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h-Stability of set differential equations

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In this paper, we introduce the notion of h-stability for set-valued differential equations. Necessary and sufficient conditions are established by using Lyapunov theory. Then, based on the obtained results, we study the h-stability of perturbed and cascaded systems. Finally, an example illustrates the proposed theorems.

Key words: h-stability, Lyapunov theory, set-valued differential, generalized Hukuhara derivative, perturbed system, cascaded systems

1. Introduction

The pressure of the various current problems arising in systems theory (control of evolution systems, viability, etc.) led to the development of the fundamental concept of differential calculus. This pressure forced numerous applied mathematicians to apply set-valued maps as evidently as the common single-valued maps.

Mathematical sciences have shown a reluctance to deal with sequences of sets and set-valued maps, despite the emergence of interesting new vistas for the applications of mathematics.

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In his book 'Topologie Generale' (the first chapter in 1940), Bourbaki disregarded set-valued maps. He rather restricted his study to single-valued maps to make them bijective. However, Kuratowski, in his important book "Topologie" (1966), gave set-valued maps their appropriate status. Despite this, set-valued maps remained confined to a period of time. Their generalization was no more than a mathematical curiosity.

On the other hand, the need to solve problems that emerged in different fields of knowledge, such as economics [24,25] and artificial intelligence [12] motivated mathematicians to analyze set-valued maps, which in turn led to diverse applications. Set-valued analysis deals with the study of the continuity of set-valued maps, the analysis of a linear or non-linear multivalued function (the existence of solutions), and differentiation, convergence, integration, and measurability of set-valued maps [13].

In 1990, Aubin and Frankowska [2] constructed a differential calculus of set-valued maps by considering the map as a graph. The definition of the derivative of a set-valued mapping on the space of nonempty convex compact subsets of Euclidean space (nonlinear space) improves their theory. In fact, there exist several procedures to differentiate between two sets [18], namely the Hukuhara difference [15] and Hukuhara derivatives.

Thereafter, Stefanini [31,32] extended the Hukuhara difference to the concept of a generalized Hukuhara difference. This generalized Hukuhara difference has been widely adopted to study interval dynamical systems in order to determine if a difference exists between any pair of intervals. It is therefore an inestimable mathematical concept for investigating the theory of interval numbers.

After that, Stefanini and Bede [33] introduced the generalized Hukuhara derivative version. Different types of derivatives can be used to investigate set-valued equations compared with the Hukuhara derivative. The Hukuhara derivative has some weaknesses that make it difficult to examine the properties of set-valued differential equations, as the Hukuhara difference does not always exist for two set-values.

The stability of set-valued mappings leads to many difficulties. There are some difficulties in defining the stability and asymptotic stability of set differential equations related to non-decreasing functions. Determining the stability and asymptotic stability of set differential equations is not a trivial problem. To resolve this problem, approaches are suggested in [11]. Although the properties of stability and asymptotic stability are not common for set differential equations, many studies [4,27,36] examining these solution properties have been published. Furthermore, the use of Lyapunov functions served to study the stability of set differential equations with Hukuhara derivative [20]. It is well known that Lyapunov's direct method and its applications play a significant role in the field of theoretical development. This method is used to investigate the asymptotic stability of dynamical systems with no need to solve any equation [23, 37]. However, the second Lyapunov method, also called Lyapunov's indirect method, is efficient for stability analysis and control system development.

There are multiple stability concepts, such as absolute stability, uniform stability, asymptotic stability, exponential stability and h-stability. Recently, there has been increasing interest in improving the h-stability theory.

H-stability was developed by Pinto [30] (1984), who ensured the stability of a weaker stable system under few perturbations [29] compared to other systems showing uniform Lipschitz stability and exponential stability. In other words, Pinto extended his study of exponential asymptotic stability to diverse reasonable systems named h-systems. The notion of h-stability is flexible enough because it combines the notion of exponential stability with uniform stability to form one framework. In a study conducted by Choi, Koo and Ryu, the concept of h-stability is more useful compared to other research on asymptotic stability and non-exponential types of stability [6, 7]. Choi, Koo and Goo [8, 9] examined the h-stability of nonlinear differential systems by using Lyapunov functions. Moreover, they studied the h-stability of nonlinear differential systems associated with the principles of exponential asymptotic stability and Lipschitz stability.

The major objective of our work is to investigate the h-stability of set differential equations. Inspired by [20, 31] and [7], we determine the conditions of h-stability that are also examined by using the generalized Hukuhara derivative together with a Lyapunov function. Based on the strengths of the Lyapunov approach, stability can be checked without solving the underlying differential equation. This approach shows that if a suitable Lyapunov function may be found, then the system will have some stability properties. Moreover, some theorems prove, at least conceptually with respect to several Lyapunov stability theorems, that the conditions given are indeed necessary. These theorems are generally known as converse theorems (see [10,16,22,35]). Converse theorems are generally the most difficult part of the theory and the first overall outcomes for nonlinear systems were achieved by Massera [28] and Kurzweil [19]. In [17], an inverse theorem for uniform exponential stability is established. It confirms that the origin is uniformly exponentially stable. Then, a Lyapunov function that satisfies some conditions exists. Inspired by the previously studied works, we present a converse Lyapunov theorem for the h-stability of set-valued differential equations. To deal with this situation, we propose that the system should be globally h-stable for a set of all non-empty, convex, and compact intervals of R. As a result, sufficient conditions guarantee the h-stability of perturbed systems and cascaded systems using the Lyapunov theory.

The remaining of this paper is structured as follows: Section 2 presents some basic definitions and notations to study the system of set-valued differential equations. Section 3 summarizes the key results of the present work. In fact, based on Lyapunov theory, a sufficient condition is given to guarantee not only the h-stability of set differential equations (Theorem 5), but also the converse h-stability theorem (Theorem 6) for set interval-valued equations on $K_c(\mathbb{R}) = \mathbb{I}$ (The necessary condition). Section 4 focuses on the h-stability of the perturbed system (Theorem 7) and the cascaded system (Theorem 8). Finally, an example is used to illustrate the feasibility of the results of our theory.

2. Basic notations

This section presents the definitions and notions needed to build set-valued differential equations with a generalized Hukuhara derivative in a metric space.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and let $K_c(\mathbb{R}^n)$ denote the space of all non-empty compact convex subsets of $\mathbb{R}^n$. Minkowski addition and scalar multiplication are defined by

$$U + V = \{u + v|\ u \in U, v \in V\},$$
$$k\, U = \{ku|\ u \in U\},$$

for two subsets U, V in $\mathbb{R}^n$ and $k \in \mathbb{R}$ and it is well known that addition is associative and commutative and with neutral element denoted by Θ_0.

Scalar multiplication gives the opposite $-U = (-1)U = \{-u|\ u \in U\}$, if $k = -1$, but in general, $U + (-U) \neq \Theta_0$ i.e. there is no difference between the two sets due to the non-linearity of $K_c(\mathbb{R}^n$. Thus, Hukuhara offers a solution to the problem described in the following definition

Definition 1 *[15] Suppose that the sets U and V belong to $\mathbb{R}^n$. If there exists a convex compact subset $W \in \mathbb{R}^n$ such that $U = V + W$ then we say the Hukuhara difference of U and V exists and it is then denoted by $U \ominus V$ i.e.*

$$U \ominus V = W \quad \Leftrightarrow \quad U = V + W. \tag{1}$$

An interesting property of $\ominus$ is that $U \ominus U = \Theta_0$ for all $U \in K_c(\mathbb{R}^n)$. The Hukuhara difference is unique, but it does not always exist unless the translation $\{w\} + V$ of V is included in U (a necessary condition for the existence of $U \ominus V$ see [20]). The following definition suggests a solution to overcome this problem.

Definition 2 *[31] Let $U,\ V \in K_c(\mathbb{R}^n)$, the generalized Hukuhara difference of U and V is defined as:*

$$U \ominus_{gH} V = W \quad \Leftrightarrow \quad \begin{cases} \quad (a) \quad U = V + W, \\ or \ \ (b) \quad V = U + (-1)W, \end{cases} \tag{2}$$

where the set $W \in K_c(\mathbb{R}^n)$ and $(-1)W$ is the opposite set of W.

