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MATERIAL SCIENCE AND NANOTECHNOLOGY

Levitation of permanent magnet ball

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Abstract. The paper presents approaches to the levitation of permanent magnet levitation in a forced inhomogeneous magnetostatic field. For a permanent magnet ball, the magnetic field distribution is derived in analytical forms using the separation of variables separation method. Distribution is given by power functions and Legendre polynomials. The force (i.e., material force) is caused by the vertical magnetization of a permanent magnet and reluctivity change at the magnet boundary. The levitation force is evaluated using the generalized Maxwell stress tensor, coenergy, material force density, and equivalent magnetic dipole methods. The levitation forces are presented in terms of both magnetic permeabilities. The stability of the equilibrium point is investigated. The frequencies of free and damped oscillations are evaluated.

Keywords: permanent magnet; levitation force; material force.

LIST OF MAIN SYMBOLS

a_n, b_n, c_n, d_n	–	constants for magnetic potentials,
b_γ	–	damping force coefficient (of viscosity Stokes force),
B_r, B_θ, B_φ	–	magnetic flux density components,
g	–	the acceleration of gravity e.g. 9.80665 m·s^{-2},
H_0	–	forced magnetic field strength (constant item),
h_1	–	forced magnetic field strength gradient at $z = 0$,
$\vec{i}_u$	–	versor for u-th coordinate,
m	–	ball mass,
$\vec{N}$	–	inhomogeneity force density (material force),
$P_n(x)$	–	Legendre polynomials,
$\vec{Q}$	–	magnetization force density (material force),
R	–	radius of the ball,
V_μ	–	magnetic scalar potential,
W_C	–	co-energy,
$\Delta\vec{M}$	–	magnetization of the permanent magnet,
γ	–	conductivity,
$\delta_\mathrm{D}(r-R)$	–	Dirac delta for singularity point $r = R$,
δ_{ik}	–	Kronecker's delta,
ε_0	–	dielectric permittivity of free space,
ε	–	dielectric permittivity,
μ_0	–	magnetic permeability of free space,
μ_out	–	magnetic permeability out of the permanent magnet,
μ	–	magnetic permeability inside the permanent magnet,
ν_0	–	reluctivity of free space,
ν	–	reluctivity,
ρ	–	radius/distance from z-axis,
σ_{uw}	–	Maxwell stress tensor for u–w axes.

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1. INTRODUCTION

Levitation driven by an inhomogeneous magnetostatic field (e.g., a gradient field) is often applied in many technological solutions [1–6]. Nowadays, technological engineering and analysis often focus on magnetostatic levitation arising in the presence of magnets for maglev trains [7] and energy harvesting [8] while contemporary medicine applies levitation for drug delivery and tissue cure [3]. Magnetic levitation allows for the simulation of conditions to advance tissue engineering for regenerative medicine [9]. The interest of both engineers and medics is focused on antibiotic treatment and some laboratory investigations [6], and there are often applied technologies that use the phenomenon of levitation in electromagnets, valves, and separators [10, 11].

The paper considers magnetostatic levitations driven by material forces acting on a permanent magnet ball. The material forces arise for two reasons. Firstly, there are magnetization forces. Secondly, of lower impact, there are inhomogeneous forces (Table 1).

It should be pointed out that magnetostatic forces can act either separately [11–13], or in combination with other electromagnetic forces [4, 14–16].

This paper investigates the magnetostatic levitation forces, the stability of equilibrium points, and the oscillation frequencies of a permanent magnet ball in a magnetostatic field. Analytical solutions are obtained by the method of separation of variable potential. The magnetic field distributions are given by power functions and Legendre polynomials. The analytical solutions reveal knowledge about the influence of magnetization and magnet parameters on the field distribution and magnetic forces.

The magnetic forces are evaluated using the following five methods

- Maxwell stress tensor generalized method [5, 17],
- both magnetization and inhomogeneity material force densities (Table 1),
- co-energy,
- magnetization current, and
- equivalent dipole (only for an isotopic ball).

Table 1
Levitations in the electromagnetic field and acting forces

Electrostatic levitations		Magnetostatic levitations		Electromagnetic levitations	
charged objects or capacitances	non-charged objects, polarized dielectrics, or dielectric permittivity changes	objects without magnets, magnetic permeability (reluctivity) changes	objects with magnets	conductive and non-magnetic objects	conductive and magnetic objects
Coulomb forces	material forces (of inhomogeneity and polarization)	material forces (of inhomogeneity)	material forces (of magnetization)	Lorentz force	Lorentz force and material forces
$\vec{f}_{\mathrm{Q}} = \rho \vec{E}$	$\vec{N}_{\varepsilon} = -\frac{1}{2} E_u E_w \operatorname{grad}(\varepsilon_{uw})$ $\vec{Q}_{\varepsilon} = \frac{1}{2} \operatorname{grad}(\Delta P_u E_u) - E_u \operatorname{grad}(\Delta P_u)$	$\vec{N}_{\mu} = \frac{1}{2} B_u B_w \operatorname{grad}(\nu_{uw})$	$\vec{Q}_{\mu} = \frac{1}{2} \operatorname{grad}(B_u \Delta M_u) - B_u \operatorname{grad}(\Delta M_u)$	$\vec{f}_{\mathrm{L}} = \vec{J} \times \vec{B}$	$\vec{f}_{\mathrm{L}} + \vec{N}_{\mu}$

