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On the control of the final speed for a class of finite-dimensional linear systems: controllability and regulation

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In this article, we extended the concept of controllability, traditionally used to control the final state of a system, to the exact control of its final speed. Inspired by Kalman's theory, we have established some conditions to characterize the control that allows the system to reach a desired final speed exactly. When the assumptions ensuring speed-controllability are not met, we adopt a regulation strategy that involves determining the control law to make the system's final speed approach as closely as possible to the predefined final speed, and this at a lower cost. The theoretical results obtained are illustrated through three examples.

Key words: speed-controllability, controllability, continuous systems, regulation, Kalman's condition.

1. Introduction

It is known that one of the most important themes in the analysis of a system is controllability. Since Kalman's results in 1963 on the controllability of linear systems with localized parameters [1], scientists have worked on various types of controllability to address increasingly complex questions arising from technological developments. These types include controllability of nonlinear systems [2–4], systems with time delays [5–7], discrete systems [8–10], fractional systems [11–13], and distributed systems [14–16].

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To contribute to this theme, we started with the standard linear system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases}$$

where $A \in \mathcal{L}(\mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, and $u \in L^2(0, T, \mathbb{R}^m)$.

Instead of seeking control $u \in L^2(0, T, \mathbb{R}^m)$ to reach a predefined desired state x_d, which was the primary objective of early works on controllability, we aimed to determine the control u (under certain conditions) that achieves the objective $\dot{x}(T) = v_d$, where v_d is the desired final speed. This led us to the search for control u such that $Bu(T) = v_d - Ax(T)$, which turned out to be ill-structured and difficult to solve. To overcome this difficulty and instead of focusing on standard linear systems, this paper addresses a class of linear systems defined by

$$\begin{cases} \dot{x}(t) = Ax(t) + \int\limits_0^t B(\theta)u(\theta)\,\mathrm{d}\theta, \quad 0 \leqslant t \leqslant T, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases}$$

where $A \in \mathcal{L}(\mathbb{R}^n)$ is the matrix of the state, $B \in L^2(0, T, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$, $u \in L^2(0, T, \mathbb{R}^m)$ is the control, $x(t)$ the state of system at instance t and x_0 is the initial state. Similar to Kalman's theory, we established conditions to characterize the control that allows the system to exactly reach the desired final speed. In the absence of conditions ensuring the speed-controllability of our system, we consider a weaker version of the problem, a regulation problem, where we aimed to find the control that allows the final velocity of the system $\dot{x}(T)$ to approximate as closely as possible a predefined desired speed v_d.

The rest of the paper is organized as follows. In Section 2, we give some preliminary results. In Section 3, taking advantage of the results of analysis and integration, we characterized the speed-controllability and we give the exact expression of the control allowing the transfer from the initial state to a desired speed. In Section 4, we study an important special case of the operator $B(.)$, and then we provide an algebraic characterization of the speed-controllability. In Section 5, we are concerned with the case when the system cannot be speed controlled. In Section 6, we provide a numerical example. Section 7 is devoted to a short conclusion.

Notation In the sequel, $L^2(0, T, \mathbb{R}^m)$ will denote the spaces of integrable square functions defined on $[0, T]$ and with values in $\mathbb{R}^m$, $\chi_{[a,b]}(t)$ is the indicator function that takes value 1 when $t \in [a, b]$ and value 0 when $t \notin [a, b]$. Transpose of a matrix A is denoted by $A^\top$. rank$(.)$ represents the rank of a matrix. Also, range$(.)$ and ker$(.)$ designated image and kernel of an operator, respectively.

2. Preliminary results

Consider the controlled system

$$\begin{cases} \dot{x}(t) = Ax(t) + \displaystyle\int_0^t B(\theta)u(\theta)\,\mathrm{d}\theta, \quad t \geqslant 0, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $A \in \mathcal{L}(\mathbb{R}^n)$ is the matrix of the state, $B \in L^2(0, T, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$, and $u \in L^2(0, T, \mathbb{R}^m)$ is the matrix of the input.

Definition 1 *The system (1) is said to be speed-controllable on* $[0; T]$ *if*

$$\forall x_0, v_d \in \mathbb{R}^n,\ \exists\, u \in L^2(0, T, \mathbb{R}^m) :\ \dot{x}_u^{x_0}(T) = v_d.$$

Theorem 1 *The speed in time T is written as*

$$\dot{x}(T) = Ae^{TA}x_0 + \mathcal{H}u, \tag{2}$$

where

$$\begin{aligned} \mathcal{H} \colon L^2(0, T, \mathbb{R}^m) &\longrightarrow \mathbb{R}^n, \\ u &\longmapsto \mathcal{H}_1 u + \mathcal{H}_2 u \end{aligned} \tag{3}$$

with

$$\mathcal{H}_1 u = \int_0^T B(\theta)u(\theta)\,\mathrm{d}\theta \tag{4}$$

and

$$\mathcal{H}_2 u = A\int_0^T e^{(T-s)A}\left(\int_0^s B(\theta)u(\theta)\,\mathrm{d}\theta\right)\mathrm{d}s. \tag{5}$$

Proof. We have

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}\left(\int_0^s B(\theta)u(\theta)\,\mathrm{d}\theta\right)\mathrm{d}s \tag{6}$$

since

$$\dot{x}(t) = Ae^{tA}x_0 + A\int_0^t e^{(t-s)A}\left(\int_0^s B(\theta)u(\theta)\,\mathrm{d}\theta\right)\mathrm{d}s + \int_0^t B(\theta)u(\theta)\,\mathrm{d}\theta$$

then

$$\dot{x}(T) = Ae^{TA}x_0 + \mathcal{H}u.$$

□

Theorem 2 *The system* (1) *is speed-controllable on* $[0;T]$ *if and only if* $range\,\mathcal{H} = \mathbb{R}^n$.

Proof.
$\Rightarrow$) Let us show that $\mathbb{R}^n \subset range\mathcal{H}$, let $x \in \mathbb{R}^n$. The system 1 is speed-controllable on $[0;T]$, then for $x_0 = 0$ and $v_d = x$, there exists a control $u \in L^2(0,T,\mathbb{R}^m)$ such that $\dot{x}_u = v_d \Rightarrow Ae^{TA}x_0 + \mathcal{H}u \Rightarrow x = \mathcal{H}u$.
$\Leftarrow$) Since $\mathcal{H}$ is surjective, therefore for $v_d - Ae^{TA}x_0 \in \mathbb{R}^n$, there exists $u \in L^2(0,T,\mathbb{R}^m)$ such that

$$\mathcal{H}u = v_d - Ae^{TA}x_0$$

then

$$Ae^{TA}x_0 + \mathcal{H}u = v_d$$

thus

$$\dot{x}_u^{x_0}(T) = v_d$$

□

Theorem 3 *The following properties are equivalent*

(i) *The system* (1) *is speed-controllable on* $[0;T]$,

(ii) $range\,\mathcal{H} = \mathbb{R}^n$,

(iii) $\ker \mathcal{H}^* = \{0\}$.