Remark 1

1. *If $U \ominus V$ exists and if $U \ominus_{gH} V$ also exists, it is unique, then* $U \ominus_{gH} V = U \ominus V$ *[31].*

2. *A necessary condition for the existence of $U \ominus_{gH} V$ is that either U contains a translation of V (as for $U \ominus V$) or V contains a translation of U. This means that for each $w \in W$, we obtain $V + \{w\} \subseteq U$ from (a) or $U + \{-w\} \subseteq V$ (b) [32].*

The case of set intervals of $\mathbb{R}$

If $n = 1$ the set of all nonempty, convex and compact intervals of $\mathbb{R}$ is denoted by $K_c(\mathbb{R}) = \mathbb{I}$.

For two arbitrary compact convex intervals, the generalized Hukuhara difference always exists [31].

If $U = [u^-, u^+] \in \mathbb{I}$, then the length (or diameter) of the interval U will be denoted by $len(U) = u^+ - u^-$.

Proposition 1 *[31, 32] The generalized Hukuhara difference $W = U \ominus_{gH} V$ of the two intervals $U = [u^-, u^+]$ and $V = [v^-, v^+]$ is:*

$$[u^-, u^+] \ominus_{gH} [v^-, v^+] = [w^-, w^+]$$
$$\Leftrightarrow \begin{cases} \quad (a) \ \ \{u^- = v^- + w^-, \quad u^+ = v^+ + w^+, \\ or \ \ (b) \ \ \{v^- = u^- - w^+, \quad v^+ = u^+ - w^-, \end{cases} \tag{3}$$

with $w^- = \min\{u^- - v^-, u^+ - v^+\}$, $w^+ = \max\{u^- - v^-, u^+ - v^+\}$.

The two conditions (a) and (b) in (2) are satisfied concurrently if and only if both lengths of the intervals are the same and $w^- = w^+$.

If $U \in \mathbb{I}$, the norm of U is defined by $\|U\| = \max\{|u^-|, |u^+|\}$. Then $(\mathbb{I}, +, ., \|.\|)$ becomes the normed quasilinear space such that $\|.\|$ is a norm on $\mathbb{I}$.

The metric structure on $\mathbb{I}$ is determined by the Hausdorff distance between two intervals: $D : \mathbb{I} \times \mathbb{I} \to \mathbb{R}_+ \cup \{0\}$. The Hausdorff distance is defined as follows:

$$D[U, V] = \max\left\{|u^- - v^-|, |u^+ - v^+|\right\},$$

where $U = [u^-, u^+]$ and $V = [v^-, v^+]$. Then, the following properties hold:

$$D[kU, kV] = |k|D[U, V], \ \ \forall k \in \mathbb{R},$$
$$D[U + A, V + A] = D[U, V],$$
$$D[U + A, V + B] \leqslant D[U, V] + D[A, B],$$

for all $U, V, A, B \in \mathbb{I}$. Clearly, the metric D is related to the norm $\|.\|$ by $D[U, \{0\}] = \|U\|$. $(\mathbb{I}, D)$ is a complete separable metric space (see [31]).

Proposition 2 *[33] For $U,\ V \in \mathbb{I}$, we get*

$$D[U,\ V] = D[U \ominus_{gH} V,\ \{0\}]. \tag{4}$$

Since $\mathbb{I}$ is a complete separable metric space, the continuity and the limits of an interval-valued function $F\colon\ ]\alpha,\ \beta[\to \mathbb{I}$, such that $F(t) = [F^-(t),\ F^+(t)]$, can be characterized in the D-metric sense (see [33]).

Based on the generalized Hukuhara difference, we make the following definition

Definition 3 *[34] Let $t \in]\alpha,\ \beta[$ and τ be such that $t+\tau \in]\alpha,\ \beta[$, then an interval-valued function $F\colon\ ]\alpha,\ \beta[\to \mathbb{I}$ has a generalized Hukuhara derivative at t as follows*

$$F'_{gH}(t) = \lim_{\tau\to 0} \frac{1}{\tau}\left[F(t+\tau) \ominus_{gH} F(t)\right]. \tag{5}$$

If $F'_{gH}(t) \in \mathbb{I}$ satisfying (5) exists, then we can say that F is generalized Hukuhara differentiable (gH-differentiable for brevity) at t.

The following outcome gives a characterization of the generalized Hukuhara differentiability of interval-valued functions.

Theorem 1 (see [34]) *Let $F\colon\ ]\alpha,\ \beta[\to \mathbb{I}$ be an interval-valued function where $F(t) = [F^-(t),\ F^+(t)]$ for all $t \in]\alpha,\ \beta[$. Suppose that the real valued functions $t \mapsto F^-(t)$ and $t \mapsto F^+(t)$ are differentiable. Then, the function $t \in]\alpha,\ \beta[\mapsto F(t)$ is generalized-Hukuhara differentiable at $t \in]\alpha,\ \beta[$ and*

$$F'_{gH}(t) = \left[\min\left\{(F^-)'(t),\ (F^+)'(t)\right\},\ \max\left\{(F^-)'(t),\ (F^+)'(t)\right\}\right], \tag{6}$$

According to Theorem 1, two cases can be distinguished.

Definition 4 *[34] Let $F\colon\ ]\alpha,\ \beta[\to \mathbb{I}$ be a generalized-Hukuhara differentiable at t in $]\alpha,\ \beta[$ function. Then we can say that:*

- *F is (i)-generalized Hukuhara differentiable at t if*

$$(i)\quad F'(t) = \left[(F^-)'(t),\ (F^+)'(t)\right] \tag{7}$$

- *F is (ii)-generalized Hukuhara differentiable at t if*

$$(ii)\quad F'(t) = \left[(F^+)'(t),\ (F^-)'(t)\right]. \tag{8}$$

Remark 2 *In [34], example 8 illustrates that $F\colon\ [\alpha,\ \beta] \to \mathbb{I}$ is generalized Hukuhara differentiable at t, which is not the case for (i) and (ii).*

So an interval-valued differential equation is defined by

$$X'_{gH}(t) = F(t,\ X(t)), \quad X(t_0) = X_0, \tag{9}$$

where $F\colon [\alpha,\ \beta] \times \mathbb{I} \longrightarrow \mathbb{I}$ with $F(t,\ X) = [F^-(t,\ X),\ F^+(t,\ X)]$, X'_{gH} is the generalized Hukuhara derivative of the interval set $X \in \mathbb{I}$, $X = [X^-,\ X^+]$ and $X_0 = [X_0^-,\ X_0^+]$.

Therefore, the continuous mapping $X\colon T_0 = [t_0,\ t_0 + a] \longrightarrow \mathbb{I}$ $(a > 0)$ is a solution of the system (9) if and only if $X\colon T_0 = [t_0,\ t_0 + a] \longrightarrow \mathbb{I}$ satisfies the following interval integral equation

$$X(t) \ominus_{gH} X_0 = \int_{t_0}^{t} F(s,\ X(s))\,\mathrm{d}s, \tag{10}$$

on some interval $T_0 = [t_0,\ t_0 + a]$ (Lemma 33 [33]).

Generalized Hukuhara difference shows that integral equation (10) is a unified expression of the following integral equations

$$X(t) \ominus X_0 = \int_{t_0}^{t} F(s,\ X(s))\,\mathrm{d}s,$$

and

$$X_0 \ominus X(t) = -\int_{t_0}^{t} F(s,\ X(s))\,\mathrm{d}s,$$

where $\ominus$ represents the usual Hukuhara difference.

If $F(t,\ \{0\}) = \{0\}$, then $X(t) = \{0\}$ is an interval-valued of stationary solution to (9).

Then, we list the already known results of problem (9) (see [3]). These results are useful for our study. In what follows, we present the two following theorems:

Theorem 2 (see [3]) *Let $F \in C[T_0 \times \bar{B}[X_0,\ r],\ \mathbb{I}]$ and as $F'_{gH}(t,\ X)$ exists and is continuous on $T_0 \times \bar{B}[X_0,\ r]$, where $\bar{B}[X_0,\ r] \subset \mathbb{I}$ the closed ball with centre X_0 and radius r.*

Then we have

$$F(t,\ X_1) \ominus_{gH} F(t,\ X_2) = \int_{0}^{1} F'_{gH}(t,\ sX_1 + (1 - s)X_2) \cdot (X_1 \ominus_{gH} X_2)\,\mathrm{d}s, \quad t \in T_0, \tag{11}$$

for all $X_1,\ X_2 \in \bar{B}[X_0,\ \beta]$ and $t \in T_0$.