The presented solutions can also be treated as default solutions to magnetostatic problems. Moreover, the analytical solution is immanently a benchmark test for numerical algorithms. The presented solution can also constitute starting points for numerical algorithms and be part of hybrid methods combining analytical and numerical procedures [1, 18].

The novelty of the presented analyses and solutions is demonstrated through consideration of:

- thorough force analyses utilizing five evaluation methods,
- material forces arising at the surface of permanent magnet imposed by not only magnetization but also by reluctivity change,
- extension (compared to [4, 5]) of imposed fields class in the form of polynomials (1),
- analysis of the vertical levitations stability, and
- evaluation of oscillation frequencies.

2. MAGNETOSTATIC LEVITATION OF PERMANENT MAGNET

Let us consider a static magnetic field in which the component along the z-axis is oriented upwards (Fig. 1).

The field wraps a homogeneous and magnetized ball, i.e., a permanent magnet. The magnetization of a permanent magnet is vertical. In such a case magnet can be raised by the phenomena of both magnetization of the magnet and the inhomogeneity of the reluctivity at the magnet boundary [18, 19].

The strength of the imposed magnetic field strength (far from the ball) along the z-axis is given by the sum of N terms as follows

$$H_z = H_0 + \sum_{k=1}^{N-1} h_k z^k, \tag{1}$$

where H_0 and h_k are real constants. For $N > 2$ 2 the imposed field gradient can be spatially variable (for $N = 2$ field gradient along the z-axis is always constant). For non-current problems, the magnetostatic field can be described by magnetic scalar potential V_μ [12, 18]. The axially symmetric magnetostatic field is independent of the longitudinal angle φ, thus

$$V_\mu(r, \theta, \phi) = V_\mu(r, \theta). \tag{2}$$

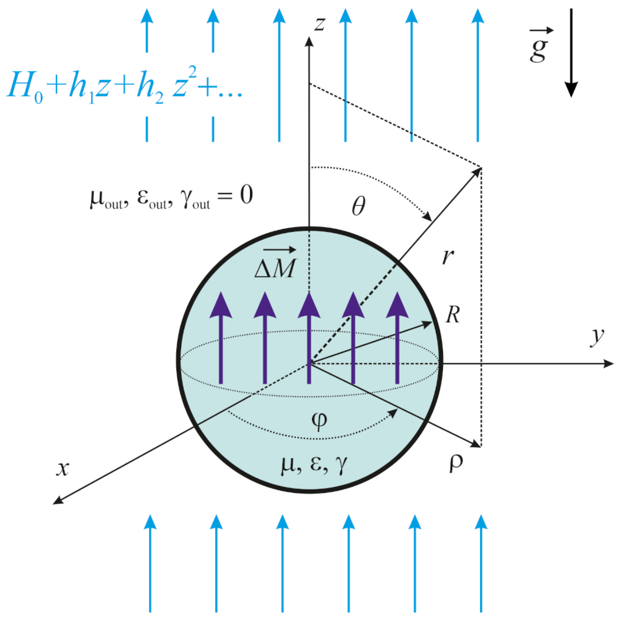

Fig. 1. A permanent magnet ball in an imposed magnetostatic field

Hence, according to the relation

$$\vec{H} = -\operatorname{grad}(V_\mu) \tag{3}$$

results in

$$H_\varphi = -\operatorname{grad}_\varphi(V_\mu) = -\frac{\partial V_\mu}{r \sin\theta \, \partial\varphi} = 0. \tag{4}$$

For a permanent magnet, the constitutional relation is given in the following form

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \left(\vec{M}_{\mathrm{prop\_H}} + \vec{M}_{\mathrm{mag}} \right) = \mu \vec{H} + \mu_0 \vec{M}_{\mathrm{mag}} = \mu \vec{H} + \mu \Delta \vec{M}. \tag{5}$$

The magnetization is the only anisotropic parameter of a permanent magnet (it is oriented along the z-axis – Fig. 1). The magnetization of a permanent magnet is constant inside and outside the ball, thus

$$\operatorname{div}\left(\Delta\vec{M}\right) = 0. \tag{6}$$

Only the magnetization at the boundary of the permanent magnet changes.

