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Practical Mittag-Leffler stability of quasi-one-sided Lipschitz fractional order systems

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This paper focuses on the global practical Mittag-Leffler feedback stabilization problem for a class of uncertain fractional-order systems. This class of systems is a larger class of nonlinearities than the Lipschitz ones. Based on the quasi-one-sided Lipschitz condition, firstly, we provide sufficient conditions for the practical observer design. Then, we exhibit that practical Mittag-Leffler stability of the closed loop system with a linear, state feedback is attained. Finally, a separation principle is established and we prove that the closed loop system is practical Mittag-Leffler stable.

Key words: fractional-order systems, Caputo derivative, quasi-one-sided Lipschitz condition, nonlinear systems, observer design, output feedback stabilization, separation principle

1. Introduction

Fractional calculus is a classic mathematical concept with a long history and is a generalization of ordinary and integral calculus with an arbitrary system. However, since there is no real background to the applications, the application of partial calculus has received little research attention for a long time. With the development of natural sciences and complex engineering applications, partial calculus and fractional differential equation theory and their applications began to attract increasing attention from physicists to engineers and became the focus of researchers' interests and its applications expanded, such as electrochemistry [15], electrode polarization [20], viscous damping [12, 19], viscoelastic regimes [3],

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electric fractal networks [6], and electromagnetic waves [11]. Moreover, it has been emphasized that fractional-order differential state equations can characterize physical systems in the real world. The in-depth study of these fractional order systems from different perspectives such as partial order control techniques is increasing significantly [9, 21, 25, 26].

The Lipschitzian nonlinear system is an important class of nonlinear systems and has drawn considerable attention in the past few decades. In fact, a major class of nonlinear systems do satisfy the Lipschitz condition either globally or locally. Moreover, incorporation of the Lipschitz condition into a linear matrix inequality offers a tractable formulation for an efficient solution of the observer design. Thus, many strategies concerning observer design have been developed for such systems [1, 22, 27]. In addition, the one-sided Lipschitz nonlinearity has been recently introduced by Hu [13, 14] to the nonlinear observer design framework. After Hu (2006), many researchers have tackled the one-sided Lipschitz class of systems. Abbazadeh and Marquez [2] have proposed a systematic approach to the design of one-sided Lipschitz observers. More recently, various schemes of observers have been proposed: in [17], an adaptive observer approach is given, in [4] an exponential observer is developed for stochastic one-sided Lipschitz systems and in [28] an unknown input observer is considered. Furthermore, a more relaxed condition, namely the quasi-one-sided Lipschitz condition, was further proposed, soon afterwards, to replace the one-sided Lipschitz condition for observer design. Due to the fact that it involves much more useful information of the nonlinear part, the quasi-one-sided Lipschitz condition is shown to be an extension of the one-sided Lipschitz condition and the Lipschitz condition and is less conservative than those two kinds of conditions.

In [16], the observer design problem for integer-order systems is investigated and the conditions that ensure the global Mittag-Leffler stability of the controlled system are introduced. When the nonlinear part is not perfectly known, as in [23], it is not possible to build an exact observer, which explains why these authors introduced the notion of a practical observer for the fractional order case. The problem of global practical Mittag-Leffler stabilization for a class of nonlinear fractional order systems is described in [23] based on the Lyapunov method and derived a linear matrix inequality.

Now, for nonlinear fractional-order systems with unknown disturbances, an interesting question naturally arises: is it possible to apply control to achieve global Mittag-Leffler stabilization when the uncertain is unknown but bounded? In our opinion, this matter still remainder unresolved. For such a problem, the main difficulty stems from the uncertainties which cannot be linearly parameterized. By combining the fractional calculus and techniques of control and using a Lyapunov function, this paper resolves the above mentioned problem and proposes a observer-based output feedback controller for a class of uncertain nonlinear fractional-order systems. More specifically, compared with the closely related works [16] and [23], the main contribution of the paper is the less conservative and more convenient sufficient condition which guarantees the practical Mittag-Leffler stability.