Suppose that the unique solution $X(t, t_0, X_0)$ of (9) exists and that $\dfrac{\partial X(t, t_0, X_0)}{\partial X_0}$ exists and is continuous. Then, this leads to the next theorem.

Theorem 3 (see [3]) *We assume that* $F \in C[T_0 \times \bar{B}[X_0, r], \mathbb{I}]$, $F'_{gH}(t, X)$ *exists and is continuous on* $T_0 \times \bar{B}[X_0, r]$ *and a unique solution* $X(t, t_0, X_0)$ *of (9) exists on* $T_0 = [t_0, t_0 + a]$, $a > 0$. *Then,*

1. $\psi(t, t_0, X_0) = \dfrac{\partial X(t, t_0, X_0)}{\partial X_0}$ *exists and* $\psi(t, t_0, X_0)$ *is a solution of*

$$U'_{gH}(t) = G(t, t_0, X_0)U(t), \tag{12}$$

where $G(t, t_0, X_0) = F'_{gH}(t, X(t, t_0, X_0))$ *such that* $\psi(t_0, t_0, X_0) = \dfrac{\partial X(t_0, t_0, X_0)}{\partial X_0} = I$ *is the identity matrix as element in* $\mathbb{I}$;

2. $\phi(t, t_0, X_0) = \dfrac{\partial X(t, t_0, X_0)}{\partial t_0}$ *exists, is a solution of (12) and satisfies:*

$$\phi(t, t_0, X_0) + \psi(t, t_0, X_0)F(t_0, X_0) = 0, \tag{13}$$

with $\phi(t_0, t_0, X_0) = -F(t_0, X_0)$.

The next outcomes (14) and (15), inspired by those examined in classic case studies [21], are still applicable to our space $\mathbb{I}$.

Remark 3 *When we integrate*

$$\frac{\partial X(t, t_0, sX_0)}{\partial s} = \frac{\partial X(t, t_0, sX_0)}{\partial X_0}.X_0 = \psi(t, t_0, sX_0).X_0$$

from $s = 0$ *into* $s = 1$. *Then the solution* $X(t_0, t_0, X_0)$ *of (9) is associated with* $\psi(t, t_0, sX_0)$ *[21] which is defined in Theorem 3 by*

$$X(t, t_0, X_0) = \left[\int_0^1 \psi(t, t_0, sX_0).X_0 \,\mathrm{d}s \right]. \tag{14}$$

The following outcomes show an expression that estimates the difference of two solutions $X(t, t_0, X_1)$ and $X(t, t_0, X_2)$ of (9). Thus, these findings will be useful in later discussions.

Theorem 4 *Let $F \in C[T_0 \times \bar{B}[X_0, r], \mathbb{I}]$ such that $F'_{gH}(t, X)$ exists and is continuous on $T_0 \times \bar{B}[X_0, r]$. Suppose that $X(t, t_0, X_1)$ and $X(t, t_0, X_2)$ are the solutions of (9) where (t, X_1) and (t, X_2), respectively, exist for $t \in T_0$. Then*

$$X(t, t_0, X_1) \ominus_{gH} X(t, t_0, X_2) = \left[\int_0^1 \psi(t, t_0, X_2) + s(X_1 - X_2).(X_1 \ominus_{gH} X_2) \mathrm{d}s \right], \qquad (15)$$

for $t \in T_0$.

The case of set valued of $\mathbb{R}^n$

If $n \geqslant 1$ the metric structure is generally given by the Hausdorff distance between two nonempty subsets U and V on $K_c(\mathbb{R}^n)$ which is defined by

$$D[U, V] = \max\{d_H(U, V), d_H(V, U)\}$$

where $d_H(U, V) = \sup\{d(u, V); u \in U\}$ is a Hausdorff separation, $d(u, V) = \inf\{\|u - v\|; v \in V\}$ is the distance from a point u to a set V and $\|.\|$ is the Euclidean norm. Then, the next properties hold:

$$D[\alpha U, \alpha V] = |\alpha| D[U, V], \quad \forall \alpha \in \mathbb{R},$$
$$D[U + A, V + A] = D[U, V],$$
$$D[U + A, V + B] \leqslant D[U, V] + D[A, C],$$

for all $U, V, W, Q \in K_c(\mathbb{R}^n)$ and $(K_c(\mathbb{R}^n), D)$ is complete separable metric space [20].

Remark 4 *Let $\|U\| = \sup\{\|u\| : u \in U\}$ be the supremum of a nonempty subset U of $K_c(\mathbb{R}^n)$, which is reached. Then*

$$D[U, \Theta] = \|U\| \qquad (16)$$

for $U \in K_c(\mathbb{R}^n)$, with Θ the zero element of $\mathbb{R}^n$.

Moreover, from (16) and (1.3.20) of [20],it obviously follows that

$$\big| \|U\| - \|W\| \big| \leqslant D[U, W],$$

for all $U, W \in K_c(\mathbb{R}^n)$ (see [3, 20]).

Let F be a multivalued (set valued) mapping from a domain Ω in $\mathbb{R}^p$ into the metric space $(K_c(\mathbb{R}^p), D)$: $F : \Omega \longrightarrow K_c(\mathbb{R}^p)$ identically equal to $F(t) \in K_c(\mathbb{R})^p$, for all $t \in \Omega$. And the next definition reveals that F is generalized Hukuhara differentiable

Definition 5 (see [20]) *The multivalued mapping $F : T_0 \to \mathbb{R}^n$ is generalized Hukuhara differentiable at a point $t \in T_0$, if there exists $D_{gH}F(t) \in \mathbb{R}^n$ such that the limits*

$$\lim_{\tau \to 0^+} \frac{F(t+\tau) \ominus_{gH} F(t)}{\tau} \tag{17}$$

and

$$\lim_{\tau \to 0^+} \frac{F(t) \ominus_{gH} F(t-\tau)}{\tau} \tag{18}$$

both exist and are equal to $D_{gH}F(t)$, where T_0 is an interval of $\mathbb{R}$.

Then, the system of set differential equations is defined by

$$D_{gH}X = F(t,\ X), \qquad X(t_0) = X_0 \in K_c(\mathbb{R}^n), \quad t_0 \geqslant 0, \tag{19}$$

where $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n),\ K_c(\mathbb{R}^n)]$ is a set-valued mapping, $D_{gH}X$ is the generalized Hukuhara derivative of the set X.

The mapping $X \in C^1[T_0,\ K_c(\mathbb{R}^n)]$ where $T = [t_0,\ t_0 + a],\ a > 0$, is said to be a solution of (19) on T_0, if it satisfies (19) on T_0.

Since $X(t)$ is continuously differentiable, we have

$$X(t) = X_0 + \int_{t_0}^{t} D_H X(s)\,\mathrm{d}s, \quad t \in T_0\,, \tag{20}$$

according to (19)

$$X(t) = X_0 + \int_{t_0}^{t} F(s,\ X(s))\,\mathrm{d}s, \quad t \in T_0\,. \tag{21}$$

If $F(t,\ \Theta_0) = \Theta_0$, then $X(t) = \Theta_0$ is a set of stationary solutions to (19). We will use this form (19) in analyzing the obtained results.

Remark 5 *If the generalized Hukuhara differences $X_1 \ominus_{gH} X_2$ and $F(t,\ X_1) \ominus_{gH} F(t,\ X_2)$ both exist for all $X_1,\ X_2 \in B[X_0,\ b] = \{X \in K_c(\mathbb{R}^n) :\ D[X,\ X_0] \leqslant b\} \subset K_c(\mathbb{R}^n)$ and $t \in T_0$, then all estimates from (11) to (15) are satisfied where $F \in C[T_0 \times B[X_0,\ b],\ K_c(\mathbb{R}^n)]$ and as $D_{gH}F(t,\ X)$ exists and is continuous on $T_0 \times B[X_0,\ b]$ (see [3, 21]).*

Let $h\colon \mathbb{R}_+ \to \mathbb{R}_+^*$ be a continuous positive bounded function. The last notion is called h-stability.