Theorem 4 *The adjoint of the operator* $\mathcal{H}$ *is given by*

$$\begin{aligned} \mathcal{H}^* : \mathbb{R}^n &\longrightarrow L^2(0,T,\mathbb{R}^m) \\ x &\longmapsto \mathcal{H}^* x \end{aligned} \tag{7}$$

with

$$(\mathcal{H}^* x)(\theta) = B^\top(\theta) e^{(T-\theta)A^\top} x. \tag{8}$$

Proof. Let $u \in L^2(0,T,\mathbb{R}^m)$ and $x \in \mathbb{R}^n$, then

$$\begin{aligned} \langle \mathcal{H}_1 u, x \rangle_{\mathbb{R}^n} &= \int_0^T \langle B(\theta)u(\theta), x \rangle \, \mathrm{d}\theta \\ &= \int_0^T \left\langle u(\theta), B^\top(\theta)x \right\rangle \mathrm{d}\theta \\ &= \left\langle u(.), B^\top(.)x \right\rangle_{L^2(0,T,\mathbb{R}^m)}, \end{aligned}$$

this leads to

$$\mathcal{H}_1^* x = B^\top(.)x. \tag{9}$$

On the other hand

$$\begin{aligned}
\langle \mathcal{H}_2 u, x\rangle &= \left\langle A\int_0^T e^{(T-s)A}\left[\int_0^s B(\theta)u(\theta)\,\mathrm{d}\theta\right]\mathrm{d}s, x\right\rangle \\
&= \left\langle \int_0^T e^{(T-s)A}\left[\int_0^s B(\theta)u(\theta)\,\mathrm{d}\theta\right]\mathrm{d}s, A^\top x\right\rangle \\
&= \int_0^T \left\langle e^{(T-s)A}\left[\int_0^s B(\theta)u(\theta)\,\mathrm{d}\theta\right], A^\top x\right\rangle \mathrm{d}s \\
&= \int_0^T \left\langle \int_0^s B(\theta)u(\theta)\,\mathrm{d}\theta, e^{(T-s)A^\top} A^\top x\right\rangle \mathrm{d}s.
\end{aligned}$$

By using Fubini's theorem, we find

$$\begin{aligned}
\langle \mathcal{H}_2 u, x\rangle &= \int_0^T \left(\left\langle \int_\theta^T B(\theta)u(\theta), e^{(T-s)A^\top} A^\top x\right\rangle \mathrm{d}s\right)\mathrm{d}\theta \\
&= \int_0^T \left\langle u(\theta), \int_\theta^T B^\top(\theta)e^{(T-s)A^\top} A^\top x\,\mathrm{d}s\right\rangle \mathrm{d}\theta,
\end{aligned}$$

thus

$$\left(\mathcal{H}_2^* x\right)(\theta) = \int_\theta^T B^\top(\theta)e^{(T-s)A^\top} A^\top x\,\mathrm{d}s \tag{10}$$

$$= B^\top(\theta)\int_\theta^T \left[e^{(T-s)A^\top} x\right]' \mathrm{d}s \tag{11}$$

$$= B^\top(\theta)\left[e^{(T-\theta)A^\top} x - x\right], \tag{12}$$

according to the two relations (9) and (12) we obtain that

$$\left(\mathcal{H}^* x\right)(\theta) = B^\top(\theta)e^{(T-\theta)A^T} x. \qquad \square$$

Remark 1 *By a simple use of Cayley-Hamelton theorem, we establish that*

$$\ker \mathcal{R} \subset \ker \mathcal{H}^* \tag{13}$$

with

$$\begin{aligned} \mathcal{R} \colon \mathbb{R}^n &\longrightarrow \left(L^2\left(0, T, \mathbb{R}^m\right)\right)^n, \\ x &\longmapsto \begin{bmatrix} B^\top(.)x \\ B^\top(.)A^\top x \\ \vdots \\ B^\top(.)\left(A^\top\right)^{n-1} x \end{bmatrix}. \end{aligned} \tag{14}$$

3. Exact determination of the control

Let Λ be the matrix of order n defined by

$$\Lambda = \mathcal{H}\mathcal{H}^*. \tag{15}$$

Remark 2 *The matrix Λ is symmetric positive.*

Indeed, let $x \in \mathbb{R}^n$, then

$$\langle \Lambda x, x \rangle = \|\mathcal{H}^* x\|^2 \geqslant 0.$$

Proposition 1 *If The system (1) is speed-controllable on $[0;T]$, then there exists a control u^* permitting the transfer from x_0 to v_d with minimum energy. Furthermore, the control u^* is given by*

$$u^* = \mathcal{H}^* z, \tag{16}$$

where $z \in \mathbb{R}^n$ such that

$$\Lambda z = v_d - Ae^{TA}x_0.$$

Proof. Let $x \in \mathbb{R}^n$ and assume that system (1) is speed-controllable on $[0;T]$, i.e., $\ker \mathcal{H}^* = \{0\}$, then we have

$$\langle \Lambda x, x \rangle = 0 \implies H^* x = 0 \implies x = 0 \tag{17}$$

now, according to (17) and remark 2, we conclude that the matrix Λ is invertible.
$v_d - Ae^{TA}x_0 \in \mathbb{R}^n \implies \exists! z \in \mathbb{R}^n$ such that

$$\Lambda z = v_d - Ae^{TA}x_0$$

we take $u^* = \mathcal{H}^* z$, then

$$Ae^{TA}x_0 + \mathcal{H}u^* = v_d \tag{18}$$

this leads to

$$\dot{x}_{u^*}^{x_0}(T) = v_d\,. \tag{19}$$

Let's show that the control u^* has a minimum energy. If there exists another control ω solves the same problem, i.e., $\dot{x}_{\omega}^{x_0}(T) = v_d$.

$$\begin{aligned}
\dot{x}_{\omega}^{x_0}(T) = v_d &\Longrightarrow \mathcal{H}\omega = \mathcal{H}u^* \\
&\Longrightarrow \mathcal{H}\left(\omega - u^*\right) = 0 \\
&\Longrightarrow \langle \mathcal{H}\left(\omega - u^*\right), z\rangle = 0 \\
&\Longrightarrow \langle \omega - u^*, \mathcal{H}^* z\rangle = 0 \\
&\Longrightarrow \langle \omega - u^*, u^*\rangle = 0 \\
&\Longrightarrow \langle \omega, u^*\rangle = \|u^*\|^2 \\
&\Longrightarrow \|u^*\|^2 \leqslant \|u^*\|\|\omega\| \\
&\Longrightarrow \|u^*\| \leqslant \|\omega\|.
\end{aligned}$$

This achieves the demonstration. □

4. An important special case

In this section we discretize the interval $[0;T]$ as , $[0;T] = \cup_{i=0}^{N-1}[t_i;t_{i+1}[$, with $t_{i+1} - t_i = h$, and we take

$$B(\theta) = \begin{cases} B_0 & \text{in } [t_0;t_1[, \\ B_1 & \text{in } [t_1;t_2[, \\ \vdots & \\ B_{N-1} & \text{in } [t_{N-1};t_N[\end{cases} = \sum_{i=0}^{N-1} \chi_{[t_i;t_{i+1}[}(\theta) B_i \tag{20}$$

with $B_1, B_2, \ldots, B_N$ are $n \times m$-order matrices. We obtain that

Proposition 2 *The adjoint of the operator $\mathcal{H}$ is given by*

$$\left(\mathcal{H}^* x\right)(\theta) = \begin{cases} B_0^\top e^{(T-\theta)A^\top} x & \textit{in } [t_0;t_1[, \\ B_1^\top e^{(T-\theta)A^\top} x & \textit{in } [t_1;t_2[, \\ \vdots & \\ B_{N-1}^\top e^{(T-\theta)A^\top} x & \textit{in } [t_{N-1};t_N[. \end{cases} \tag{21}$$

4.1. Characterization of the speed-controllability

In order to characterize speed-controllability, we cite the following result.