More precisely, the main result of this paper is to generalize the idea investigated in [23] for the purpose of establishing the design of observer. We introduce the notion of practical Mittag-Leffler stability of nonlinear fractional-order systems and so, the paper deals with a large class of nonlinear fractional-order systems whose the nonlinear function is not necessarily Lipschitz. The nonlinear parts satisfy the quasi-one-sided Lipschitz condition, which is less conservative than the one-sided Lipschitz condition, while the uncertain part is bounded. Using Lyapunov theory, we study the problem of designing an observer-based output feedback controller in order to practically Mittag-Leffler stabilize the closed-loop system. Uncertain delimited and sufficient conditions are given to insure the practical Mittag-Leffler stability of the proposed observer. Finally, a separation principle is established, so that we implement the control law with estimates states given by the practical observer and we prove that the closed loop system is practical Mittag-Leffler stable.

The paper is organized as follows: In Section 2, basic definitions are provided and the system description is given. The design of the proposed observer by constructing Lyapunov functions is presented in Section 3. Moreover, the required assumptions and the statement of the main results are provided. Section 5 illustrates the validity of our design method in the selected numerical example. Conclusions are drawn in Section 6.

2. System description and preliminary

In this section, some definitions and results related to the fractional calculus are presented. The literature contains different definitions of the fractional derivative [18, 24]. In this paper, the Caputo definition is adopted

Definition 1 [7] *Given an interval $[a, b]$ of $\mathbb{R}$, the Riemann-Liouville fractional integral of a function $x \in L^1([a, b])$ of order $\alpha > 0$ is defined by*

$$I_a^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau)\,\mathrm{d}\tau, \quad t \in [a, b],$$

where Γ is the Gamma function. For $\alpha = 0$, $I_a^0 := I$, the identity operator.

Definition 2 [7] *Given an interval $[a, b]$ of R, the Caputo fractional derivative of a function x of order $\alpha > 0$ is defined by*

$$^C D_a^\alpha x(t) = I_a^{m-\alpha} x^{(m)}(t), \quad t \in [a, b],$$

where $0 < m-1 < \alpha \leqslant m$. *When* $0 < \alpha < 1$, *then the Caputo fractional derivative of order* α *of an absolutely continuous function* x *on* $[a, b]$ *reduces to*

$$ {}^{C}D_{t_0}^{\alpha}x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} x'(\tau) d\tau, \quad t \in [a, b]. \tag{1}$$

The Mittag-Leffler function plays the role of exponential function in the fractional calculus and arises naturally in the expression of solution of fractional order differential equations.

Definition 3 *The Mittag-leffler function* $E_{\alpha}(z)$ *and the generalized Mittag-leffler function* $E_{\alpha,\beta}(z)$ *are defined as:*

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0.$$

For $\alpha = 1$, *we have the exponential series. Similarly,*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0.$$

Definition 4 *The system*

$$\begin{cases} {}^{C}D_{t_0}^{\alpha}x(t) = g(t, x), & t \geqslant t_0, \\ x(t_0) = x_0, \end{cases} \tag{2}$$

is said to be globally uniformly practically Mittag-Leffler stable if there exist positive scalars b, λ *and* r *such that the trajectory passing through any initial state* x_0 *at any initial time* t_0 *evaluated at time* t *satisfies:*

$$\|x(t)\| \leqslant \left[m(x_0) E_{\alpha}(-\lambda(t-t_0)^{\alpha}) \right]^{b} + r\,, \qquad \forall t \geqslant t_0, \tag{3}$$

with $m(0) = 0$, $m(x) \geqslant 0$ *and* m *is locally Lipschitz.*

Lemma 1 *[5] Suppose that:*

$${}^{C}D_{t_0}^{\alpha}m(t) \leqslant \lambda m(t) + d, \quad m(t_0) = m_0, \quad t \geqslant t_0 \geqslant 0,$$

where $\lambda, d \in \mathbb{R}$. *Then one has*

$$m(t) \leqslant m(t_0) E_{\alpha}(\lambda(t-t_0)^{\alpha}) + d(t-t_0)^{\alpha} E_{\alpha,\alpha+1}\Big(\lambda(t-t_0)^{\alpha}\Big), \ \forall t \geqslant t_0 \geqslant 0.$$

Moreover, if $\lambda < 0$, then

$$m(t) \leqslant m(t_0)E_\alpha\big(\lambda(t-t_0)^\alpha\big) + dM, \qquad \forall t \geqslant t_0 \geqslant 0,$$

where $M = \sup\limits_{s\geqslant 0}\big(s^\alpha E_{\alpha,\alpha+1}(\lambda s^\alpha)\big).$

Lemma 2 *[8] Let x be a vector of differentiable functions. Then, for any time instant $t \geqslant t_0$ and for any scalar $\alpha \in (0,1)$, the following relation holds:*

$$\frac{1}{2}{}^C D^\alpha_{t_0}(x^T(t)Px(t)) \leqslant x^T(t)P^C D^\alpha_{t_0}x(t), \tag{4}$$

where $P \in \mathbb{R}^{n\times n}$ is a constant square symmetric positive definite matrix.