Definition 6 (see [30]) *The stationary solution $X(t) = \Theta_0$ to equations (19) is (hS) h-stable if there is a positive, bounded and continuous function h on $\mathbb{R}_+$ and constants $C \geqslant 1$, $\delta \in \bar{\mathbb{R}}_+^*$ such that*

$$D[X(t,\ t_0,\ W_0),\ \Theta_0] \leqslant CD[W_0,\ \Theta_0]h(t)h^{-1}(t_0), \tag{22}$$

for $t \geqslant t_0 \geqslant 0$ and $D[W_0,\ \Theta_0] \leqslant \delta$ and (GhS) globally h-stable if, in (hS), $\delta < \infty$.

Note that Lyapunov function is an important factor in stability theory and it changes the set differential equations to scalar differential equations (see [23]). The total derivatives $D^+V_{(F)}(t,\ X)$ with respect to the system (19) are defined as follows:

$$D^+V_{(F)}(t,\ X) = \lim_{\sigma \to 0^+} \sup \frac{1}{\sigma}[V(t+\sigma,\ X+\sigma F(t,\ X)) - V(t,\ X)], \tag{23}$$

for $(t, X) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n)$, where $V \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n),\ \mathbb{R}_+]$ [22].

Since $X(t) = X(t,\ t_0,\ X_0)$ is a solution of (19),

$$D^+V(t,\ X(t)) = \lim_{\sigma \to 0^+} \sup \frac{1}{\sigma}[V(t+\sigma,\ X(t+\sigma)) - V(t,\ X)]. \tag{24}$$

If $X \mapsto V(t,\ X)$ is Lipschitzian (in X) for each $t \in \mathbb{R}_+$, then

$$D^+V_{(F)}(t,\ X) = D^+V(t,\ X(t)) \tag{25}$$

see Yoshizawa [(1.7), p. 3, [37]].

We note that $V_{(19)}(t,\ X)$ and $V_{(9)}(t,\ X)$ are relative to the systems (19) and (9), respectively.

3. Major results

The most important findings deal with Converse h-stability theorem for set-valued mappings. These results are obtained using a Lyapunov-Like function.

3.1. h-stability of set-valued mappings

Our first issue focuses on sufficient conditions for the h-stability of the system (19) based on Lyapunov's second method.

Theorem 5 *Assume that the positive function $h(t)$ is continuously differentiable and bounded on $\mathbb{R}_+$. Further, suppose there exist a constant $c \geqslant 1$ and a function $V(t,\ X)$ satisfying the following properties:*

i) $V \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n),\ \mathbb{R}_+]$ *and* $V(t,\ X)$ *is Lipschitzian in* $X \in K_c(\mathbb{R}^n)$ *for each* $t \in \mathbb{R}_+$*,*

ii) $D[X,\ \{0\}] \leqslant V(t,\ X) \leqslant cD[X,\ \{0\}]$*,* $(t,\ X) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n)$*,* $c \geqslant 1$*,*

iii) $D^+V_{(19)}(t,\ X) \leqslant \dfrac{h'(t)}{h(t)}V(t,\ X), (t,\ X) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n).$

Then, the stationary solution $X(t) = \Theta_0$ *of (19) is* h*-stable.*

Proof. Let $X(t) = X(t,\ t_0,\ W_0)$ be any solution of (19) and $t \geqslant t_0 \geqslant 0$. From (iii), we obtain

$$V(t,\ X(t)) \leqslant V(t_0,\ W_0) \exp\left(\int_{t_0}^{t} \frac{h'(s)}{h(s)} \mathrm{d}s\right) = V(t_0,\ W_0)\frac{h(t)}{h(t_0)}.$$

From (ii) we have

$$D[X(t),\ \Theta_0] \leqslant V(t,\ X(t)) \leqslant V(t_0,\ W_0)\frac{h(t)}{h(t_0)} \leqslant cD[W_0,\ \Theta_0]\frac{h(t)}{h(t_0)}.$$

Then

$$D[X(t),\ \Theta_0] \leqslant cD[W_0,\ \Theta_0]h(t)h^{-1}(t_0), \qquad D[W_0,\ \Theta_0] \leqslant \delta,$$

for $t \geqslant t_0 \geqslant 0$, $c > 0$ and $\delta > 0$.

This completes the proof. □

From Theorem 5, we obtain the following Corollary

Corollary 1 *Assume that the positive function* $h(t)$ *is continuously differentiable and bounded on* $\mathbb{R}_+$*. Further, suppose there exist constants* $c,\ p \geqslant 1$ *and function* $V(t,\ X)$ *satisfying the properties below.*

i) $V \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n),\ \mathbb{R}_+]$ *and* $V(t,\ X)$ *is Lipschitzian in* $X \in K_c(\mathbb{R}^n)$ *for each* $t \in \mathbb{R}_+$*,*

ii) $D[X,\ \{0\}]^p \leqslant V(t,\ X) \leqslant cD[X,\ \{0\}]^p$*,* $(t,\ X) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n)$*,* $c \geqslant 1$*,*

iii) $D^+V_{(19)}(t,\ X) \leqslant \dfrac{h'(t)}{h(t)}V(t,\ X), (t,\ X) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n).$

Then, the stationary solution $X(t) = \Theta_0$ *of (19) is* $h^{\frac{1}{p}}$*-stable.*

Proof. Let $X(t) = X(t, t_0, W_0)$ be any solution of (19) and $t \geqslant t_0 \geqslant 0$. From (iii), we obtain

$$V(t, X(t)) \leqslant V(t_0, W_0)\exp\left(\int_{t_0}^{t} \frac{h'(s)}{h(s)}\mathrm{d}s\right) = V(t_0, W_0)\frac{h(t)}{h(t_0)}.$$

From (ii) we have

$$D[X(t), \Theta_0]^p \leqslant V(t, X(t)) \leqslant V(t_0, W_0)\frac{h(t)}{h(t_0)} \leqslant cD[W_0, \Theta_0]^p\frac{h(t)}{h(t_0)}.$$

Then

$$D[X(t), \Theta_0] \leqslant c^{\frac{1}{p}}D[W_0, \Theta_0]h^{\frac{1}{p}}(t)h^{-\frac{1}{p}}(t_0), \qquad D[W_0, \Theta_0] \leqslant \delta,$$

for $t \geqslant t_0 \geqslant 0$, $c^{\frac{1}{p}} > 0$ and $\delta > 0$.

Then the stationary solution $X(t) = \Theta_0$ of (19) is $h^{\frac{1}{p}}$-stable. □

Remark 6

1. *If $h(t) = c$ where c is a positive constant, then the system (19) is uniformly Lipschitz stable i.e. there exist $k > 0$ and $\delta > 0$ such that*

$$D[X(t, t_0, X_0), \Theta_0] \leqslant kD[X_0, \Theta_0],$$

where $D[X_0, \Theta_0] \leqslant \delta, \quad t \geqslant t_0 \geqslant 0$.

2. *If $h(t) = e^{-\gamma t}$, where γ is a positive constant, then the system (19) is exponentially stable.*

3. *If the function $t \mapsto h(t)$ is strictly decreasing and tends to 0 when $t \to +\infty$, then the system (19) is asymptotically stable.*

3.2. Converse h-stability theorems

In the previous section, we proved that the existence of a Lyapunov function generates a sufficiently strong condition for the partially desired h-stability.

Now, the converse question is posed. We propose to investigate whether global h-stability involves the existence of Lyapunov functions of the type given in Theorem 5. Such issues are resolved in a classical case by Lakshmikantham and Leela [22] and Choi, Koo and Ryu [7].

In this part, the main challenge is to construct a Lyapunov- Like function when the system is globally h-stable. There are two obstructions that prove this converse result.

First, because of the existence of the generalized Hukuhara difference on $K_c(\mathbb{R}^n)$, we prove the converse theorem only for $n = 1$, namely $\mathbb{I}$, since it always exists in this case.

Second, the following statement ($\mathbf{H}_1$), which is in usual cases a consequence of h-stability, must be regarded as an assumption. In the linear framework, this assumption is automatically satisfied.