Gauss's law for the magnetic field is as follows

$$\operatorname{div}\vec{B} = \operatorname{div}\left(\mu\vec{H} + \mu\Delta\vec{M}\right) = \mu\operatorname{div}\left(\vec{H}\right) = 0, \tag{7}$$

and (4) leads to a partial differential equation for the inside or outside region of the magnet

$$\frac{\partial}{r^2\partial r}\left(r^2\frac{\partial V_\mu}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V_\mu}{\partial\theta}\right) = 0. \tag{8}$$

It should be emphasized that for the inhomogeneous divergence-free field (7) there is an immanent radial component. This fact is considered in the following stability analysis (Section 3).

The variable separation method in the form of

$$V_\mu(r,\theta) = \mathrm{R}(r)\,\mathrm{P}(\theta). \tag{9}$$

Facilitates the separation of (8) into two independent equations

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R(r)}{\mathrm{d}r}\right) - n(n+1)R(r) = 0, \tag{10}$$

$$n(n+1)\mathrm{P}(\theta) + \frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}(\mathrm{P}(\theta))}{\mathrm{d}\theta}\right) = 0, \tag{11}$$

where n is an integer [10, 20]. The solution of (10) inside the ball takes the form of

$$R_n(r) = c_n r^n + d_n r^{-n-1}. \tag{12}$$

For a positive integer n the functions r^n are increasing, but r^{-n-1} are decreasing. Hence, it must be set $d_n = 0$ to obtain a limited solution for the magnetic field strength inside the ball. Solutions of (11) are well-known Legendre polynomials $\mathrm{P}_n(\cos(\theta))$ [20]. The magnetic scalar potential solutions for $n = 1, 2, \ldots, N$ create a general solution in the form of a sum as follows

$$V_\mu(r,\theta) = \sum_{n=1}^{N} c_n r^n \mathrm{P}_n(\cos(\theta)), \tag{13}$$

which is a series developed to satisfy all boundary conditions. Namely, the continuity of the tangential magnetic field strength H_θ and the normal magnetic flux density B_r on the surface of the ball constitute two conditions. The third condition for constants a_n, b_n, and c_n results from the fact that the distribution of the magnetic field strength must be described by (1) at a large distance from the permanent magnet ball. Hence, the constants c_n $(n = 1, 2, \ldots, N)$ are as follows

$$c_n = a_n\frac{(2n+1)}{(\mu_{\mathrm{out}}/\mu)n + n + 1} + \delta_{n1}\frac{\mu_{\mathrm{out}}\Delta M_1}{2\mu + \mu_{\mathrm{out}}}, \tag{14}$$

where constants $a_1 = -H_0$, $a_n = -h_{n-1}/n$ $(n = 2, \ldots, N)$, and

$$b_n = a_n R^{2n+1}\left(\frac{2n+1}{(\mu_{\mathrm{out}}/\mu)n + n + 1} - 1\right) + \delta_{n1}R^3\frac{\mu_{\mathrm{out}}\Delta M_1}{2\mu + \mu_{\mathrm{out}}}. \tag{15}$$

Outside the permanent magnet (12) and (13) lead to the solution

$$V_{\mu,\mathrm{out}}(r,\theta) = \sum_{n=1}^{N}\left(a_n r^n + b_n r^{-n-1}\right)\mathrm{P}_n(\cos(\theta)). \tag{16}$$

The magnetic scalar potential (13) and (16) lead to magnetic field strength (Fig. 2) via the gradient formula (3) and to the magnetic flux density via formula (5), respectively.

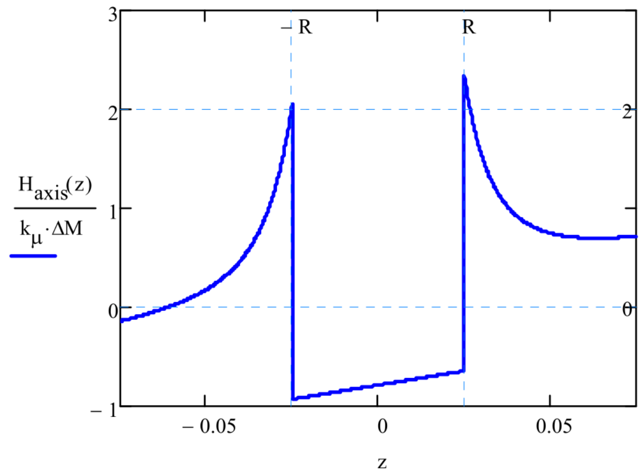

Fig. 2. Exemplary relative magnetic field strength distribution along the z-axis ($k_\mu = 1/(1+2\mu_{\mathrm{out}}/\mu)$)

The magnetic field strength and flux density determine the force acting on the permanent magnet ball, i.e., magnetostatic levitation force that may lift the magnet. Magnetostatic forces physically act at the boundary of a permanent magnet ball, because only there do both the changes in magnetization and isotropic reluctivity occur. These parameters change abruptly at the boundary (as the Heaviside step function).