Notation 1 *Throughout the paper, A^T means the transpose of A. $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximal and minimal real eigenvalues of a matrix A respectively. $P > 0$ means that the matrix P is symmetric positive definite matrix. I is an appropriately dimensioned identity matrix. $\langle .,.\rangle$ is the inner product in $\mathbb{R}^n$, i.e., given $x, y \in \mathbb{R}^n$, then $\langle x,y\rangle = x^T y$ and $\|\ \|$ being the Euclidean-norm in $\mathbb{R}^n$ defined by $\|x\| = \sqrt{\langle x,x\rangle}$.*

In this paper, we consider the following nonlinear fractional order system, for all $t \geqslant t_0$

$$\begin{cases} {}^C D^\alpha_{t_0}x(t) &= Ax(t) + Bu(t) + f(t,x(t)) + B\varepsilon(t), \\ y(t) &= Cx(t), \end{cases} \tag{5}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, $\varepsilon : \mathbb{R}_+ \to \mathbb{R}^p$ is an unknown disturbance, $f(t,x(t))$ represents the nonlinear dynamics associated with the state vector with $f(t,0) = 0$, and A, B and C correspond to the linear constant matrices of a system of appropriate dimensions.

Definition 5 *[13] The function $f(t,x)$ is quasi-one-sided Lipschitz in $\mathbb{R}^n$ with a one-sided Lipschitz constant matrix, that is,*

$$\langle Pf(t,x) - Pf(t,\hat{x}), x-\hat{x}\rangle \leqslant (x-\hat{x})^T N(x-\hat{x}), \forall x,\hat{x} \in \mathbb{R}^n, t \in \mathbb{R}, \tag{6}$$

where P is a symmetric positive-definite matrix and N is a real symmetric matrix.

Remark 1 *Note that when $N = \rho I_n$ with ρ is a constant, the condition (6) becomes the one-sided Lipschitz condition. It is then an extension of the one-sided Lipschitz. Moreover, it is clear that a Lipschitz function with Lipschitz constant L is also a quasi one-sided Lipschitz function for any matrix $P > 0$ with one-sided constant matrix $N = L\lambda_{max(P)}I_n$.*

Remark 2 *The matrix N in the quasi one-sided Lipschitz condition (6) is not necessary to be a positive or negative definite matrix. The inequality (6) only requires that the matrix N is symmetric.*

3. Separation principle

The following assumption is introduced to design the proposed observer:

Assumption 1 *The unknown disturbance ε is an essentially bounded function, i.e.*

$$\exists \delta_\varepsilon > 0 \textit{ such that } \|\varepsilon\| \triangleq \textit{ ess} \sup_{t \geqslant 0} \|\varepsilon(t)\| \leqslant \delta_\varepsilon. \tag{7}$$

Assumption 2 *The pairs (A, C) is observable.*

Assumption 3 *The pairs (A, B) is stabilizable.*

Remark 3 *Since (A, C) is observable, then there exists a gain matrix L such that for all positive definite symmetric matrix Q, there exists a positive definite symmetric matrix P which is solution of the Lyapunov equation*

$$(A - LC)^T P + P(A - LC) = -Q. \tag{8}$$

3.1. Observer design

In this paragraph, we study the designing of an observer in order to have the states of the uncertain nonlinear system. In practice, we cannot do the direct measurement of all states of the system. In what follows, we try to build an observer in order to ensure the stability of the system. We propose the following stat observer:

$$\begin{cases} {}^C D^\alpha_{t_0} \hat{x}(t) & = A\hat{x}(t) + Bu(t) + f(t, \hat{x}(t)) + L(y(t) - C\hat{x}(t)), \\ \hat{y}(t) & = C\hat{x}(t), \end{cases} \tag{9}$$

where $L = [l_1, \ldots, l_n]^T$ a gain matrix such that $A_L := A - LC$ is Hurwitz and the nonlinear function $f(t, x(t))$ satisfying the condition (6).