($\mathbf{H}_1$): There exist a constant $C \geqslant 1$ and a continuous positive function h (respectively continuous, bounded) on $\mathbb{R}_+$, so that, if $D[X_0,\ \{0\}]$ is sufficiently small such that the system (12) is h-stable, then

$$D\big[\psi(t,\ t_0,\ X_0),\ \{0\}\big] \leqslant Ch(t)h^{-1}(t_0), \tag{26}$$

for $t \geqslant t_0 \geqslant 0$ and the fundamental matrix $\psi(t,\ t_0,\ X_0)$ is a solution of the system (12).

Now, we formulate the conserve theorem to provide the necessary conditions for globally h-stable (GhS).

Theorem 6 *Suppose that the stationary solution $X = \{0\}$ of the system (9) is globally h-stable and that the solution $U = \{0\}$ of the system (12) is globally h-stable. Assume further that $h'(t)$ exists and is continuous on $\mathbb{R}_+$.*

Then, there exists a function $(t,\ X) \mapsto V(t,\ X)$ fulfilling the following properties:

i) $V \in C[\mathbb{R}_+ \times \mathbb{I},\ \mathbb{R}_+]$ *and* $X \mapsto V(t,\ X)$ *is Lipschitzian in* $X \in \mathbb{I}$ *for each* $t \in \mathbb{R}_+$,

ii) $D[X,\ \{0\}] \leqslant V(t,\ X) \leqslant cD[X,\ \{0\}]$, $(t,\ X) \in \mathbb{R}_+ \times \mathbb{I}$, $c \geqslant 1$,

iii) $D^+V_{(9)}(t,\ X) \leqslant \dfrac{h'(t)}{h(t)}V(t,\ X), (t,\ X) \in \mathbb{R}_+ \times \mathbb{I}$.

Proof. Set up a Lyapunov function

$$V(t,\ X) = \sup_{\tau \geqslant 0} \big\{D[X(t+\tau,\ t,\ X),\ \{0\}]h^{-1}(t+\tau)h(t)\big\},$$

where $X(t,\ t_0,\ X_0)$ is a solution of (9) for $(t,\ X) \in \mathbb{R}_+ \times \mathbb{I}$.

Based on the global h-stability of (9) we get

$$D[X(t,\ t_0,\ X_0),\ \{0\}] \leqslant CD[X_0,\ \{0\}]h(t)h^{-1}(t_0), \qquad D[X_0,\ \{0\}] < \infty.$$

Moreover, we get

$$\sup_{\tau \geqslant 0} \big\{D[X(t+\tau,\ t,\ X),\ \{0\}]h^{-1}(t+\tau)h(t)\big\} \geqslant D[X(t,\ t,\ X),\ \{0\}] = D[X,\ \{0\}],$$

and

$$V(t,\ X) \leqslant cD[X,\ \{0\}]h(t+\tau)h^{-1}(t)h^{-1}(t+\tau)h(t) = cD[X,\ \{0\}].$$

Then (ii) is satisfied, and $V(t, X)$ is defined on $\mathbb{R}_+ \times \mathbb{I}$.

Let $(t,\ X_1),\ (t,\ X_2) \in \mathbb{R}_+ \times \mathbb{I}$.

$$\begin{aligned}
&|V(t,\ X_1) - V(t,\ X_2)| \\
&\quad = |\sup_{\tau\geqslant 0}\{D[X(t+\tau,\ t,\ X_1),\ \{0\}]h^{-1}(t+\tau)h(t)\} \\
&\qquad - \sup_{\tau\geqslant 0}\{D[X(t+\tau,\ t,\ X_2),\ \{0\}]h^{-1}(t+\tau)h(t)\}| \\
&\quad \leqslant \sup_{\tau\geqslant 0}\{|D[X(t+\tau,\ t,\ X_1),\ \{0\}] - D[X(t+\tau,\ t,\ X_2),\ \{0\}]|h^{-1}(t+\tau)h(t)\} \\
&\quad \leqslant \sup_{\tau\geqslant 0}\{|D[X(t+\tau,\ t,\ X_1),\ X(t+\tau,\ t,\ X_2)]|h^{-1}(t+\tau)h(t)\}.
\end{aligned}$$

By using the estimates (4) and (15) we have

$$\begin{aligned}
|V(t,\ X_1)-V(t,\ X_2)| &\leqslant \sup_{\tau\geqslant 0}\big\{|D[X(t+\tau,\ t,\ X_1),\ X(t+\tau,\ t,\ X_2)]|h^{-1}(t+\tau)h(t)\big\} \\
&\leqslant \sup_{\tau\geqslant 0}\big\{|D[X(t+\tau,\ t,\ X_1)\ominus_{gH} X(t+\tau,\ t,\ X_2),\ \{0\}]|h^{-1}(t+\tau)h(t)\big\} \\
&\leqslant \sup_{\tau\geqslant 0}\left\{D\left[\int_0^1 \psi(t+\tau,\ t,\ X_2+s(X_1\ominus_{gH} X_2)).(X_1\ominus_{gH} X_2)\,\mathrm{d}s,\ \{0\}\right]\right. \\
&\qquad \left. h^{-1}(t+\tau)h(t)\right\}.
\end{aligned}$$

Since for every $X_1,\ X_2 \in \bar{B}[X_0,\ r] \subset \mathbb{I}$ with $D[U,\ \{0\}] = \|U\|$ for all $U \in \mathbb{I}$, we obtain

$$\begin{aligned}
&D\left[\int_0^1 \psi(t+\tau,\ t,\ X_2+s(X_1\ominus_{gH} X_2)).(X_1\ominus_{gH} X_2)\,\mathrm{d}s,\ \{0\}\right] \\
&= \left\|\int_0^1 \psi(t+\tau,\ t,\ X_2+s(X_1\ominus_{gH} X_2)).(X_1\ominus_{gH} X_2)\,\mathrm{d}s\right\| \\
&\leqslant \sup_{Y\in\bar{B}[X_0,\ r]} |\psi(t+\tau,\ t,\ Y)|\|X_1\ominus_{gH} X_2\| \\
&\leqslant \sup_{Y\in\bar{B}[X_0,\ r]} |\psi(t+\tau,\ t,\ Y)|D[X_1\ominus_{gH} X_2,\ \{0\}].
\end{aligned}$$

Based on global h-stability properties such that $\psi(t+\tau,\ t,\ X_0)$ satisfies $(\mathbf{H}_1)$, we have

$$\begin{aligned}
&|V(t,\ X_1) - V(t,\ X_2)| \\
&\qquad \leqslant \sup_{\tau\geqslant 0}\{C\ D[X_1 \ominus_{gH} X_2,\ \{0\}]\ h^{-1}(t+\tau)h(t)h^{-1}(t)h(t+\tau)\} \\
&\qquad \leqslant C\ D[X_1,\ X_2].
\end{aligned}$$

It follows that $X \mapsto V(t,\ X)$ is Lipschitzian in X for each t.

We shall now verify the continuity of $(t, X) \mapsto V(t, X)$ on $\mathbb{R}^+ \times \mathbb{I}$. Let (t, X_2) be in $\mathbb{R}^+ \times \mathbb{I}$ and let $\epsilon \geqslant 0$. Then,

$$\begin{aligned}
|V(t+\epsilon,\ X_1) - V(t,\ X_2)| \leqslant\ & |V(t+\epsilon,\ X_1) - V(t+\epsilon,\ X_2)| \\
&+ |V(t+\epsilon,\ X_2) - V(t+\epsilon,\ X(t+\epsilon,\ t,\ X_2))| \\
&+ |V(t+\epsilon,\ X(t+\epsilon,\ t,\ X_2)) - V(t,\ X_2)|.
\end{aligned}$$