The magnetostatic force is evaluated using the following five methods.

a) Maxwell stress tensor [4, 5, 10, 18]

$$\vec{\sigma}_z = -H_z\vec{B} + \vec{i}_z\left(\vec{H}\vec{B}\right)/2, \tag{17}$$

the method leads to the force

$$F_z = F_{\mathrm{M}z} + \Delta F_z, \tag{18}$$

where integral over the sphere surrounding the permanent magnet equals to

$$F_{\mathrm{M}z} = -2\pi\int_0^\pi\left(-\frac{B_z B_r}{\mu_{\mathrm{out}}} + e_\mu\cos\theta\right)R^2\sin\theta\,\mathrm{d}\theta, \tag{19}$$

and ΔF_z is the volume integral of the Maxwell stress tensor generalized method [5]. The additional summand ΔF_z can be easily calculated using the following formula

$$\Delta F_z = 2\pi \int_0^{\pi} \int_0^{R} (\sigma_{\theta r} - \sigma_{r\theta}) \sin^2(\theta)\, r \,\mathrm{d}r \,\mathrm{d}\theta, \tag{20}$$

and it vanishes if reluctivity is an isotropic parameter and the magnetization is constant $\Delta F_z = 0$ [4, 5].

b) The co-energy method [10, 18] requires to calculate the derivative of magnetic co-energy as follows

$$F_{\mathrm{Cz}} = \frac{\partial W_{\mathrm{C}}}{\partial z}. \tag{21}$$

c) Physically, the magnetostatic levitation force is driven by the magnetization force that acts on the magnets and the inhomogeneity force that acts where the reluctivity changes (at the magnet boundary). Both forces constitute the so-called material force F_{Nz}.

Mathematically, the material force [10, 15, 21] describes the magnetization force density

$$\vec{Q}_\mu = \frac{1}{2} \mathrm{grad}\,(B_u \Delta M_u) - B_u \mathrm{grad}(\Delta M_u), \tag{22}$$

and the inhomogeneity force density

$$\vec{N}_\mu = \frac{1}{2} B_u B_w \mathrm{grad}(\nu_{uw}), \tag{23}$$

where the summation of dummy indices (i.e., summation of repeated indices) is assumed. In the case under consideration, the reluctivity changes (in a stepwise manner) only at the boundary of the permanent magnet. The inhomogeneity force density is zero both inside and outside a homogeneous magnet.

The magnetizing force $F_{\Delta Mz}$ is integral over the region V (including the entire permanent magnet) as follows

$$F_{\Delta\mathrm{Mz}} = \int_V Q_{\mu r} \cos\theta \,\mathrm{d}V = 2\pi \int_0^{\pi} Q_{\mu r} \cos\theta R^2 \sin\theta \,\mathrm{d}\theta. \tag{24}$$

Subsequently, force density (22) leads to a relation

$$F_{\Delta\mathrm{Mz}} = \int_V \left\{ \frac{1}{2} \mathrm{grad}_r (B_r \Delta M_r) + \frac{1}{2} \mathrm{grad}_r (B_\theta \Delta M_\theta) \right\} \cos\theta \,\mathrm{d}V - \int_V \left\{ B_r \mathrm{grad}_r (\Delta M_r) + B_\theta \mathrm{grad}_r (\Delta M_\theta) \right\} \cos\theta \,\mathrm{d}V. \tag{25}$$

The step-way changes in magnetization at the boundary of a permanent magnet are given by the Heaviside step function $\Delta M_u(r, R) = \Delta M_u \mathbf{1}(r - R)$ where $u = r, \theta$. Hence, $\Delta M(r, R) = \Delta M \mathbf{1}(r - R)$ and $\mathrm{grad}_r (X \Delta M_u \mathbf{1}(r - R)) = \Delta X \Delta M_u \delta(r - R)$ where

$$\int_0^{R+} \delta(r - R)\,\mathrm{d}r = 1. \tag{26}$$

Moreover, it is satisfied as follows

$$\int_V X \mathrm{grad}_r (\Delta M_u) \cos\theta \,\mathrm{d}V = 2\pi \int_0^{\pi} X_{\mathrm{av}} \Delta M_u \cos\theta R^2 \sin\theta \,\mathrm{d}\theta, \tag{27}$$

where X_{av} is the average value of X over the radial integration interval. Hence, (25) can be rewritten as follows

$$F_{\Delta\mathrm{Mz}} = 2\pi \int_0^{\pi} \left\{ \frac{1}{2} \Delta B_\theta \Delta M_\theta \right\} \cos\theta R^2 \sin\theta \,\mathrm{d}\theta - 2\pi \int_0^{\pi} \{ B_{r,\mathrm{av}} \Delta M_r \} \cos\theta R^2 \sin\theta \,\mathrm{d}\theta - 2\pi \int_0^{\pi} \{ B_{\theta,\mathrm{av}} \Delta M_\theta \} \cos\theta R^2 \sin\theta \,\mathrm{d}\theta, \tag{28}$$

where $\Delta M_r = -\Delta M \cos\theta$, $\Delta M_\theta = +\Delta M \sin\theta$. The radial flux density is continuous at the boundary, thus $B_{r,\mathrm{av}} = B_r$. However, the tangential component of the magnetic flux density is not continuous, so its average value $B_{\theta,\mathrm{av}} = (B_{\theta,\mathrm{in}} + B_{\theta,\mathrm{out}})/2$.