Theorem 1 *Consider the nonlinear system* (5) *with the quasi-one-sided Lipschitz condition* (6), *under Assumption 1 and Assumption 2. If*

$$\lambda_{\min}(Q) - 2\lambda_{\max}(N) - 1 > 0, \tag{10}$$

then the error dynamic $e = \hat{x} - x$ is globally uniformly practically Mittag-Leffler stable.

Proof. Let us consider the following Lyapunov function candidate:

$$V(e(t)) = e^T(t)Pe(t). \tag{11}$$

The dynamics of the observer error is expressed as follows:

$$ {}^{C}D^{\alpha}_{t_0} e(t) = (A - LC)e(t) + f(t,\hat{x}(t)) - f(t,x(t)) - B\varepsilon(t). \tag{12} $$

So

$$ \begin{aligned} {}^{C}D^{\alpha}_{t_0} V(e(t)) &\leqslant e^T(t)\big(A_L^T P + PA_L\big)e(t) + 2e^T(t)P\big(f(t,\hat{x}(t)) - f(t,x(t))\big) \\ &\quad - 2e^T(t)PB\varepsilon(t) \\ &\leqslant -e^T(t)Qe(t) + 2e^T(t)P\big(f(t,\hat{x}(t)) - f(t,x(t))\big) - 2e^T(t)PB\varepsilon(t). \end{aligned} $$

Then using (6), one can have

$$ \begin{aligned} {}^{C}D^{\alpha}_{t_0} V(e(t)) &\leqslant -e^T(t)Qe(t) + 2e^T(t)Ne(t) - 2e^T(t)PB\varepsilon(t) \\ &\leqslant -(\lambda_{\min}(Q) - 2\lambda_{\max}(N))\|e(t)\|^2 - 2e^T(t)PB\varepsilon(t) \\ &\leqslant -(\lambda_{\min}(Q) - 2\lambda_{\max}(N))\|e(t)\|^2 + 2\|P\|\|B\|\|\varepsilon(t)\|\|e(t)\| \\ &\leqslant -(\lambda_{\min}(Q) - 2\lambda_{\max}(N))\|e(t)\|^2 + 2\delta_\varepsilon\|P\|\|B\|\|e(t)\|. \end{aligned} \tag{13} $$

Let $\mu = \delta_\varepsilon\|P\|\|B\|$. Using the fact that

$$ 2\mu\|e(t)\| \leqslant \mu^2 + \|e(t)\|^2. \tag{14} $$

By Equation (13) and (14), we obtain,

$$ {}^{C}D^{\alpha}_{t_0} V(e(t)) \leqslant -\big(\lambda_{\min}(Q) - 2\lambda_{\max}(N) - 1\big)\|e(t)\|^2 + \mu^2. \tag{15} $$

Since P is symmetric positive definite then, for all $e \in \mathbb{R}^n$,

$$ \lambda_{\min}(P)\|e(t)\|^2 \leqslant V(e(t)) \leqslant \lambda_{\max}(P)\|e(t)\|^2. \tag{16} $$

Using (15) and (16), we can obtain,

$$ {}^{C}D^{\alpha}_{t_0} V(e(t)) \leqslant -\lambda V(e(t)) + r $$

with $\lambda = \dfrac{\lambda_{\min}(Q) - 2\lambda_{\max}(N) - 1}{\lambda_{\max}(P)}$ and $r = \mu^2$.

Then, it follows from Lemma 1 that

$$ V(e(t)) \leqslant V(e_0)E_\alpha\big(-\lambda(t - t_0)^\alpha\big) + rM, \qquad \forall t \geqslant t_0 \geqslant 0, \tag{17} $$

where $M = \sup\limits_{s\geqslant 0}(s^\alpha E_{\alpha,\alpha+1}(-\lambda s^\alpha))$.