We begin with the third term,

$$\begin{aligned}
&|V(t+\epsilon,\ X(t+\epsilon,\ t,\ X_2)) - V(t,\ X_2)| \\
&= |\sup_{\tau\geqslant 0}\{D[X(t+\epsilon+\tau,\ t+\epsilon,\ X_2),\ \{0\}]h^{-1}(t+\epsilon+\tau)h(t)\} \\
&\quad - \sup_{\tau\geqslant 0}\{D[X(t+\tau,\ t,\ X_2),\ \{0\}]h^{-1}(t+\tau)h(t)\}| \\
&= |\sup_{\tau\geqslant \epsilon}\{D[X(t+\tau,\ t,\ X_2),\ \{0\}]h^{-1}(t+\tau)h(t+\epsilon)\} \\
&\quad - \sup_{\tau\geqslant 0}\{D[X(t+\tau,\ t,\ X_2),\ \{0\}]h^{-1}(t+\tau)h(t)\}|.
\end{aligned}$$

Let

$$b(\epsilon) = \sup_{\tau\geqslant \epsilon}\{D[X(t+\tau,\ t,\ X_2),\ \{0\}]h^{-1}(t+\tau)h(t+\epsilon)\},$$

we note that $b(\epsilon)$ is nondecreasing and $b(\epsilon) \to b(0)$ as $\epsilon \to 0$, such that $D[X(t+\tau,\ t,\ X_2),\ \{0\}]h^{-1}(t+\tau)h(t)$ is a bounded continuous function, for $\tau \geqslant 0$. Then

$$|V(t+\epsilon,\ X(t+\epsilon,\ t,\ X_2)) - V(t,\ X_2)| = |b(\epsilon) - b(0)|,$$

tends to zero as $\epsilon \to 0$.

We observe that $X(t+\epsilon,\ t,\ X)$ is continuous in ϵ and $X \mapsto V(t+\epsilon,\ X)$ is Lipschitzian in X for all ϵ, the first two terms of the preceding inequality are small enough where $D[X_1,\ X_2]$ and ϵ are small enough.

This implies that $V(t, X)$ is continuous.

Based on the definition of h-stabilities and uniqueness of solutions, we obtain

$$\begin{aligned}
&D^+V(t, X(t)) \\
&\quad = \lim_{\sigma\to 0^+} \sup \frac{1}{\sigma}[V(t+\sigma, X(t+\sigma)) - V(t, X)] \\
&\quad = \lim_{\sigma\to 0^+} \sup \frac{1}{\sigma}[\sup_{\tau\geqslant 0}\{D[X(t+\sigma+\tau, t+\sigma, X), \{0\}]h^{-1}(t+\sigma+\tau)h(t)\} \\
&\qquad - \sup_{\tau\geqslant 0}\{D[X(t+\tau, t, X), \{0\}]h^{-1}(t+\tau)h(t)\}] \\
&\quad = \lim_{\sigma\to 0^+} \sup \frac{1}{\sigma}[\sup_{\tau\geqslant \sigma}\{D[X(t+\tau, t, X), \{0\}]h^{-1}(t+\tau)h(t+\sigma)\} \\
&\qquad - \sup_{\tau\geqslant 0}\{D[X(t+\tau, t, X), \{0\}]h^{-1}(t+\tau)h(t)\}] \\
&\quad \leqslant \lim_{\sigma\to 0^+} \sup \frac{1}{\sigma}[\sup_{\tau\geqslant 0}\{D[X(t+\tau, t, X), \{0\}]h^{-1}(t+\tau)h(t) \\
&\qquad (h(t+\sigma)h^{-1}(t) - 1)\}] \\
&\quad \leqslant \lim_{\sigma\to 0^+} \sup \frac{1}{\sigma}(h(t+\sigma)h^{-1}(t) - 1)V(t, X), \\
&\quad \leqslant \frac{h'(t)}{h(t)}V(t, X).
\end{aligned}$$

Since, for $\sigma > 0$,

$$\begin{aligned}
V(t+\sigma, X+\sigma F(t, X)) - V(t, X) &\leqslant |V(t+\sigma, X+\sigma F(t, X)) - V(t+\sigma, X(t+\sigma, t, X))| \\
&\quad + |V(t+\sigma, X(t+\sigma, t, X)) - V(t, X)| \\
&\leqslant LD[X+\sigma F(t, X) \ominus_{gH} X(t+\sigma, t, X), \{0\}] \\
&\quad + |V(t+\sigma, X(t+\sigma, t, X)) - V(t, X)|,
\end{aligned}$$

it follows that

$$D^+V_{(9)}(t, X) \leqslant \frac{h'(t)}{h(t)}V(t, X).$$

Then the theorem is proven. □

The comparison result is true even if $F(t, X)$ is linear in X. The following corollary substantiates this statement.

Corollary 2 *If the stationary solution $X = \{0\}$ of the system (9) is globally h-stable and $F(t, X)$ is linear in X, then, $h'(t)$ exists and is continuous on $\mathbb{R}_+$.*

Then, there exists a function $V(t, X)$ fulfilling the following properties:

i) $V \in C[\mathbb{R}_+ \times \mathbb{I}, \mathbb{R}_+]$ *and* $V(t, X)$ *is Lipschitzian in* X *for each* $t \in \mathbb{R}_+$*,*

ii) $D[X, \{0\}] \leqslant V(t, X) \leqslant cD[X, \{0\}]$*,* $(t, X) \in \mathbb{R}_+ \times \mathbb{I}$*,* $c \geqslant 1$*,*

iii) $D^+V_{(9)}(t, X) \leqslant \dfrac{h'(t)}{h(t)}V(t, X), (t, X) \in \mathbb{R}_+ \times \mathbb{I}$*.*

Proof. Set up a Lyapunov function

$$V(t, X) = \sup_{\tau \geqslant 0} \left\{D[X(t+\tau, t, X), \{0\}]h^{-1}(t+\tau)h(t)\right\},$$

where $X(t, t_0, X_0)$ is a solution of (9) for $(t, X) \in \mathbb{R}_+ \times \mathbb{I}$.

Let us show that $X \mapsto V(t, X)$ satisfies the Lipschitz condition.

Let $(t, X_1), (t, X_2) \in \mathbb{R}_+ \times \mathbb{I}$. By using the estimates (4) and the properties of globally h-stable we have

$$\begin{aligned}
&\left|V(t, X_1) - V(t, X_2)\right| \\
&\quad = \Big|\sup_{\tau \geqslant 0} \left\{D[X(t+\tau, t, X_1), \{0\}]h^{-1}(t+\tau)h(t)\right\} \\
&\qquad - \sup_{\tau \geqslant 0} \left\{D[X(t+\tau, t, X_2), \{0\}]h^{-1}(t+\tau)h(t)\right\}\Big| \\
&\quad \leqslant \sup_{\tau \geqslant 0} \left\{|D[X(t+\tau, t, X_1), \{0\}] - D[X(t+\tau, t, X_2), \{0\}]|h^{-1}(t+\tau)h(t)\right\} \\
&\quad \leqslant \sup_{\tau \geqslant 0}\{|D[X(t+\tau, t, X_1), X(t+\tau, t, X_2)]|h^{-1}(t+\tau)h(t)\} \\
&\quad \leqslant \sup_{\tau \geqslant 0}\{|D[X(t+\tau, t, X_1) \ominus_{gH} X(t+\tau, t, X_2), \{0\}]|h^{-1}(t+\tau)h(t)\} \\
&\quad \leqslant \sup_{\tau \geqslant 0}\{|D[X(t+\tau, t, X_1 \ominus_{gH} X_2), \{0\}]|h^{-1}(t+\tau)h(t)\} \\
&\quad \leqslant \sup_{\tau \geqslant 0}\{C\, D[X_1 \ominus_{gH} X_2, \{0\}]\, h^{-1}(t+\tau)h(t)h^{-1}(t)h(t+\tau)\} \\
&\quad \leqslant C\, D[X_1, X_2].
\end{aligned}$$

The relationships between (ii) and (iii) can be proved in the same way as the proof of Theorem 6. The proof is fully complete. □

Remark 7

1. *If* $h(t) = c$ *where* c *is a positive constant, then we have a converse theorem for the stability of the system (9).*

2. *If $h(t) = e^{-\gamma t}$, where γ is a positive constant, then we get a converse theorem for the exponential stability of the system (9).*

3. *If the function $h(t)$ is strictly decreasing and tends to 0 when t to $+\infty$, then we obtain a converse theorem for the asymptotic stability of the system (9).*

4. Perturbed and Cascaded Systems

In this section, we notice that perturbed and cascaded systems are globally h-stable by using a converse theorem. In particular, we show that this type of system is globally h-stable.