Proposition 3 *The kernel of the operator $\mathcal{H}^*$ is given by*

$$\ker \mathcal{H}^* = \ker L \tag{22}$$

with L the matrix defined by

$$L = \left[\begin{array}{c} B_0^\top \\ B_0^\top A^\top \\ \vdots \\ B_0^\top \left(A^\top\right)^{n-1} \\ \hline B_1^\top \\ B_1^\top A^\top \\ \vdots \\ B_1^\top \left(A^\top\right)^{n-1} \\ \hline \vdots \\ \hline B_{N-1}^\top \\ B_{N-1}^\top \left(A^\top\right)^{n-1} \end{array}\right]. \tag{23}$$

Proof.
$\Leftarrow$) We use the Cayley Hamilton theorem, we find that

$$\begin{aligned} x \in \ker L &\Longrightarrow \begin{cases} B_i^\top x = 0 \\ B_i^\top A^\top x = 0 \\ \vdots \\ B_i^\top \left(A^\top\right)^{n-1} x = 0 \end{cases} \qquad \forall i \in \{0, 1, \dots, N-1\}. \\ &\Longrightarrow B_i^\top e^{sA^\top} x = 0;\ \ \forall i \in \{0, 1, \dots, N-1\}, \forall s \in \mathbb{R}. \\ &\Longrightarrow x \in \ker \mathcal{H}^*. \end{aligned}$$

$\Rightarrow$) By also using the Cayley-Hamilton theorem, we obtain that

$$\begin{aligned} x \in \ker \mathcal{H}^* &\Longrightarrow B_i^\top e^{(T-\theta)A^\top} x = 0; && \forall \theta \in [t_i; t_{i+1}[,\ \forall i \in \{0, 1, \dots, N-1\} \\ &\Longrightarrow B_i^\top e^{(T-\theta)A^\top} \left(A^\top\right)^k x = 0; && \forall \theta \in [t_i; t_{i+1}[,\ \forall i \in \{0, 1, \dots, N-1\}, \\ & && \forall k \in \mathbb{N} \end{aligned}$$

$$\Longrightarrow B_i^\top e^{(T-\theta)A^\top} e^{rA^\top} x = 0; \qquad \forall\theta \in [t_i; t_{i+1}[, \ \forall i \in \{0,1,\ldots,N-1\}, \ \forall r \in \mathbb{N}$$

$$\Longrightarrow B_i^\top e^{sA^\top} x = 0; \qquad \forall i \in \{0,1,\ldots,N-1\}, \ \forall s \in \mathbb{R}.$$

We derive this relation and we replace each times s by 0, then

$$B_i^\top \left(A^\top\right)^p x = 0; \quad \forall p \in \mathbb{N} \Longrightarrow x \in \ker L.$$ □

Thus, we characterize the speed-controllability by the following result.

Proposition 4 *The following properties are equivalent*

(i) The system (1) *is speed-controllable on* $[0;T]$.

(ii) $rank(L) = n$*, with L is the matrix defined by* (23) .

4.2. Exact determination of the control

Proposition 5 *In this case the control* u^* *permitting the transfer from* x_0 *to* v_d *with minimum energy, is given by*

$$u^* = \mathcal{H}^* z \tag{24}$$

with z is the solution of the equation

$$\Lambda z = v_d - Ae^{TA}x_0 \tag{25}$$

and Λ *is defined by*

$$\Lambda x = \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} e^{(T-\theta)A} B_i B_i^\top e^{(T-\theta)A^\top} x \, \mathrm{d}\theta. \tag{26}$$

Proof. We have

$$\Lambda = \mathcal{H}\mathcal{H}^*$$

then

$$\Lambda x = \int_0^T B(\theta)\left(\mathcal{H}^* x\right)(\theta)\,\mathrm{d}\theta + A\int_0^T e^{(T-s)A}\left[\int_0^s B(\theta)\left(\mathcal{H}^* x\right)(\theta)\,\mathrm{d}\theta\right]\mathrm{d}s$$
$$= \sum_{i=0}^{N-1}\int_{t_i}^{t_i+1} B_i B_i^\top e^{(T-\theta)A^\top} x\,\mathrm{d}\theta + A\int_0^T\left(\int_0^s e^{(T-s)A} B(\theta)\left(\mathcal{H}^* x\right)(\theta)\,\mathrm{d}\theta\right)\mathrm{d}s.$$

We use the integration theorem of Fubini, we obtain that

$$\begin{aligned}\Lambda x &= \sum_{i=0}^{N-1} \int_{t_i}^{t_i+1} B_i B_i^\top e^{(T-\theta)A^\top} x \, \mathrm{d}\theta + A \int_0^T \left(\int_\theta^T e^{(T-s)A} B(\theta) \left(\mathcal{H}^* x\right)(\theta) \, \mathrm{d}s \right) \mathrm{d}\theta \\ &= \sum_{i=0}^{N-1} \int_{t_i}^{t_i+1} B_i B_i^\top e^{(T-\theta)A^\top} x \, \mathrm{d}\theta + \int_0^T \left(\int_\theta^T \left(A e^{(T-s)A} \right) \mathrm{d}s \right) B(\theta) \left(\mathcal{H}^* x\right)(\theta) \, \mathrm{d}\theta.\end{aligned}$$

We use the fact that

$$\left(e^{(T-s)A}\right)' = \left(-A e^{(T-s)A}\right)$$

we find that

$$\begin{aligned}\Lambda x &= \sum_{i=0}^{N-1} \int_{t_i}^{t_i+1} B_i B_i^\top e^{(T-\theta)A^\top} x \, \mathrm{d}\theta + \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \left[-e^{(T-s)A} \right]_\theta^T B_i B_i^\top e^{(T-\theta)A^\top} x \, \mathrm{d}\theta \\ &= \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \left(I_n + \left(e^{(T-\theta)A} - I_n \right) \right) B_i B_i^\top e^{(T-\theta)A^\top} x \, \mathrm{d}\theta \\ &= \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} e^{(T-\theta)A} B_i B_i^\top e^{(T-\theta)A^\top} x \, \mathrm{d}\theta.\end{aligned}$$

From the proposition 1 we can conclude the result. □

Example 1 Let the matrix A, B_0, B_1, B_2 and the parameters T, h, n, m, and N defined as follows

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_d = \begin{pmatrix} 100 \\ 100 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$T = 60, \quad h = \frac{T}{3}, \quad n = 2, \quad m = 1, \quad N = 3.$$

In this case we have

$$B(\theta) = \begin{cases} B_0, & \theta \in [0, 20[\,, \\ B_1, & \theta \in [20, 40[\,, \\ B_2, & \theta \in [40, 60[\,. \end{cases}$$

The matrix L is given by

$$L = \begin{pmatrix} B_0^\top \\ B_0^\top A \\ B_1^\top \\ B_1^\top A \\ B_2^\top \\ B_2^\top A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 3 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$$

since $\operatorname{rank}(L) = 2$, then the considered system is speed-controllable. In this example the matrix Λ is given by

$$\begin{aligned} \Lambda x &= \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} e^{(T-\theta)A} B_i B_i^\top e^{(T-\theta)A^\top} x \, \mathrm{d}\theta \\ &= \int_0^h e^{(T-\theta)A} B_0 B_0^\top e^{(T-\theta)A^\top} x \, \mathrm{d}\theta + + \int_h^{2h} e^{(T-\theta)A} B_1 B_1^\top e^{(T-\theta)A^\top} x \, \mathrm{d}\theta \\ &\quad + \int_{2h}^{3h} e^{(T-\theta)A} B_2 B_2^\top e^{(T-\theta)A^\top} x \, \mathrm{d}\theta \end{aligned}$$

or

$$B_0 B_0^\top = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e^{(T-\theta)A} = e^{(T-\theta)A^\top} = \begin{pmatrix} e^{T-\theta} & 0 \\ 0 & 1 \end{pmatrix}$$

this leads to

$$\begin{aligned} e^{(T-\theta)A} \;\; B_0 \;\; B_0^\top e^{(T-\theta)A^\top} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} e^{T-\theta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e^{T-\theta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} e^{2(T-\theta)} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} \int_0^h e^{(T-\theta)A} B_0 B_0^\top e^{(T-\theta)A^\top} x \, \mathrm{d}\theta &= \int_0^h \begin{pmatrix} e^{2(T-\theta)} & 0 \\ 0 & 0 \end{pmatrix} \mathrm{d}\theta x \\ &= \begin{pmatrix} \frac{1}{2} e^{2T} e^{-2h} \left(e^{2h} - 1\right) & 0 \\ 0 & 0 \end{pmatrix} x. \end{aligned}$$

