0 \\ 0 & 0 & -675.7 \end{bmatrix},$$

$E \in R^{9\times 9}$ is uncertainty matrix.

By choosing $Q = I$, $a = 80$ and using Theorems 1 and 2, we obtain $E_u = 0.0675(\theta = 0)$. Applying the judgment criteria for such discrete uncertain systems from [11], we obtain $\sigma_1(E) < 0.0006$, where P is calculated through 'lyap' in MATLAB. We can conclude that, at least in some cases, the judgment conditions given in Theorem 2 are more lenient compared to the existing judgment criteria.

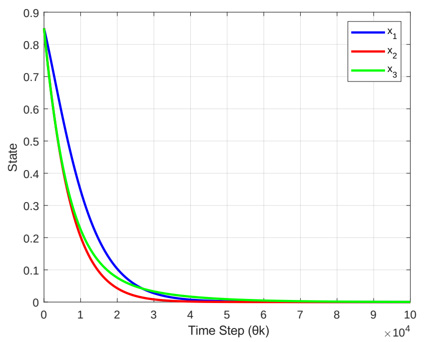

Fig. 7. State vector evolution for the system (31) with E_1

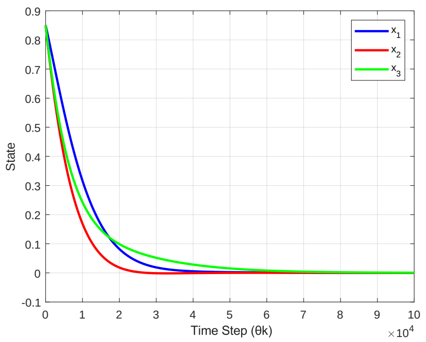

Fig. 8. State vector evolution for the system (31) with E_2

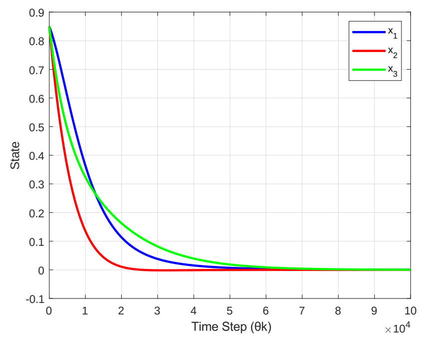

Fig. 9. State vector evolution for the system (31) with E_3

5. CONCLUSIONS

In this paper, by utilizing a special similarity transformation, a bilinear transformation with parameters, and matrix inequalities, we developed a new upper bound for the unified algebraic Lyapunov equation (UALE), which requires the weakest conditions among all existing results. Subsequently, we obtained two new lower bounds using similar methods. Both theoretical proofs and numerical simulations demonstrate that these lower bounds are tighter and involve fewer constraints than existing ones. Finally, we applied the upper bound derived for the UALE to propose a new robust stability criterion for unified systems described by the delta operator with unstructured perturbations.

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