It is worth noting that, in this application of the converse Theorem 6, the inequality (26) is automatically verified and must not be considered as an additional hypothesis.

4.1. h-Stability of Perturbed Systems

We now focus on the relationships between solutions of the unperturbed system (9) and solutions of the following perturbed system:

$$Y'_{gH} = F(t,\ Y) + G(t,\ Y), \qquad Y(t_0) = Y_0\,, \tag{27}$$

where $F,\ G \in C[\mathbb{R}_+ \times \mathbb{I},\ \mathbb{I}]$ and $F(t,\ \{0\}) = \{0\}$.

The same problem has been tackled in [1] using a common framework.

Specifically, we provide sufficient conditions to extend the global h-stability of the system (9) to the global $\widetilde{h}$-stability of the perturbed system (27) and we provide an explicit relation between h and $\widetilde{h}$.

Theorem 7 *Suppose first that the stationary solution $X = \{0\}$ of (9) is globally h-stable. Further, assume that the solution $U = \{0\}$ of the system (12) is globally h-stable, and that*

$$D[G(t,\ Y),\ \{0\}] \leqslant \lambda(t) D[Y,\ \{0\}], \quad t \geqslant t_0 \geqslant 0, \tag{28}$$

where $\lambda \in L_1(\mathbb{R}_+)$.

Then, the stationary solution $Y = \{0\}$ of (27) is globally $\widetilde{h}$-stable where

$$\widetilde{h}(t) = h(t) e^{L \int_0^t \lambda(s)\,\mathrm{d}s}.$$

Proof. Based on Theorem 6, there exist functions $(t, X) \mapsto V(t, X)$ and $t \mapsto h(t)$ satisfying the three properties indicated in that theorem. We obtain

$$\begin{aligned}
D^+V_{(27)}(t, Y) &= \lim_{\sigma\to 0^+} \sup \frac{1}{\sigma}\left[V(t+\sigma, Y+\sigma(F(t, Y)+G(t, Y)) - V(t, Y)\right] \\
&= \lim_{\sigma\to 0^+} \sup \frac{1}{\sigma}\big[V(t+\sigma, Y+\sigma(F(t, Y)+G(t, Y)) \\
&\quad - V(t+\sigma, Y+\sigma F(t, Y)) + V(t+\sigma, Y+\sigma F(t, Y)) - V(t, Y)\big] \\
&\leqslant \lim_{\sigma\to 0^+} \sup \frac{1}{\sigma}\left[V(t+\sigma, Y+\sigma F(t, Y)) - V(t, Y)\right] \\
&\quad + \lim_{\sigma\to 0^+} \sup \frac{1}{\sigma}\big[V(t+\sigma, Y+\sigma(F(t, Y)+G(t, Y)) \\
&\quad - V(t+\sigma, Y+\sigma F(t, Y))\big] \\
&\leqslant D^+V_{(9)}(t, Y) \\
&\quad + \lim_{\sigma\to 0^+} \sup \frac{1}{\sigma}\big[LD[Y+\sigma(F(t, Y)+G(t, Y)), Y+\sigma F(t, Y)]\big] \\
&\leqslant D^+V_{(9)}(t, Y) + LD[G(t, Y), \{0\}] \\
&\leqslant \frac{h'(t)}{h(t)}V(t, Y) + L\lambda(t)D[Y, \{0\}] \\
&\leqslant \left[\frac{h'(t)}{h(t)} + L\lambda(t)\right]V(t, Y),
\end{aligned}$$

where L is the Lipschitz constant of the function V.

Integrating between t_0 and t and using the property (ii) of $V(t, Y)$ leads to

$$\begin{aligned}
V(t, X(t)) &\leqslant V(t_0, Y_0)e^{\int_{t_0}^{t} \frac{h'(s)}{h(s)} + L\lambda(s)\,\mathrm{d}s} \\
&= V(t_0, Y_0)h(t)h^{-1}(t_0)e^{L\int_{t_0}^{t} \lambda(s)\,\mathrm{d}s} \\
&\leqslant cD[Y_0, \{0\}]h(t)h^{-1}(t_0)e^{L\int_{t_0}^{t} \lambda(s)\,\mathrm{d}s}.
\end{aligned}$$

Then, for $t \geqslant t_0$, we get

$$D\big[Y(t, t_0, Y_0), \{0\}\big] \leqslant cD[Y_0, \{0\}]\widetilde{h}(t)\widetilde{h}^{-1}(t_0),$$

where $\widetilde{h}(t) = h(t)e^{L\int_0^t \lambda(s)\,\mathrm{d}s}$ is a continuous, bounded, and positive function on $\mathbb{R}_+$ and $c \geqslant 1$. This implies that $Y = \{0\}$ of (27) is globally $\widetilde{h}$-stable. □

4.2. h-Stability of Cascaded Systems

Following [5] and [14], we now consider the cascaded systems of the form:

$$X'_{1_gH} = F_1(t,\ X_1) + G(t,\ X_1,\ X_2), \tag{29}$$

$$X'_{2_gH} = F_2(t,\ X_2), \tag{30}$$

where $X_1 \in \mathbb{I}_{\not\!{k}},\ X_2 \in \mathbb{I}_2$ and $X = [X_1,\ X_2]^\top \in \mathbb{I}$, are the states of the closed-loop system, $F_1,\ \ F_2$ and G are piecewise continuous in t and locally Lipschitz in $X_1,\ \ X_2$ and $X = [X_1,\ X_2]^\top$, respectively. Suppose that $F_1(t,\ \{0\}) = \{0_{\mathbb{I}_{\not\!{k}}}\}$ and $F_2(t,\ \{0\}) = \{0_{\mathbb{I}_2}\}$.

In addition, assume that the system

$$X'_{1_gH} = F_1(t,\ X_1) \tag{31}$$

and system (30) are globally h-stable, to guarantee the global h-stability of the cascaded system (29)–(30).

Let $X_1(t,\ t_0,\ X_{10})$ and $X_2(t,\ t_0,\ X_{20})$ be the unique solutions of (31) and (30) respectively, such that $X_1(t,\ t_0,\ X_{10}) = X_{10}$ and $X_2(t,\ t_0,\ X_{20}) = X_{20}$ are the initial conditions.

Note that $X(t,\ t_0,\ X_0)$ the unique solution of the cascaded system (29)-(30) satisfies

$$X(t_0,\ t_0,\ X_0) = X(t_0) = X_0 = (X_{10},\ X_{20}) \in \mathbb{I}_1 \times \mathbb{I}_2.$$

We take into account its associated systems

$$U'_{1_gH} = W_1(t,\ t_0,\ X_1)U_1(t), \tag{32}$$

$$U'_{2_gH} = W_2(t,\ t_0,\ X_2)U_2(t), \tag{33}$$

respectively where $W_i(t,\ t_0,\ X_i) = F'_{i_gH}(t,\ X(t,\ t_0,\ X_i)),\ i = 1, 2.$

The following results and techniques are similar to those obtained in [14].

Theorem 8 *We assume that the two following conditions hold:*

1. *The subsystems (30) and (31) are globally h-stable and the stationary solutions $U_1 = \{0\}$ and $U_2 = \{0\}$ of (32) and (33), respectively are globally h-stable.*

2. *For any $X_1 \in \mathbb{I}_1$, $X_2 \in \mathbb{I}_2$ the estimates*

$$D[G(t,\ X_1,\ X_2),\ \{0\}] \leqslant \gamma(t)(D[X_1,\ \{0\}] + D[X_2,\ \{0\}]) \tag{34}$$

hold for all $t \geqslant 0$, with $\displaystyle\int_0^{+\infty} \gamma(s)\mathrm{d}s < +\infty.$

Then, the cascaded system (29)–(30) is globally $\widetilde{h}$-stable where

$$\widetilde{h}(t) = h(t)\, e^{L\int_0^t \gamma(s)\,\mathrm{d}s}, \quad L \geqslant 0.$$

Proof. Based on Theorem 6, there exist two Lyapunov functions $(t,\ X_1) \mapsto V_1(t,\ X_1)$ and $(t,\ X_2) \mapsto V_2(t,\ X_2)$ satisfying the three properties below, for $i = 1,\ 2$

i) $V_i \in C[\mathbb{R}_+ \times \mathbb{I}_i,\ \mathbb{R}_+]$ and $V_i(t,\ X_i)$ is Lipschitzian in X_i for each $t \in \mathbb{R}_+$,

ii) $D[X_i,\ \{0\}] \leqslant V_i(t,\ X_i) \leqslant c_i D[X_i,\ \{0\}]$, $(t,\ X_i) \in \mathbb{R}_+ \times \mathbb{I}_i$, $c_i \geqslant 1$,

iii) $D^+V_i(t,\ X_i) \leqslant \dfrac{h'(t)}{h(t)} V_i(t,\ X_i)$, $(t,\ X_i) \in \mathbb{R}_+ \times \mathbb{I}_i$.