On the other words

$$
\begin{aligned}
e^{(T-\theta)A}B_1B_1^{\top}e^{(T-\theta)A^{\top}} &= \begin{pmatrix} e^{T-\theta} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 9 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e^{T-\theta} & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 9e^{2(T-\theta)} & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

this implies

$$
\begin{aligned}
\int_h^{2h} e^{(T-\theta)A}B_1B_1^{\top}e^{(T-\theta)A^{\top}}x\,\mathrm{d}\theta &= \int_h^{2h} \begin{pmatrix} 9e^{2(T-\theta)} & 0 \\ 0 & 0 \end{pmatrix}\mathrm{d}\theta x \\
&= \begin{pmatrix} \dfrac{9}{2}e^{2T}e^{-4h}\left(e^{2h}-1\right) & 0 \\ 0 & 0 \end{pmatrix} x
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{2h}^{3h} e^{(T-\theta)A}B_2B_2^{\top}e^{(T-\theta)A^{\top}}x\,\mathrm{d}\theta &= \int_{2h}^{3h} \begin{pmatrix} e^{T-\theta} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{T-\theta} & 0 \\ 0 & 1 \end{pmatrix}\mathrm{d}\theta\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{1}{2}e^{2T}e^{-6h}\left(e^{2h}-1\right) & \dfrac{e^T}{e^{3h}}\left(e^h-1\right) \\ \dfrac{e^T}{e^{3h}}\left(e^h-1\right) & h \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\end{aligned}
$$

therefore

$$
\begin{aligned}
\Lambda = &\begin{pmatrix} \dfrac{1}{2}e^{2T}e^{-2h}\left(e^{2h}-1\right) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \dfrac{9}{2}e^{2T}e^{-4h}\left(e^{2h}-1\right) & 0 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} \dfrac{1}{2}e^{2T}e^{-6h}\left(e^{2h}-1\right) & \dfrac{e^T}{e^{3h}}\left(e^h-1\right) \\ \dfrac{e^T}{e^{3h}}\left(e^h-1\right) & h \end{pmatrix} = \begin{pmatrix} \begin{array}{l} \frac{1}{2}e^{2T}e^{-2h}\left(e^{2h}-1\right) \\ +\frac{9}{2}e^{2T}e^{-4h}\left(e^{2h}-1\right) \\ +\frac{1}{2}e^{2T}e^{-6h}\left(e^{2h}-1\right) \end{array} & \dfrac{e^T}{e^{3h}}\left(e^h-1\right) \\ \dfrac{e^T}{e^{3h}}\left(e^h-1\right) & h \end{pmatrix}.
\end{aligned}
$$

Now, we solve the following equation

$$
\Lambda z = v_d - Ae^{TA}x_0 ,
$$

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where

$$a = \frac{-2he^{6h}}{\widetilde{a}}, \qquad b = \frac{-2e^{T}e^{3h} + 2e^{T}e^{4h}}{\widetilde{b}},$$

$$c = \frac{-2e^{T}e^{3h} + 2e^{T}e^{h}e^{3h}}{\widetilde{c}}, \qquad d = \frac{2e^{T}e^{3h} - 2e^{T}e^{h}e^{3h}}{2e^{T} - 2e^{T}e^{h}} \times \frac{k}{\widetilde{d}}$$

with

$$\begin{aligned}\widetilde{a} = {} & 2e^{2T}e^{2h} - 4e^{2T}e^{h} + 2e^{2T} + he^{2T}e^{-2h}e^{2(3h)} \\ & + 9he^{2T}e^{2(3h)}e^{-4h} + he^{2T}e^{2(3h)}e^{-6h} \\ & - he^{2T}e^{-2h}e^{2h}e^{2(3h)} - 9he^{2T}e^{2h}e^{2(3h)}e^{-4h} \\ & - he^{2T}e^{2h}e^{2(3h)}e^{-6h},\end{aligned}$$

$$\begin{aligned}\widetilde{b} = {} & 2e^{2T}e^{2h} - 4e^{2T}e^{h} + 2e^{2T} \\ & + he^{2T}e^{-2h}e^{2(3h)} + 9he^{2T}e^{2(3h)}e^{-4h} \\ & + he^{2T}e^{2(3h)}e^{-6h} - he^{2T}e^{-2h}e^{2h}e^{2(3h)} \\ & - 9he^{2T}e^{2h}e^{2(3h)}e^{-4h} - he^{2T}e^{2h}e^{2(3h)}e^{-6h},\end{aligned}$$

$$\begin{aligned}\widetilde{c} = {} & 2e^{2T}e^{2h} - 4e^{2T}e^{h} + 2e^{2T} \\ & + he^{2T}e^{-2h}e^{2(3h)} + 9he^{2T}e^{2(3h)}e^{-4h} \\ & + he^{2T}e^{2(3h)}e^{-6h} - he^{2T}e^{-2h}e^{2h}e^{2(3h)} \\ & - 9he^{2T}e^{2h}e^{2(3h)}e^{-4h} - he^{2T}e^{2h}e^{2(3h)}e^{-6h},\end{aligned}$$

$$\begin{aligned}\widetilde{d} = {} & 2e^{2T}e^{2h} - 4e^{2T}e^{h} + 2e^{2T} \\ & + he^{2T}e^{-2h}e^{2(3h)} + 9he^{2T}e^{2(3h)}e^{-4h} \\ & + he^{2T}e^{2(3h)}e^{-6h} - he^{2T}e^{-2h}e^{2h}e^{2(3h)} \\ & - 9he^{2T}e^{2h}e^{2(3h)}e^{-4h} - he^{2T}e^{2h}e^{2(3h)}e^{-6h},\end{aligned}$$

$$\begin{aligned}k = {} & e^{2T}e^{-2h}e^{3h} + 9e^{2T}e^{3h}e^{-4h} \\ & + e^{2T}e^{3h}e^{-6h} - e^{2T}e^{-2h}e^{2h}e^{3h} \\ & - 9e^{2T}e^{2h}e^{3h}e^{-4h} - e^{2T}e^{2h}e^{3h}e^{-6h}.\end{aligned}$$

The control u permitting the transfere from $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to $v_d = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$ is given by

$$u(t) = (H^* z)(t) = \sum_{i=0}^{N} B_i^\top e^{(T-t)A^\top} z \chi_{[t_i, t_{i+1}[}(t).$$

Example 2 In this example we take $n = 2, m = 2, N = 2$ and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We obtain that

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix},$$

since rank$L = 1$, then the system is not speed-controllable .

5. The regulation problem

In this section, we deal with the case where the system is not speed-controllable on $[0, T]$. We are concerned with determining the control minimizing the quadratic functional

$$\begin{aligned} \mathcal{J}(u) = {} & \langle \dot{x}(T) - v_d, M(\dot{x}(T) - v_d) \rangle_{\mathbb{R}^n} \\ & + \int_0^T \langle \dot{x}(r) - \rho(r), G(\dot{x}(r) - \rho(r)) \rangle_{\mathbb{R}^n} \mathrm{d}r \\ & + \int_0^T \langle u(r), Ru(r) \rangle_{\mathbb{R}^m} \mathrm{d}r. \end{aligned} \tag{27}$$

with M and G two positive symmetric matrices, and R is a positive definite symmetric matrix.

Proposition 6 *The speed at time t can be written as*

$$\dot{x}(t) = Ae^{tA} x_0 + (\mathcal{K}u)(t), \tag{28}$$

where

$$\begin{aligned} \mathcal{K}\colon L^2(0, T, \mathbb{R}^m) & \longrightarrow L^2(0, T, \mathbb{R}^n), \\ u & \longmapsto \mathcal{K}u = \mathcal{K}_1 u + \mathcal{K}_2 u \end{aligned} \tag{29}$$

with

$$(\mathcal{K}_1 u)(t) = A\int_0^t e^{(t-s)A}\left(\int_0^s B(\theta)u(\theta)\,\mathrm{d}\theta\right)\mathrm{d}s \tag{30}$$

and

$$(\mathcal{K}_2 u)(t) = \int_0^t B(\theta)u(\theta)\,\mathrm{d}\theta. \tag{31}$$

Proposition 7 *The adjoint of the operator $\mathcal{K}$ is given by*

$$\begin{aligned} \mathcal{K}^* : L^2(0,T,\mathbb{R}^n) &\longrightarrow L^2(0,T,\mathbb{R}^m) \\ y &\longmapsto \mathcal{K}^* y \end{aligned} \tag{32}$$

with

$$(\mathcal{K}^* y)(\theta) = B^\top(\theta)\int_\theta^T e^{(t-\theta)A^\top} y(t)\,\mathrm{d}t. \tag{33}$$