The inhomogeneity force appears at the magnet boundary and leads to the following force formula

$$F_{\mathrm{Nz}} = \pi \int_0^{\pi} \left\{ \left(\frac{1}{\mu_{\mathrm{out}}} - \frac{1}{\mu} \right) B_r^2 + (\mu - \mu_{\mathrm{out}}) H_\theta^2 \right\} \cos\theta R^2 \sin\theta \,\mathrm{d}\theta. \tag{29}$$

The total force – the so-called material force – arising along the z-axis is equal to

$$F_z = F_{\Delta\mathrm{Mz}} + F_{\mathrm{Nz}}. \tag{30}$$

d) The magnetostatic force can be also calculated for an isotropic permanent magnet ball using an equivalent magnetic dipole [10, 18]. Namely, the second term in (16) has the same form as the field distant from a dipole with the equivalent moment, as follows

$$m_{\mathrm{eff}1} = 4\pi \mu_{\mathrm{out}} b_1, \tag{31}$$

and analogously for the higher terms of the dipole. The force acting on a dipole is given by the formula

$$F_{\mathrm{Dz}} = 4\pi \mu_{\mathrm{out}} \sum_{k=1}^{N-1} b_k h_k. \tag{32}$$

According to formula (15), it follows that levitation force is a polynomial of ball radius odd powers, i.e., $R^3, \ldots, R^{2N-1}$ – Fig. 3.

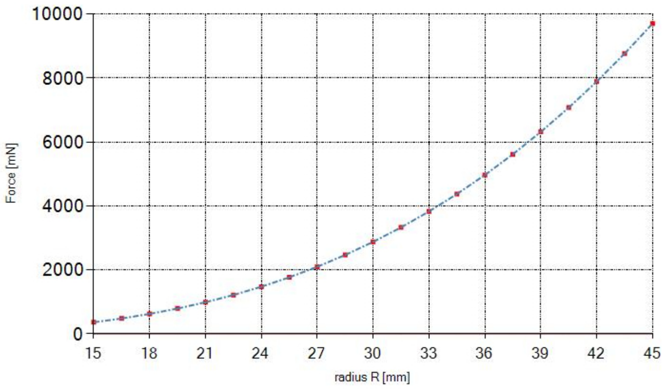

Fig. 3. Magnetostatic levitation force for a permanent magnet ball evaluated by Maxwell stress tensor method (points), material force (dash-dot line) vs. ball radius $R = (15 \div 45)$ mm ($\mu_{r\mathrm{rel}} = 5$, $\mu_{\mathrm{outrel}} = 1$, $H_0 = 7\,\mathrm{kA{\cdot}m^{-1}}$, $h_1 = 1.8\,\mathrm{MA{\cdot}m^{-2}}$, $h_2 = 0$, $h_3 = 0$, $\Delta M_z = 500\,\mathrm{kA{\cdot}m^{-1}}$)

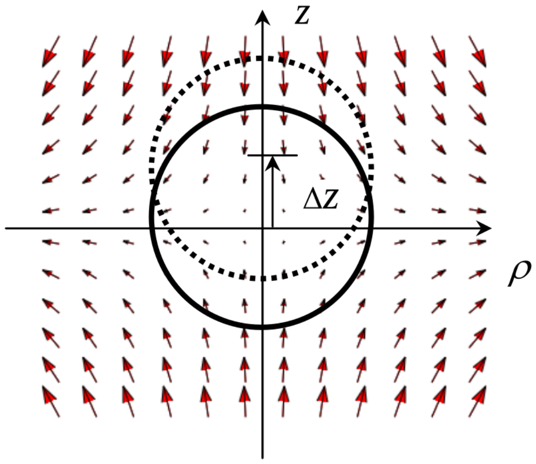

Fig. 4. Infinitesimal displacement $\Delta z \to 0$ of the ball along the z-axis

e) The method of equivalent magnetizing currents (in free space) based on the basic formula [10, 18, 19]

$$\mathrm{curl}\left(\vec{M}_{\mathrm{mag}}\right) = \vec{J}_\mu\,, \tag{33}$$

where according to (5) is satisfied $\mu_0 \vec{M}_{\mathrm{mag}} = \mu \Delta \vec{M}$. For a given magnetization in the spherical coordinate system, the material current density (33) is as follows

$$\vec{J}_\mu = \vec{i}_\varphi \frac{\mu}{\mu_0} \Delta M \sin(\theta) \delta_{\mathrm{D}}(r - R). \tag{34}$$