(16) implies that on the one hand,

$$ \lambda_{\min}(P)\|e(t)\|^2 \leqslant V(e(t)) \tag{18} $$

and on the other hand,

$$V(e_0) \leqslant \lambda_{\max}(P)\|e_0\|^2. \tag{19}$$

So, from (17), (18) and (19), it follows that

$$\lambda_{\min}(P)\|e(t)\|^2 \leqslant \lambda_{\max}(P)\|e_0\|^2 E_\alpha(-\lambda(t-t_0)^\alpha) + rM, \qquad \forall t \geqslant t_0,$$

finally

$$\|e(t)\| \leqslant \left[\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\|e_0\|^2 E_\alpha(-\lambda(t-t_0)^\alpha)\right]^{\frac{1}{2}} + \mu\sqrt{\frac{M}{\lambda_{\min}(P)}}, \qquad \forall t \geqslant t_0.$$

The last inequality is in the form of (3), so the error dynamic (12) is globally uniformly practically Mittag-Leffler stable. □

3.2. Global stabilization by state feedback

In this subsection, we establish a condition for the globally uniformly practically Mittag-Leffler stability of the nonlinear system (5). The state feedback controller is given by

$$u = Kx, \tag{20}$$

where $K = [k_1, \ldots, k_n]$ such that $A_K := A + BK$ is Hurwitz. Let S be the symmetric positive definite solution of the Lyapunov equation:

$$A_K^T S + S A_K = -Q_1, \tag{21}$$

for all positive definite symmetric matrix Q_1. We suppose that the nonlinear function $f(t, x(t))$ satisfies the following condition:

$$\langle Sf(t,x) - Sf(t,\hat{x}), x - \hat{x}\rangle \leqslant (x-\hat{x})^T N_1(x-\hat{x}), \forall x, \hat{x} \in \mathbb{R}^n, t \in \mathbb{R}, \tag{22}$$

where N_1 is a real symmetric matrix.

Theorem 2 *Consider the nonlinear system (5) with the quasi-one-sided Lipschitz condition (22), under Assumption 1 and Assumption 3. If*

$$\lambda_{\min}(Q_1) - 2\lambda_{\max}(N_1) - 1 > 0, \tag{23}$$

then the origin of the closed loop system (5) by the feedback (20) is globally uniformly practically Mittag-Leffler stable.

Proof. The closed loop system is given by

$${}^C D_{t_0}^\alpha x(t) = (A + BK)x(t) + f(t, x(t)) + B\varepsilon(t). \tag{24}$$

Let us choose a Lyapunov functional candidate as follows

$$W(x) = x^T Sx. \tag{25}$$

The α derivative of W along the trajectories of (24) is given by

$$\begin{aligned} {}^C D^\alpha_{t_0} W(x(t)) &\leqslant x^T(t)(A_K^T S + SA_K)x(t) + 2x^T(t)Sf(t,x(t)) + 2x^T(t)SB\varepsilon(t) \\ &\leqslant -x^T(t)Q_1 x(t) + 2x^T(t)Sf(t,x(t)) + 2x^T(t)SB\varepsilon(t), \end{aligned}$$

Since $f(t,0) = 0$, then using (22), one can have:

$$\begin{aligned} {}^C D^\alpha_{t_0} W(x(t)) &\leqslant -x^T(t)Q_1 x(t) + 2x^T(t)N_1 x(t) - 2x^T(t)SB\varepsilon(t) \\ &\leqslant -(\lambda_{\min}(Q_1) - 2\lambda_{\max}(N_1))\|x(t)\|^2 - 2x^T(t)SB\varepsilon(t) \\ &\leqslant -(\lambda_{\min}(Q_1) - 2\lambda_{\max}(N_1))\|x(t)\|^2 + 2\|S\|\|B\|\|\varepsilon(t)\|\|x(t)\| \\ &\leqslant -(\lambda_{\min}(Q_1) - 2\lambda_{\max}(N_1))\|x(t)\|^2 + 2\delta_\varepsilon\|S\|\|B\|\|x(t)\|. \end{aligned} \tag{26}$$

Let $\mu_1 = \delta_\varepsilon\|S\|\|B\|$. Using the fact that

$$2\mu_1\|x(t)\| \leqslant \mu_1^2 + \|x(t)\|^2. \tag{27}$$

By Equation (26) and (27), we obtain,

$${}^C D^\alpha_{t_0} W(x(t)) \leqslant -(\lambda_{\min}(Q_1) - 2\lambda_{\max}(N_1) - 1)\|x(t)\|^2 + \mu_1^2.$$