Proof. We have

$$\begin{aligned} \langle \mathcal{K}_2 u, y\rangle_{L^2(0,T,\mathbb{R}^n)} &= \int_0^T \langle (\mathcal{K}_2 u)(t), y(t)\rangle\,\mathrm{d}t \\ &= \int_0^T \left\langle \int_0^t B(\theta)u(\theta)d\theta, y(t)\right\rangle \mathrm{d}t = \int_0^T \left(\int_0^t \langle B(\theta)u(\theta), y(t)\rangle\,\mathrm{d}\theta\right)\mathrm{d}t \\ &= \int_0^T \left(\int_\theta^T \langle B(\theta)u(\theta), y(t)\rangle\,\mathrm{d}t\right)\mathrm{d}\theta = \int_0^T \left\langle u(\theta), \int_\theta^T B^\top(\theta)y(t)dt\right\rangle \mathrm{d}\theta. \end{aligned}$$

Thus

$$(\mathcal{K}_2^* y)(\theta) = \int_\theta^T B^\top(\theta)y(t)\,\mathrm{d}t. \tag{34}$$

Hence

$$(\mathcal{K}_2^* y)(.) = B^\top(.)\int_.^T y(t)\,\mathrm{d}t. \tag{35}$$

On the other hand, we have

$$\begin{aligned}\langle \mathcal{K}_1 u, y\rangle_{L^2(0,T,\mathbb{R}^n)} &= \int_0^T \left\langle A \int_0^t e^{(t-s)A} \left[\int_0^s B(\theta)u(\theta)\,\mathrm{d}\theta \right] \mathrm{d}s, y(t) \right\rangle \mathrm{d}t \\ &= \int_0^T \left\langle \int_0^t Ae^{(t-s)A} \left(\int_0^s B(\theta)u(\theta)\,\mathrm{d}\theta \right) \mathrm{d}s, y(t) \right\rangle \mathrm{d}t.\end{aligned}$$

According to Fubini's integration theorem, we obtain the following

$$\begin{aligned}\langle \mathcal{K}_1 u, y\rangle_{L^2(0,T,\mathbb{R}^n)} &= \int_0^T \left\langle \int_0^t \left(\int_\theta^t Ae^{(t-s)A} B(\theta)u(\theta)\,\mathrm{d}s \right) \mathrm{d}\theta, y(t) \right\rangle \mathrm{d}t \\ &= \int_0^T \left\langle \int_0^t \left[-e^{(t-s)A} \right]_\theta^t B(\theta)u(\theta)\,\mathrm{d}\theta, y(t) \right\rangle \mathrm{d}t \\ &= \int_0^T \left\langle \int_0^t \left(e^{(t-\theta)A} - I \right) B(\theta)u(\theta)\,\mathrm{d}\theta, y(t) \right\rangle \mathrm{d}t \\ &= \int_0^T \left(\int_0^t \left\langle \left(e^{(t-\theta)A} - I \right) B(\theta)u(\theta), y(t) \right\rangle \mathrm{d}\theta \right) \mathrm{d}t.\end{aligned}$$

By a second use of the Fubini integration theorem, we have

$$\begin{aligned}\langle \mathcal{K}_1 u, y\rangle &= \int_0^T \left(\int_\theta^T \left\langle \left(e^{(t-\theta)A} - I \right) B(\theta)u(\theta), y(t) \right\rangle \mathrm{d}t \right) \mathrm{d}\theta \\ &= \int_0^T \left(\int_\theta^T \left\langle \left(e^{(t-\theta)A} - I \right) B(\theta)u(\theta), y(t) \right\rangle \mathrm{d}t \right) \mathrm{d}\theta \\ &= \int_0^T \left(\int_\theta^T \left\langle u(\theta), B^\top(\theta) \left(e^{(t-\theta)A^\top} - I \right) y(t) \right\rangle \mathrm{d}t \right) \mathrm{d}\theta \\ &= \int_0^T \left\langle u(\theta), B^\top(\theta) \int_\theta^T \left(e^{(t-\theta)A^\top} - I \right) y(t)\,\mathrm{d}t \right\rangle \mathrm{d}\theta.\end{aligned}$$

Which results in

$$\left(\mathcal{K}_1^* y\right)(\theta) = B^\top(\theta) \int_\theta^T \left(e^{(t-\theta)A^\top} - I\right) y(t)\,\mathrm{d}t. \tag{36}$$

If we use Eq. (35) and Eq. (36), we conclude

$$\left(\mathcal{K}^* y\right)(\theta) = B^\top(\theta) \int_\theta^T e^{(t-\theta)A^\top} y(t) dt. \tag{37}$$

□

Proposition 8 *The functional $\mathcal{J}$ could be rewritten as*

$$\begin{aligned} \mathcal{J}(u) = &\left\langle Ae^{TA}x_0 - v_d, M\left(Ae^{TA}x_0 - v_d\right)\right\rangle_{\mathbb{R}^n} \\ &+ \left\langle Ae^{\cdot A}x_0 - \rho, G\left(Ae^{\cdot A}x_0 - \rho\right)\right\rangle_{L^2(0,T,\mathbb{R}^n)} + \bar{\mathcal{J}}(u), \end{aligned} \tag{38}$$

where

$$\begin{aligned} \bar{\mathcal{J}}(u) = &\left\langle u, \left(\mathcal{H}^* M \mathcal{H} + \mathcal{K}^* G \mathcal{H} + R\right) u\right\rangle_{L^2(0,T,\mathbb{R}^m)} \\ &+ 2\left\langle u, \mathcal{H}^* M\left(Ae^{TA}x_0 - v_d\right) + \mathcal{K}^* G\left(Ae^{\cdot A}x_0 - \rho\right)\right\rangle. \end{aligned} \tag{39}$$

Proof. We have

$$\begin{aligned} &\left\langle \dot{x}(T) - v_d, M\left(\dot{x}(T) - v_d\right)\right\rangle_{\mathbb{R}^n} \\ &\quad = \left\langle Ae^{TA}x_0 + \mathcal{H}u - v_d, M\left(Ae^{TA}x_0 + \mathcal{H}u - v_d\right)\right\rangle_{\mathbb{R}^n} \\ &\quad = \left\langle Ae^{TA}x_0 - v_d, M\left(Ae^{TA}x_0 - v_d\right)\right\rangle_{\mathbb{R}^n} \\ &\qquad + 2\left\langle \mathcal{H}u, M\left(Ae^{TA}x_0 - v_d\right)\right\rangle_{\mathbb{R}^n} + \left\langle u, \mathcal{H}^* M \mathcal{H} u\right\rangle_{L^2(0,T,\mathbb{R}^m)} \\ &\quad = \left\langle Ae^{TA}x_0 - v_d, M\left(Ae^{TA}x_0 - v_d\right)\right\rangle_{\mathbb{R}^n} \\ &\qquad + 2\left\langle u, \mathcal{H}^* M\left(Ae^{TA}x_0 - v_d\right)\right\rangle_{L^2(0,T,\mathbb{R}^m)} + \left\langle u, \mathcal{H}^* M \mathcal{H} u\right\rangle_{L^2(0,T,\mathbb{R}^m)} \end{aligned} \tag{40}$$

and

$$\begin{aligned}
\int_0^T \langle \dot{x}(r) - \rho(r), G(\dot{x}(r) - \rho(r)) \rangle \,\mathrm{d}r &= \langle \dot{x} - \rho, G(x - \rho) \rangle_{L^2(0,T,\mathbb{R}^n)} \\
&= \left\langle Ae^{\cdot A}x_0 - \rho, G\left(Ae^{\cdot A}x_0 - \rho\right)\right\rangle_{L^2(0,T,\mathbb{R}^n)} \\
&\quad + 2\left\langle \mathcal{K}u, G\left(Ae^{\cdot A}x_0 - \rho\right)\right\rangle_{L^2(0,T,\mathbb{R}^n)} \\
&\quad + \langle u, \mathcal{K}^* G \mathcal{K} u \rangle_{L^2(0,T,\mathbb{R}^m)} \\
&= \left\langle Ae^{\cdot A}x_0 - \rho, G\left(Ae^{\cdot A}x_0 - \rho\right)\right\rangle_{L^2(0,T,\mathbb{R}^n)} \\
&\quad + 2\left\langle u, \mathcal{K}^* G\left(Ae^{\cdot A}x_0 - \rho\right)\right\rangle_{L^2(0,T,\mathbb{R}^m)} + \langle u, \mathcal{K}^* G \mathcal{K} u \rangle_{L^2(0,T,\mathbb{R}^m)}
\end{aligned} \tag{41}$$

if we combine Eq. (40) and Eq. (41) we obtain the desired result. □

Remark 3 *It is clear that*

$$\begin{gathered}
\textit{minimizing the function } \mathcal{J} \\
\Updownarrow \\
\textit{minimizing the function } \bar{\mathcal{J}}.
\end{gathered} \tag{42}$$

According to this remark, we are interested in the problem of minimization of the functional $\bar{\mathcal{J}}$.