Hence, the force given by the Lorentz force density

$$\vec{f}_{\mathrm{J}\mu} = \vec{J}_\mu \times \mu_0 \vec{H} \tag{35}$$

leads to the force acting on the magnet along the z-axis as follows

$$F_{\mathrm{J}\mu z} = -2\pi \frac{\mu}{\mu_0} \Delta M \int\limits_0^\pi \left\{ H_r \sin\theta + H_{\theta,\mathrm{av}} \cos\theta \right\} R^2 \sin^2\theta \,\mathrm{d}\theta. \tag{36}$$

The total force equals to

$$F_z = F_{\mathrm{J}\mu z} + F_{\mathrm{N}z}\,. \tag{37}$$

The forces evaluated for magnetostatic levitation using the methods of Maxwell stress tensor F_z, co-energy F_{Cz}, material force $F_{\Delta M z} + F_{Nz}$, equivalent dipole F_{Dz}, and equivalent magnetizing currents $F_{\mu z} + F_{Nz}$ always lead to the same results

$$F_z = F_{\mathrm{C}z} = F_{\Delta \mathrm{M}z} + F_{\mathrm{N}z} = F_{\mathrm{D}z} = F_{\mathrm{J}\mu z} + F_{\mathrm{N}z}\,. \tag{38}$$

3. STABILITY. FREE AND DAMPED OSCILLATIONS

If the magnetic force lifts the permanent magnet ball ($F_z = mg$), then let us virtually move it by a displacement Δz along the z-axis (Fig. 4). Thus, the axial component of the magnetic field strength in the ball changes as follows

$$H_z = H_0 + h_1(z + \Delta z) + \ldots = H_0' + h_1 z + \ldots, \tag{39}$$

where the changed value of the constant item is equal to

$$H_0' = H_0 + h_1 \Delta z. \tag{40}$$

The imposed field (1) horizontal component is as follows

$$B_\rho = -\frac{1}{2}\rho \frac{\partial B_z}{\partial z}\,, \tag{41}$$

and results simply from (1), (5) and Maxwell relation

$$\mathrm{div}\left(\vec{B}\right) = 0, \tag{42}$$

for an axially-symmetric field $B_\varphi = 0$.

For the gradient field ($N = 2$) or when the second term in (1) dominates, relation (41) takes the following form

$$B_\rho = -\frac{1}{2}\rho \mu h_1\,. \tag{43}$$

As a consequence, the total force acting on the displaced stationary ($\Delta \dot{z} = 0$) magnet ball is as follows

$$F(\Delta z) = F_0 - k\Delta z, \tag{44}$$

where the elastic k constant graphically represents the slope of the curve $F(\Delta z)$ – Fig. 5. If the elastic constant is positive $k > 0$, then the levitation equilibrium point $\Delta z = 0$ is stable.

Additionally, in Figs. 6 and 7 the levitation forces are presented vs. broad relative permeability changes ±50%. The forces increase as each permeability rises and $\mu > \mu_{\mathrm{out}}$.

For checking, a global average relative error is defined as follows

$$\mathrm{err} = \frac{1}{n+1} \sum_{i=0}^{n} \left| \frac{(F_{\Delta M zi} + F_{\mathrm{N}zi}) - F_{\mathrm{M}zi}}{F_{\mathrm{M}zi}} \right|. \tag{45}$$

Exemplary, for forces in Fig. 7 ($n = 20$ intervals per curve) the average relative error is equal to 1.6E–11.

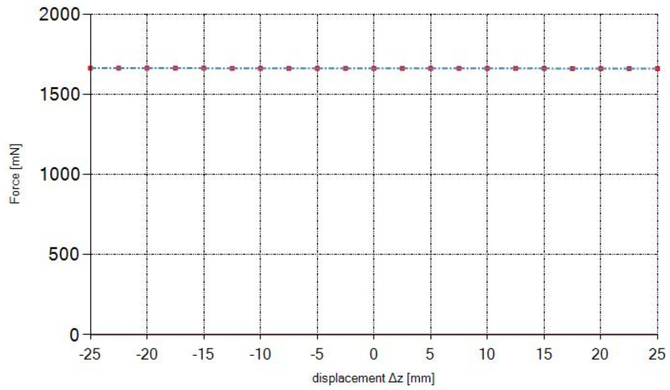

Fig. 5. Magnetostatic levitation force for a permanent magnet ball evaluated by Maxwell stress tensor method (points), material force (dash-dot line) vs. magnet displacement Δz [mm] ($R = 25$ mm, $\mu_{\text{rel}} = 5$, $\mu_{\text{outrel}} = 1$, $H_0 = 7$ kA·m^{-1}, $h_1 = 1.8$ MA·m^{-2}, $h_2 = 0$, $h_3 = 0$, $\Delta M_z = 500$ kA·m^{-1})