Since

$$\lambda_{\min}(S)\|x(t)\|^2 \leqslant W(x(t)) \leqslant \lambda_{\max}(S)\|x(t)\|^2.$$

As in the proof of Theorem 1, we have,

$${}^C D^\alpha_{t_0} W(x(t)) \leqslant -\lambda W(x(t)) + r,$$

with $\lambda_1 = \frac{\lambda_{\min}(Q_1)-2\lambda_{\max}(N_1)-1}{\lambda_{\max}(S)}$ and $r_1 = \mu_1^2$. By the same way we obtain:

$$\|x(t)\| \leqslant \left[\frac{\lambda_{\max}(S)}{\lambda_{\min}(S)}\|x(t_0)\|^2 E_\alpha(-\lambda(t-t_0)^\alpha)\right]^{\frac{1}{2}} + \mu_1\sqrt{\frac{M}{\lambda_{\min}(S)}}, \quad , \forall t \geqslant t_0.$$

So, the closed loop system (5) by the feedback (20) is globally uniformly practically Mittag-Leffler stable. □

Remark 4 *For the same class of systems* (5) *with* $\alpha = 1$, *the control problem of quasi-one-sided Lipschitz nonlinear systems are studied in [10]. However, those state feedback controllers cannot be immediately applied to systems considered in this paper, whereas the converse is true.*

4. Observer-based control stabilization

The design of the observer-based controller is established in the subsection. We implement the control law with estimate states. The observer-based controller is given by:

$$u = K\hat{x}, \tag{28}$$

where $\hat{x}$ is provided by the observer (9).

Theorem 3 *Suppose that conditions (6) and (22) and Assumption 1 to Assumption 3 are satisfied. Moreover the conditions (10) and (23) hold. Then the origin of the closed loop system (5) by the feedback (28) is globally uniformly practically Mittag-Leffler stable.*

Proof. The closed loop system in the $(e(t), x(t))$ coordinates can be represented by:

$$\begin{aligned} {}^{C}D^{\alpha}_{t_0}x(t) &= A_K x(t) + BKe(t) + f(t, x(t)) + B\varepsilon(t), \\ {}^{C}D^{\alpha}_{t_0}e(t) &= A_L e(t) + f(t, \hat{x}(t)) - f(t, x(t)) - B\varepsilon(t). \end{aligned} \tag{29}$$

Let

$$U(e, x) = \theta V(e) + W(x),$$

where V and W are given by (11) and (25) respectively and $\theta > 0$ is to be determined. Using the above results, we get

$$\begin{aligned} {}^{C}D^{\alpha}_{t_0}U(e(t), x(t)) \leqslant\ & -\theta(\lambda_{\min}(Q) - 2\lambda_{\max}(N) - 1)\|e(t)\|^2 + \theta\mu^2 \\ & -(\lambda_{\min}(Q_1) - 2\lambda_{\max}(N_1) - 1)\|x(t)\|^2 + \mu_1^2 \\ & +2||S||\ ||K||\ ||B||\ ||e(t)||\ ||x(t)||. \end{aligned}$$

Now using the fact that for all $\theta_1 > 0$, we have

$$2||x(t)||\ ||e(t)|| \leqslant \theta_1||x(t)||^2 + \frac{1}{\theta_1}||e(t)||^2,$$

we deduce that

$$\begin{aligned} {}^{C}D^{\alpha}_{t_0}U(e(t), x(t)) - \theta\mu^2 - \mu_1^2 \leqslant & -\theta\nu(Q, N)\|e(t)\|^2 - \nu(Q_1, N_1)||x(t)||^2 \\ & + \theta_1||S||||K||||B||||x(t)||^2 + \frac{1}{\theta_1}||S||||K||||B||||e(t)||^2, \end{aligned}$$

where

$$\begin{cases} \nu(Q, N) = \lambda_{\min}(Q) - 2\lambda_{\max}(N) - 1, \\ \nu(Q_1, N_1) = \lambda_{\min}(Q_1) - 2\lambda_{\max}(N_1) - 1. \end{cases}$$