Proposition 9 *The functional $\bar{\mathcal{J}}$ can be written as*

$$\bar{\mathcal{J}}(u) = \mathcal{B}(u,u) + 2\ell(u) \tag{43}$$

with $\mathcal{B}$ the continuous, symmetric and coercive bilinear form defined by

$$\begin{aligned}
\mathcal{B}\colon L^2(0,T,\mathbb{R}^m) \times L^2(0,T,\mathbb{R}^m) &\longrightarrow \mathbb{R} \\
(u,v) &\longmapsto \langle u, (\mathcal{H}^* M \mathcal{H} + \mathcal{K}^* G \mathcal{K} + R)\, v \rangle
\end{aligned} \tag{44}$$

and ℓ the continuous linear form, defined by

$$\begin{aligned}
\ell : L^2(0,T,\mathbb{R}^m) &\longrightarrow \mathbb{R} \\
u &\longmapsto \left\langle u, \mathcal{H}^* M\left(Ae^{TA}x_0 - v_d\right) + \mathcal{K}^* G\left(Ae^{\cdot A}x_0 - \rho\right)\right\rangle
\end{aligned} \tag{45}$$

Proposition 10 *There exists a unique $u^* \in L^2(0,T,\mathbb{R}^m)$ minimizing the function $\bar{\mathcal{J}}$ and moreover*

$$u^*(t) = -C^{-1}B^\top(t)p(t). \tag{46}$$

With

$$C = (\mathcal{H}^* M \mathcal{H} + \mathcal{K}^* G \mathcal{K} + R) \tag{47}$$

and

$$p(t) = e^{(T-t)A^\top} M \left(Ae^{TA}x_0 - v_d\right) + \int_t^T e^{(s-t)A^\top} G \left(Ae^{sA}x_0 - \rho(s)\right) \mathrm{d}s. \tag{48}$$

Proof. We use Eq. (43) and the fact that $\mathcal{B}$ is continuous and coercive bilinear form and ℓ is continuous linear form, so by the Lax-Milgram theorem [17] there exists a unique $u^* \in L^2(0,T,\mathbb{R}^m)$ such that

$$\mathcal{B}(v,u^*) = -\ell(v), \quad \forall v \in L^2(0,T,\mathbb{R}^m).$$

Which leads to

$$\begin{aligned}\left\langle (\mathcal{H}^* M \mathcal{H} + \mathcal{K}^* G \mathcal{K} + R)\, u^*, v\right\rangle = -\Big\langle \mathcal{H}^* M &\left(e^{TA}x_0 - v_d\right) \\ +\mathcal{K}^* G \left(Ae^{\cdot A}x_0 - \rho\right), v\Big\rangle \;\; &\forall v \in L^2\left(0,T,\mathbb{R}^m\right).\end{aligned}$$

Thus

$$(\mathcal{H}^* M \mathcal{H} + \mathcal{K}^* G \mathcal{K} + R)\, u^* = -\left(\mathcal{H}^* M \left(e^{TA}x_0 - v_d\right) + + \mathcal{K}^* G \left(Ae^{\cdot A}x_0 - \rho\right)\right)$$

therefore

$$\begin{aligned}u^*(.) &= -C^{-1}\left[\mathcal{H}^* M \left(e^{TA}x_0 - v_d\right) + \mathcal{K}^* G \left(Ae^{\cdot A}x_0 - \rho\right)\right] \\ &= -C^{-1}\left[B^\top(.)e^{(T-.)A^\top} M\left(Ae^{T}x_0 - v_d\right) + B^\top(.)\int_{.}^{T} e^{(s-.)A^\top} G\left(Ae^{sA}x_0 - \rho(s)\right)\mathrm{d}s\right] \\ &= -C^{-1}\left[B^\top(.)p(.)\right]\end{aligned}$$

with $p(.)$ defined as follows

$$p(t) = e^{(T-t)A^\top} M \left(Ae^{TA}x_0 - v_d\right) + \int_t^T e^{(s-t)A^\top} G \left(Ae^{sA}x_0 - \rho(s)\right) \mathrm{d}s$$

and the operator C is given by

$$C = \mathcal{H}^* M \mathcal{H} + \mathcal{K}^* G \mathcal{K} + R.$$

Moreover, since the bilinear form B is symmetric, then u^* is the minumum of the function $\bar{\mathcal{J}}$ [17]. □

Proposition 11 *The function $p(.)$ is the solution to the following equation*

$$\begin{cases} -\dot{p}(t) = A^\top p(t) + G\left(Ae^{tA}x_0 - \rho(t)\right), & t \in [0, T[, \\ p(T) = M\left(Ae^{TA}x_0 - v_d\right). \end{cases} \tag{49}$$

Proof. We take

$$d(t) = p(T-t), \quad t \in [0, T]$$

then

$$d(0) = p(T) = M\left(Ae^{TA}x_0 - v_d\right)$$

and

$$\begin{aligned} \dot{d}(t) &= -\dot{p}(T-t) \\ &= A^\top p(T-t) + G\left(Ae^{(T-t)A}x_0 - \rho(T-t)\right) \\ &= A^\top d(t) + G\left(Ae^{(T-t)A}x_0 - \rho(T-t)\right) \end{aligned}$$

this leads to

$$d(t) = e^{tA^\top} d(0) + \int_0^t e^{(t-s)A^T} G\left(Ae^{(T-s)A}x_0 - \rho(T-s)\right) \mathrm{d}s$$

thus

$$p(T-t) = e^{tA^\top} M\left(Ae^\top x_0 - v_d\right) + \int_0^t e^{(t-s)A^T} G\left(Ae^{(T-s)A}x_0 - \rho(T-s)\right) \mathrm{d}s.$$

We put $\theta = T - t$, we obtain that

$$\begin{aligned} p(\theta) &= e^{(T-\theta)A^\top} M\left(Ae^{TA}x_0 - v_d\right) + \int_0^{T-\theta} e^{(T-\theta-s)A^T} G\left(Ae^{(T-s)A}x_0 - \rho(T-s)\right) \mathrm{d}s \\ &= e^{(T-\theta)A^\top} M\left(Ae^{TA}x_0 - v_d\right) + \int_T^{\theta} e^{(r-\theta)A^T} G\left(Ae^{rA}x_0 - \rho(r)\right) (-\mathrm{d}r) \\ &= e^{(T-\theta)A^T} M\left(Ae^{TA}x_0 - v_d\right) + \int_\theta^{T} e^{(r-\theta)A^T} G\left(Ae^{rA}x_0 - \rho(r)\right) \mathrm{d}r \end{aligned}$$

and therefore

$$p(t) = e^{(T-t)A^\top} M \left(Ae^{TA}x_0 - v_d \right) + \int_t^T e^{(r-t)A^\top} G \left(Ae^{rA}x_0 - \rho(r) \right) \mathrm{d}r.$$

□

Remark 4 *The optimum of the function $\mathcal{J}$ is*

$$u^*(t) = -C^{-1}B^\top(t)d(T-t), \tag{50}$$

where $d(.)$ is the solution of the adjoint equation

$$\begin{cases} \dot{d}(t) = A^\top d(t) + G\left(Ae^{(T-t)A}x_0 - \rho(T-t)\right); \quad t \in [0, T[, \\ d(0) = M\left(Ae^{TA}x_0 - v_d\right). \end{cases} \tag{51}$$