Let us define the function $U(t,\ X) = V_1(t,\ X_1) + V_2(t,\ X_2)$ with $X = [X_1,\ X_2]^\top \in \mathbb{I}_1 \times \mathbb{I}_2$. The derivative of $U(t,\ X)$ with respect to the systems (29) and (30) is given by

$$\begin{aligned}
D^+U(t,\ X) &= D^+V_{1(29)}(t,\ X_1) + D^+V_{2(30)}(t,\ X_2)\\
&\leqslant D^+V_{1(31)}(t,\ X_1) + D^+V_{2(30)}(t,\ X_2) + LD[G(t,\ X_1,\ X_2),\ \{0\}]\\
&\leqslant \frac{h'(t)}{h(t)} V_1(t,\ X_1) + \frac{h'(t)}{h(t)} V_2(t,\ X_2)\\
&\quad + L\gamma(t)\big(D[X_1,\ \{0\}] + D[X_2,\ \{0\}]\big)\\
&\leqslant \left(\frac{h'(t)}{h(t)} + L\gamma(t)\right)(V_1(t,\ X_1) + V_2(t,\ X_2))\\
&\leqslant \left(\frac{h'(t)}{h(t)} + L\gamma(t)\right) U(t,\ X),
\end{aligned}$$

where L is the Lipschitz constant of the function V.

Let

$$\begin{aligned}
\varphi(t) &= U(t,\ X(t)) e^{-\int_{t_0}^t \frac{h'(s)}{h(s)} + L\gamma(s)\,\mathrm{d}s}\\
&= U(t,\ X(t)) \frac{h(t_0)}{h(t)} e^{-L\int_{t_0}^t \gamma(s)\,\mathrm{d}s}, \quad \text{for } t \geqslant t_0\,.
\end{aligned}$$

Then

$$\begin{aligned}
D^+\varphi(t) &= \left[D^+U(t,\ X(t))\frac{h(t_0)}{h(t)} - U(t,\ X(t))\frac{h(t_0)h'(t)}{h^2(t)}\right.\\
&\quad \left. -L\gamma(t)U(t,\ X(t))\frac{h(t_0)}{h(t)}\right] e^{-L\int_{t_0}^{t}\gamma(s)\,\mathrm{d}s}\\
&= \left[D^+U(t,\ X(t)) - U(t,\ X(t))\left(\frac{h'(t)}{h(t)} + L\gamma(t)\right)\right]\frac{h(t_0)}{h(t)}e^{-L\int_{t_0}^{t}\gamma(s)\,\mathrm{d}s}\\
&\leqslant \left[U(t,\ X(t))\left(\frac{h'(t)}{h(t)} + L\gamma(t)\right) - U(t,\ X(t))\left(\frac{h'(t)}{h(t)} + L\gamma(t)\right)\right]\\
&\quad \cdot \frac{h(t_0)}{h(t)}e^{-L\int_{t_0}^{t}\gamma(s)\,\mathrm{d}s} = 0.
\end{aligned}$$

Integrating between t_0 and t:

$$\varphi(t) - \varphi(t_0) \leqslant 0,$$

then

$$U(t,\ X(t))\frac{h(t_0)}{h(t)}e^{-L\int_{t_0}^{t}\gamma(s)\,\mathrm{d}s} \leqslant U(t_0,\ X(t_0)),$$

which implies that

$$U(t,\ X(t)) \leqslant U(t_0,\ X(t_0))\frac{h(t)}{h(t_0)}e^{L\int_{t_0}^{t}\gamma(s)\,\mathrm{d}s},$$

Thanks to the following inequalities

$$\begin{aligned}
D[X_1(t_0),\ \{0\}] &\leqslant D[X(t_0),\ \{0\}],\\
D[X_2(t_0),\ \{0\}] &\leqslant D[X(t_0),\ \{0\}],\\
D[X(t),\ \{0\}] &\leqslant D[X_1(t),\ \{0\}] + D[X_2(t),\ \{0\}],
\end{aligned}$$

we deduce that

$$\begin{aligned}
D[X(t,\ t_0,\ X(t_0)),\ \{0\}] &\leqslant (c_1D[X_1(t_0),\ \{0\}] + c_2D[X_2(t_0),\ \{0\}])\widetilde{h}(t)\widetilde{h}^{-1}(t_0)\\
&\leqslant CD[X(t_0),\ \{0\}]\widetilde{h}(t)\widetilde{h}^{-1}(t_0),
\end{aligned}$$

where $\widetilde{h}(t) = h(t)e^{L\int_0^t\gamma(s)\,\mathrm{d}s}$, $t \geqslant t_0$, is a positive, continuous and bounded function defined on $\mathbb{R}_+$ and $C = \max\{c_1,\ c_2\} \geqslant 1$.

This completes the proof of Theorem. □

5. Example

Example 1 Consider the differential system shown below as investigated in [26]

$$D_{gH}X = \mu(t)X, \qquad X(t_0) = X_0 \in K_c(\mathbb{R}^n), \tag{35}$$

where $\mu \in L^1(\mathbb{R}_+)$, a real function $\mu(t) > 0$ on $\mathbb{R}_+$.

Selecting the Lyapunov function

$$V(t,\ X(t)) = D[X(t),\ \Theta_0], \quad t \geqslant t_0$$

we have;

$$D[X(t),\ \Theta_0] \leqslant V(t,\ X(t)) \leqslant 2D[X(t),\ \Theta_0],$$

for all $(t,\ X_1(t)),\ (t,\ X_2(t)) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n)$ we get

$$\begin{aligned}\big|V(t,\ X_1(t)) - V(t,\ X_2(t))\big| &= \big|D[X_1(t),\ \Theta_0] - D[X_2(t),\ \Theta_0]\big| \\ &\leqslant D[X_1(t),\ X_2(t)],\end{aligned}$$

and

$$\begin{aligned}V(t+\sigma,\ X(t)+\sigma\mu(t)X(t)) &= D[X(t)+\sigma\mu(t)X(t),\ \Theta_0] \\ &\leqslant D[X(t),\ \Theta_0] + \sigma D[\mu(t)X(t),\ \Theta_0] \\ &\leqslant D[X(t),\ \Theta_0] + \sigma\mu(t)D[X(t),\ \Theta_0],\end{aligned}$$

then the derivative of V satisfies

$$D^+V(t,\ X(t)) \leqslant \mu(t)D[X(t),\ \Theta_0] = \frac{h'(t)}{h(t)}V(t,\ X(t)),$$

with the positive function $h(t) = e^{\int_0^t \mu(s)\,\mathrm{d}s}$ is bounded defined on $\mathbb{R}_+$.

Consequently, all the assumptions in Theorem 5 are satisfied and thus the system (35) is h-stable.

It should be noted that, in [26], Martynyuk demonstrates the stability for the same system on $K_c(\mathbb{R}^n)$, while we show here its h-stability when it is not asymptotically stable.

6. Conclusion

This paper studies the h-stability of set-valued differential equations. Based on Lyapunov theory, sufficient conditions for the h-stability of set-valued differential equations are proved. These results are even more general than exponential stability and asymptotic stability. As the system is globally h-stable, we prove the necessary conditions of h-stability on $\mathbb{I}$ set of all non-empty, convex, and compact intervals of $\mathbb{R}$ as well as the converse theorem due to the existence of the Lyapunov function. The results of this theorem indicate that the global h-stability of perturbed and cascaded systems is established. Our future works will put special focus on the extension of the converse theorem in the general case.

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