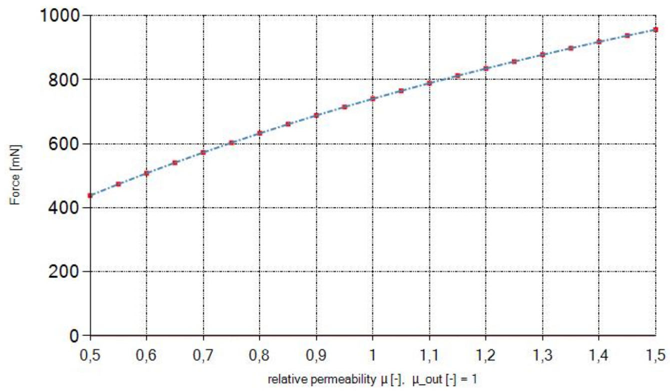

Fig. 6. Magnetostatic levitation force for permanent magnet ball evaluated by Maxwell stress tensor method (points), material force (dash-dot line) vs. magnet relative magnetic permeability μ [–] ($R = 25$ mm, $\mu_{r\text{rel}} = (0.5\div1.5)$, $\mu_{\text{outrel}} = 1$, $H_0 = 7$ kA·m^{-1}, $h_1 = 1.8$ MA·m^{-2}, $h_2 = 0$, $h_3 = 0$, $\Delta M_z = 500$ kA·m^{-1})

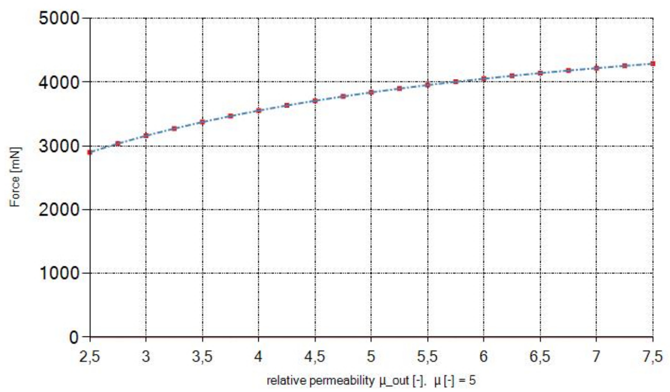

Fig. 7. Magnetostatic levitation force for permanent magnet ball evaluated by Maxwell stress tensor method (points), material force (dash-dot line) vs. outer region relative magnetic permeability μ_{out} [–] ($R = 25$ mm, $\mu_{r\text{rel}} = 5$, $\mu_{\text{outrel}} = (2.5\div7.5)$, $H_0 = 7$ kA·m^{-1}, $h_1 = 1.8$ MA·m^{-2}, $h_2 = 0$, $h_3 = 0$, $\Delta M_z = 500$ kA·m^{-1})

The movement of the ball leads to a change in the magnetic field inside the ball ($0 < r \le R$). Hence, according to Faraday law, the electric field is the induced electric field E_{ind} and satisfies the relation

$$2\pi r E_{\text{ind}} = -\pi r^2 \mu h_1 \Delta\dot{z}. \tag{46}$$

If the permanent magnet conductivity is positive ($\gamma > 0$), the induced currents of density γE_{ind} are subject to Lorentz forces of density as follows

$$\Delta f_{\text{Lz}} = -\gamma E_{\text{ind}} B_\rho = -\frac{1}{4}\pi\gamma\mu^2 h_1^2 r^2 \sin(\theta)\Delta\dot{z}, \tag{47}$$

hence the total Lorentz force along the z-axis equals ($b_\gamma > 0$)

$$\Delta F_{\text{Lz}} = \int_V \Delta f_{\text{Lz}}\,\mathrm{d}V = -\frac{1}{20}\pi^2\gamma\mu^2 h_1^2 R^5 \Delta\dot{z} = -b_\gamma \Delta\dot{z}. \tag{48}$$

The equation of motion of a ball in a uniform gravitational field g takes the form

$$m\Delta\ddot{z} = F_0 - mg - k\Delta z - b_\gamma \Delta\dot{z}, \tag{49}$$

and is decisive for the stability of the equilibrium point [3, 13, 22]. Some viscosity and damping forces provided by an outer medium can also incorporated to the right-hand side of (49). The eddy currents inside the ball are immanently related to the power losses as follows

$$\Delta p_{\text{ind}} = \int_V \gamma E_{\text{ind}}^2\,\mathrm{d}V = 2\pi \int_0^\pi \int_0^R \gamma E_{\text{ind}}^2 r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta, \tag{50}$$

and taking into account (46) one finally obtains

$$\Delta p_{\text{ind}} = \frac{1}{5}\pi\gamma (G_1 \Delta\dot{z})^2 R^5. \tag{51}$$