Now, select $\theta_1 = \dfrac{\nu(Q_1,N_1)}{2||S||||K||||B||}$, we obtain

$$
\begin{aligned}
{}^{C}D^{\alpha}_{t_0}U(e(t),x(t)) &\leqslant -\left\{\theta\nu(Q,N) - \frac{2||S||^2||K||^2||B||^2}{\nu(Q_1,N_1)}\right\}\|e(t)\|^2 \\
&\quad - \frac{\nu(Q_1,N_1)}{2}||x(t)||^2 + \theta\mu^2 + \mu_1^2 \\
&\leqslant -\left\{\theta\nu(Q,N) - \frac{2||S||^2||K||^2||B||^2}{\nu(Q_1,N_1)}\right\}\|e(t)\|^2 + \theta\mu^2 + \mu_1^2.
\end{aligned}
$$

Finally we select θ such that

$$\theta\nu(Q,N) - \frac{2||S||^2||K||^2||B||^2}{\nu(Q_1,N_1)} > 0,$$

So, there exists $\lambda > 0$ such that

$${}^{C}D^{\alpha}_{t_0}U(e(t),x(t)) \leqslant -\lambda U(e(t),x(t)) + r_2,$$

with $r_2 = \theta\mu^2 + \mu_1^2$.

As in the proofs of Theorem 1 and Theorem 2, we deduce that the origin of the closed loop system (5) by the feedback (28) is globally uniformly practically Mittag-Leffler stable. □

Remark 5 *In [16,23] the authors consider a class of nonlinear systems by means of quasi-one-sided Lipschitz condition. This paper is the extension of that one proposed by [16, 23] to a class of uncertain systems. We derived a separation principle for the class of systems given by (5). Compare with [23], the system (5) is more general, and the sufficient conditions in Theorem 3 are less restrictive to ensure the global uniformly practically Mittag-Leffler stability.*

5. Numerical example

Let us consider the system (5), where

$$
A = \begin{bmatrix} 1.4 & -1 & 0 \\ 2.6 & 2.5 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}, \quad C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},
$$

$$
f(t,x) = \begin{bmatrix} 0 & 0 & -x_3^{1/3} \end{bmatrix}^{\mathrm{T}} \quad \text{and} \quad \varepsilon(t) = \begin{bmatrix} 1.2\sin 10t & 0 \end{bmatrix}^{T}.
$$

By a similar argument as that in Example 4.2 in [29], it is easy to check that $f(t,x)$ is not a Lipschitz function. Select

$$K = \begin{bmatrix} -42.44 & -13.1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 0 & 0 & -1 \\ 12.9 & -42.25 & 0 \end{bmatrix}^T,$$

A_L and A_K are Hurwitz. We also choose the matrices

$$Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The solutions of the Lyapunov equations (8) and (21) are given by

$$P = \begin{bmatrix} 0.5203 & 1.7284 & 0 \\ 1.7284 & 7.5328 & 0 \\ 0 & 0 & 0.5000 \end{bmatrix}, \qquad S = \begin{bmatrix} 0.1611 & -0.8532 & 0 \\ -0.8532 & 7.5328 & 0 \\ 0 & 0 & 1.000 \end{bmatrix}.$$

Let us now check the quasi-one-sided Lipschitz condition (6) and (22).

For any $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^{\mathrm{T}}$ and $\hat{x} = \begin{bmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \end{bmatrix}^{\mathrm{T}}$ with $x_3 \neq \hat{x}_3$, the mean-value theorem yields a nonzero. Let $\xi \in (\min\{x_3, \hat{x}_3\}, \max\{x_3, \hat{x}_3\})$, such that

$$\begin{aligned} \langle Pf(t,x) - Pf(t,\hat{x}), x - \hat{x} \rangle &= \frac{1}{2}\left[-x_3^{1/3} - \left(-\hat{x}_3^{1/3}\right)\right](x_3 - \hat{x}_3) \\ &= -\frac{1}{6}\xi^{-2/3}\left|x_3 - \hat{x}_3\right|^2 \leqslant 0 \end{aligned}$$

and

$$\begin{aligned} \langle Sf(t,x) - Sf(t,\hat{x}), x - \hat{x} \rangle &= \left[-x_3^{1/3} - \left(-\hat{x}_3^{1/3}\right)\right](x_3 - \hat{x}_3) \\ &= -\frac{1}{3}\xi^{-2/3}\left|x_3 - \hat{x}_3\right|^2 \leqslant 0. \end{aligned}$$

Then, $f(t,x)$ obeys the quasi-one-sided Lipschitz condition (6) and (22) by taking $N = 0$ and $N_1 = 0$.