6. Numerical approximation

We discretize the interval $[0, T]$ at points $t_i = i\Delta$ $(i = 0, 1, ..., N)$, where Δ is the time step such that $t_N = T$. Next, we denote $d(t_i) = d_i$, then according to Euler's schema

$$\begin{cases} d_0 = M\left(Ae^{TA}x_0 - v_d\right) \\ d_{i+1} = \Delta\left[A^\top d_i + G\left(Ae^{(T-t_i)}x_0 - \rho(T-t_i)\right)\right] + d_i \\ i = 0, 1, \ldots, N-1. \end{cases}$$

$$\begin{aligned} u_i^* &= -C^{-1}B^\top(t_i)\, d\,(T-t_i) \\ &= -C^{-1}B^\top(t_i) d\,((t_0+N\Delta)-(t_0+i\Delta)) \\ &= -C^{-1}B^\top(t_i)\, d\,((N-i)\Delta) \\ &= -C^{-1}B_i^\top d\,((N-i)\Delta) \\ &= -C^{-1}B_i^\top d_{N-i}, \end{aligned}$$

$i \in \{0, 1, ..., N-1\}$.

Example 3 First of all, let us determine the inverse of the following operator

$$C = \mathcal{H}^* M \mathcal{H} + \mathcal{K}^* G \mathcal{K} + R$$

we have

$$C = R\left(I + R^{-1}\left(\mathcal{H}^*M\mathcal{H} + \mathcal{K}^*G\mathcal{K}\right)\right).$$

If we assume that $\left\|R^{-1}\left(\mathcal{H}^*M\mathcal{H} + K^*GK\right)\right\| \leqslant 1$ then $I+R^{-1}\left(\mathcal{H}^*M\mathcal{H} + K^*GK\right)$ is invertible and the following result holds

$$\left(I + R^{-1}\left(\mathcal{H}^*M\mathcal{H} + \mathcal{K}^*G\mathcal{K}\right)\right)^{-1} = \sum_{k=0}^{+\infty} P^k$$

with

$$P = R^{-1}\left(\mathcal{H}^*M\mathcal{H} + \mathcal{K}^*G\mathcal{K}\right)$$

and P is defined by

$$Py = R^{-1}\left(\mathcal{H}^*M\mathcal{H} + \mathcal{K}^*G\mathcal{K}\right)y, \quad y \in L^2(0, T, \mathbb{R}^m)$$

in this case we obtain

$$C^{-1} = (I + P)^{-1}R^{-1} = \sum_{k=0}^{+\infty} P^k R^{-1}.$$

We have

$$\left(\mathcal{H}^*M\mathcal{H} + \mathcal{K}^*G\mathcal{K}\right)y = \mathcal{H}^*\left(M\mathcal{H}y\right) + K^*\left(GKy\right)$$

or

$$\mathcal{H}^*\left(M\mathcal{H}y\right) = \sum_{k=0}^{N-1} B_k^\top e^{(T-.)A^\top} M\mathcal{H}y$$

on the other hand by using Fubini's theorem we obtain

$$\mathcal{H}u = \int_0^T e^{(T-s)A}B(s)u(s)\,\mathrm{d}s = \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} e^{(T-s)A}B_k u(s)\,\mathrm{d}s$$

this leads to

$$\mathcal{H}^*\left(M\mathcal{H}y\right)(\theta) = \sum_{k=0}^{N-1} B_k^\top e^{(T-\theta)A^\top} M \times \left(\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} e^{(T-s)A}B_k y(s)\,\mathrm{d}s\right) \quad \text{and}$$

$$\mathcal{K}^*(GKy) = B^\top(.) \int_.^T e^{(T-s)A^\top}(GKy)(s)\,\mathrm{d}s$$

or

$$
\begin{aligned}
(\mathcal{K}y)(t) &= \int_0^t Ae^{(t-s)A}\left(\int_0^s B(\theta)y(\theta)\,\mathrm{d}\theta\right)\mathrm{d}s + \int_0^t B(\theta)y(\theta)\,\mathrm{d}\theta \\
&= \int_0^t Ae^{(t-s)A}\left(\sum_{i=0}^{N-1} B_i \int_0^s 1_{[t_i;t_{i+1}[}(\theta)y(\theta)\,\mathrm{d}\theta\right)\mathrm{d}s \\
&\quad + \sum_{i=0}^{N-1} B_i \int_0^t 1_{[t_i;t_{i+1}[}(\theta)y(\theta)\,\mathrm{d}\theta.
\end{aligned}
$$

Now, we take the same parameters and matrices as defined in example 2. Then

$$
\begin{aligned}
(\mathcal{K}y)(t) &= \int_0^t Ae^{(t-s)A}\left(\sum_{i=0}^{1} B_i \int_0^s 1_{[t_i;t_{i+1}[}(\theta)y(\theta)\,\mathrm{d}\theta\right)\mathrm{d}s \\
&\quad + \sum_{i=0}^{1} B_i \int_0^t 1_{[t_i;t_{i+1}[}(\theta)y(\theta)\,\mathrm{d}\theta \\
&= \int_0^t \left(\int_0^s \begin{pmatrix} e^{t-s} \\ 0 \end{pmatrix} 1_{[t_0;t_1[}(\theta)y(\theta)\,\mathrm{d}\theta\right)\mathrm{d}s \\
&\quad + \int_0^t \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \int_0^s 1_{[t_1;t_2[}(\theta)y(\theta)\,\mathrm{d}\theta)\right)\mathrm{d}s \\
&\quad + B_0 \int_0^t 1_{[t_0;t_1[}(\theta)y(\theta)\,\mathrm{d}\theta + B_1 \int_0^t 1_{[t_1;t_2[}(\theta)y(\theta)\,\mathrm{d}\theta \\
&= \begin{pmatrix} \displaystyle\int_0^t \left(\int_0^s e^{t-s} 1_{[t_0;t_1[}(\theta)y(\theta)\,\mathrm{d}\theta\right)\mathrm{d}s \\ 0 \end{pmatrix} + \begin{pmatrix} \displaystyle\int_0^t 1_{[t_0;t_1[}(\theta)y(\theta)\,\mathrm{d}\theta \\ \displaystyle\int_0^t 1_{[t_1;t_2[}(\theta)y(\theta)\,\mathrm{d}\theta \end{pmatrix}.
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
(\mathcal{K}^{*}G\mathcal{K}y)\,(t) &= \sum_{i=0}^{1} B_i^{\top} 1_{[t_i,t_{i+1}[}(t) \int_t^T e^{(T-s)A^{\top}} G(Ky)(s)\,\mathrm{d}s \\
&= \left(B_0^{\top} 1_{[t_0,t_1[}(t) + B_1^{\top} 1_{[t_1,t_2[}(t)\right) \times \int_t^T \begin{pmatrix} e^{(T-s)} & 0 \\ 0 & 0 \end{pmatrix} G(Ky)(s)\,\mathrm{d}s \\
&= \left(1_{[t_0,t_1[}(t) \;\; 1_{[t_1,t_2[}(t)\right) \times \int_t^T \begin{pmatrix} e^{(T-s)} & 0 \\ 0 & 0 \end{pmatrix} G(Ky)(s)\,\mathrm{d}s \\
&= \int_t^T \left(1_{[t_0,t_1[}r(t) e^{(T-s)} \;\; 0\right) \times \begin{pmatrix} \displaystyle\int_0^s \left(\int_0^r e^{s-r} 1_{[t_0;t_1[}(\theta) y(\theta)\,\mathrm{d}\theta\right) \mathrm{d}r \\ \displaystyle + \int_0^s 1_{[t_0;t_1[}(\theta) y(\theta)\,\mathrm{d}\theta \\ \displaystyle\int_0^s 1_{[t_1;t_2[}(\theta) y(\theta)\,\mathrm{d}\theta \end{pmatrix} \mathrm{d}s \\
&= \int_t^T 1_{[t_0,t_1[}(t) e^{(T-s)} \begin{bmatrix} \displaystyle\int_0^s \left(\int_0^r e^{s-r} 1_{[t_0;t_1[}(\theta) y(\theta)\,\mathrm{d}\theta\right) \mathrm{d}r \\ \displaystyle + \int_0^s 1_{[t_0;t_1[}(\theta) y(\theta)\,\mathrm{d}\theta \end{bmatrix} \mathrm{d}s
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{H}^{*}\left(M\mathcal{H}y\right)(\theta) &= \sum_{k=0}^{N-1} B_k^{\top} e^{(T-\theta)A^{\top}} M \left(\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} e^{(T-s)A} B_k y(s)\,\mathrm{d}s\right) \\
&= B_0^{\top} e^{(T-\theta)A^{\top}} M \left(\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} e^{(T-s)A} B_k y(s)\,\mathrm{d}s\right) \\
&\quad + B_1^{\top} e^{(T-\theta)A^{\top}} M \left(\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} e^{(T-s)A} B_k y(s)\,\mathrm{d}s\right)
\end{aligned}
$$