Moreover, (49) indicates the oscillation frequency of the value

$$f_{\text{osc}} = \frac{\omega_{\text{osc}}}{2\pi} = \frac{1}{2\pi}\sqrt{\omega_0^2 - \beta^2}, \tag{52}$$

where the free oscillation frequency is equal to

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{k/m}, \tag{53}$$

and the damping coefficient

$$\beta = \frac{b_\gamma}{2m} = \frac{\pi^2\gamma}{40m}\mu^2 h_1^2 R^5. \tag{54}$$

The condition of equilibrium stability requires two inequalities $F_0 = \Delta F(\Delta z = 0) > 0$ and $k > 0$. According to (32) the stability of magnet levitation (Fig. 8) is guaranteed in the gradient field ($N = 2$) when it is satisfied

$$b_1 h_1 > 0. \tag{55}$$

This condition can be easily derived by physical reasoning. There are two types of material forces acting, i.e., magnetization ($\Delta M_z \neq 0$) and inhomogeneity ($\mu \neq \mu_{\text{out}}$). If $\mu > \mu_{\text{out}}$, then inhomogeneity forces act outward on the boundary sphere of the ball (Fig. 8).

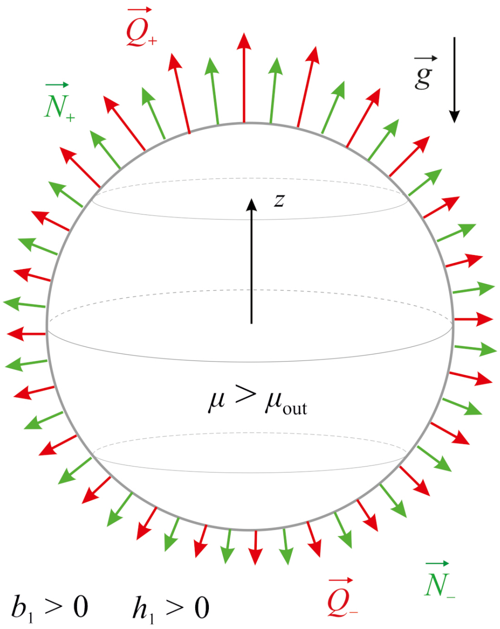

Fig. 8. Material forces acting on permanent magnet ball for $\mu > \mu_{\text{out}}$

If the permeability of the magnet ball is greater than the magnetic permeability of the surrounding region, i.e., $\mu > \mu_{\text{out}}$, and the magnitudes of the magnetic field vectors are greater in the upper hemisphere than in the lower hemisphere (imposed gradient field $h_1 > 0$), then the total inhomogeneity force acts upwards (towards the z-axis) (if $\mu < \mu_{\text{out}}$ this force acts against the z-axis). Subsequently, the magnetization forces act on the boundary sphere and lift the ball for $\Delta M_z > 0$ ($b_1 > 0$), which leads to condition (55).

The same can be concluded if both $h_1 < 0$ and $b_1 < 0$.

If the magnetostatic levitation force along the z-axis exceeds the gravitational force mg (regarding the buoyant force of the outer medium), then the magnet can be lifted and its equilibrium is stable.

4. CONCLUSIONS

The permanent magnet is placed in an inhomogeneous magnetostatic field, e.g., a gradient field (1). The magnetization of the magnet is vertical (5). The analytical solutions to the problem of levitation of a permanent magnet ball were presented (16).

The distribution of magnetic fields for the levitation problem was obtained by the method of separation of variables in the spherical coordinate system (9).

The levitation was evaluated using five methods: Maxwell stress tensor, co-energy, material force density (i.e., magnetization and inhomogeneity forces), magnetizing currents, and equivalent dipole (17)–(32). All five methods yield the same results (45).

The obtained solution is valid in a wide range of parameters, e.g., the magnetic permeability of the magnet and its surroundings, radius, and distribution of the imposed field, which is not easy to achieve using numerical methods. Hence, the presented analytical solutions can be used to test numerical algorithms.

The presented analyses led to the following conclusions:

- The analytical solutions of the field and force distributions are rapid tools for design and calculations.
- The permanent magnet movement, stability (if levitating), and oscillation frequencies are predicted by the model developed for a wide range of parameters (magnetic permeabilities, radius, and imposed field shape – i.e., polynomial coefficients).
- The obtained analytical solutions can be used as benchmark tasks for numerical algorithms.
- They can be incorporated into hybrid analytical-numerical algorithms for magnetic field analysis.

Further investigation can be focused on the levitation of permanent magnets of different shapes.

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