We plot, in Figure 1, the simulation results of the output feedback control with the initial condition $x(0) = \begin{bmatrix} 0.5 & -0.6 & 1 \end{bmatrix}^{\mathrm{T}}$, $\hat{x}(0) = \begin{bmatrix} 0.7 & -0.4 & 1, 1 \end{bmatrix}^{\mathrm{T}}$ and $\alpha = 0.9$. From Figure 1, we can see that all of the real states, the estimate and the estimation errors are globally uniformly practically Mittag-Leffler stable.

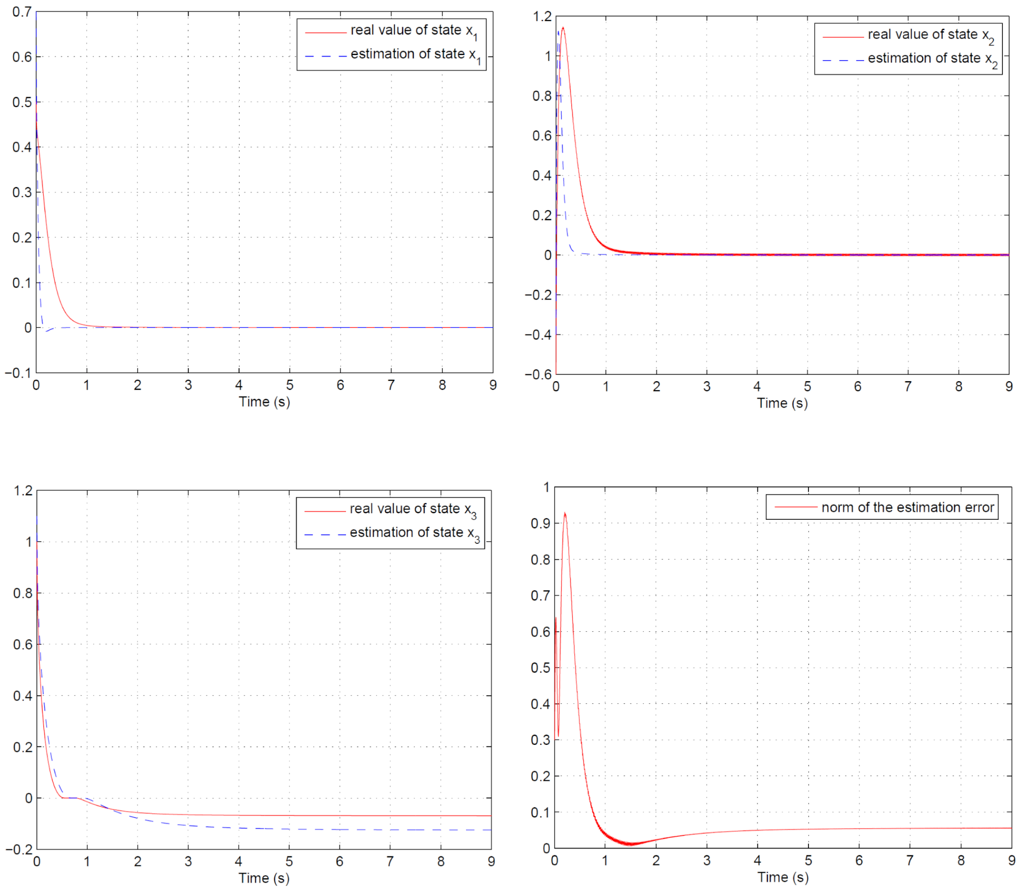

Figure 1: The simulation results via output feedback control

6. Conclusions

In this paper, we have proposed a separation principle for a class of fractional-order systems. The nonlinearities of this class of systems satisfy the quasi-one-sided Lipschitz condition while the uncertain term is bounded. Since the origin is not supposed to be an equilibrium point, we proposed a practical observer, a state feedback, and we proved that the observer based controller asserts practical Mittag-Leffler stability of the closed loop system. Nevertheless, conditions are less restrictive which makes the design process simple and tractable. Finally, as example, we applied the present results to a nonlinear fractional-order system with a disturbance.

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