[cont.]

$$= \left(e^{(T-\theta)} \;\; 0\right) \sum_{k=0}^{1} \int_{t_k}^{t_{k+1}} e^{(T-s)A} B_k y(s)\,\mathrm{d}s$$

$$+ \left(0 \;\; e^{(T-\theta)}\right) \left(\sum_{k=0}^{1} \int_{t_k}^{t_{k+1}} e^{(T-s)A} b B_k y(s)\,\mathrm{d}s \right)$$

$$= \sum_{k=0}^{1} \int_{t_k}^{t_{k+1}} \left(e^{(T-s)+(T-\theta)} \;\; 0\right) B_k y(s)\,\mathrm{d}s$$

$$= \int_{t_0}^{t_1} e^{(T-s)+(T-\theta)} y(s)\,\mathrm{d}s$$

$$= \int_{t_0}^{t_1} e^{(2T-(s+\theta)} y(s)\,\mathrm{d}s$$

therefore

$$\left(\mathcal{H}^* M \mathcal{H} + \mathcal{K}^* G \mathcal{K}\right)(y)(t)$$

$$= 1_{[t_0,t_1[}(t) \int_{t}^{T} e^{(T-s)} \left[\begin{array}{c} \int_0^s \left(\int_0^r e^{s-r} 1_{[t_0;t_1[}(\theta) y(\theta)\,\mathrm{d}\theta \right) \mathrm{d}r \\ + \int_0^s 1_{[t_0;t_1[}(\theta) y(\theta)\,\mathrm{d}\theta \end{array} \right] \mathrm{d}s$$

$$+ \int_{t_0}^{t_1} e^{2T-(s+\theta)} y(s)\,\mathrm{d}s$$

$$= 1_{[t_0,t_1[}(t) \int_{t}^{T} e^{T-s} \left(\int_0^s (Fy)(r)\,\mathrm{d}r + (Ly)(s) \right) \mathrm{d}s$$

$$+ \int_{t_0}^{t_1} e^{(2T-(s+\theta)} y(s)\,\mathrm{d}s$$

where

$$(Fy)(r) = \int_0^r e^{s-r} 1_{[t_0;t_1[}(\theta) y(\theta)\,\mathrm{d}\theta, \quad \forall r \in [0, s] \tag{52}$$

and

$$(Ly)(s) = \int_0^s 1_{[t_0;t_1[}(\theta)y(\theta)\,\mathrm{d}\theta, \quad \forall s \in [t,T] \tag{53}$$

Let $\mathcal{P} = \mathcal{K}^* G\mathcal{K} + \mathcal{H}^* M\mathcal{H}$ then

$$\mathcal{P}y(t) = 1_{[t_0,t_1[}(t) \int_t^T e^{T-s}\left(\int_0^s (Fy)(r)\,\mathrm{d}r + (Ly)(s)\right)\mathrm{d}s$$
$$+ \int_{t_0}^{t_1} e^{2T-(s+\theta)} y(s)\,\mathrm{d}s.$$

Remark 5 *If $t \notin [t_0,t_1]$ then we have*

$$\mathcal{P}(y)(t) = \int_{t_0}^{t_1} e^{(2T-(s+\theta)} y(s)\,\mathrm{d}s \tag{54}$$

In this example we select R such that

$$\|R^{-1}\| = \frac{1}{2\|\mathcal{P}\|}$$

then the operator C is invertible.

Thus for $y \in L^2(0,T,\mathbb{R}^m)$ we have

$$\begin{aligned} C^{-1}(y) &= (I+P)^{-1}\, R^{-1}(y) \\ &= \sum_{k=0}^{+\infty} P^k R^{-1}(y) \\ &= \sum_{k=0}^{+\infty} \left(R^{-1}\mathcal{P}\right)^k R^{-1}(y) \\ &= \sum_{k=0}^{+\infty} R^{-k-1}\mathcal{P}^k(y) \end{aligned}$$

since $R \in \mathbb{R}_+^*$. $C^{-1}(y)$ is defined by

$$C^{-1}(y)(t) = \sum_{k=0}^{+\infty} R^{-k-1}\mathcal{P}^k(y)(t).$$

Thus

$$u^*(t) = -C^{-1}\left(B^\top(t)p(t)\right) = \sum_{k=0}^{+\infty} R^{-k-1}\mathcal{P}^k\left(B^\top(t)p(t)\right).$$

It is easy to see that

$$\mathcal{P}(B^\top p)(t) = 1_{[t_0,t_1[}(t)\int_t^T e^{T-s}\left(\begin{array}{l}\int_0^s F\left(B^\top p\right)(r)\,\mathrm{d}r\\ +L\left(B^\top p\right)(s)\end{array}\right)\mathrm{d}s$$
$$+\, e^{2T-\theta}\int_{t_0}^{t_1} e^{-s}B^\top(s)p(s)\,\mathrm{d}s$$

where

$$B^\top(t)p(t) = \left(1 - 101e^{T-t}\right)1_{[t_0,t_1[}(t) \tag{55}$$

then

$$F\left(B^\top p\right)(r) = \begin{cases} \dfrac{1}{e^{2r}}\left(101e^Te^s + re^re^s - 101e^Te^re^s\right), & \text{if } r \leqslant t_1,\\ \dfrac{e^s}{e^{t_1}}e^{-r}\left(101e^T - 101e^Te^{t_1} + t_1e^{t_1}\right), & \text{if } r > t_1\end{cases}$$

and

$$L\left(B^\top p\right)(s) = \begin{cases} s + 101e^{T-s} - 101e^T, & s \leqslant t_1,\\ t_1 - 101e^T + 101e^{T-t_1}, & t_1 \leqslant s\end{cases}$$

and

$$\int_{t_0}^{t_1} e^{-s}B^\top(s)p(s)\,\mathrm{d}s = \frac{1}{2e^{2t_0}e^{2t_1}}\left(e^{t_0} - e^{t_1}\right)\left(101e^Te^{t_0} + 101e^Te^{t_1} - 2e^{t_0}e^{t_1}\right).$$

Now, we are in a position to present our conclusion.

7. Conclusion

In this article, we focus on the exact controllability of the speed of a linear system with localized parameters. Due to topological reasons, it turns out that the classical Kalman theory cannot be applied to systems governed by the standard equation:

$$\begin{cases}\dot{x}(t) = Ax(t) + Bu(t),\\ x(0) = x_0 \in \mathbb{R}^n.\end{cases}$$

To overcome this difficulty, we devote this paper to the controllability of the "speed" variable for a class of systems described by

$$\begin{cases} \dot{x}(t) = Ax(t) + \int\limits_0^t B(\theta)u(\theta)\,\mathrm{d}\theta, \quad 0 \leqslant t \leqslant T, \\ x(0) = x_0 \in \mathbb{R}^n. \end{cases}$$

We have established conditions to ensure the existence of the optimal control for transferring the speed to a desired value. In the event that these conditions are not satisfied, we have considered a relatively weak version of the problem, namely the optimization of a quadratic function whose objective is to minimize the separation between the final velocity of the system and the desired velocity at least cost.

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