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A REVIEW OF ENERGY-BASED MULTIAXIAL FATIGUE FAILURE CRITERIA

The paper contains a review of energy-based multiaxial fatigue failure criteria for cyclic and random loading. The criteria for cyclic loading have been divided into three groups, depending on the kind of strain energy density per cycle which is assumed as a damage parameter. They are: a) criteria based on elastic strain energy for high-cycle fatigue, b) criteria based on plastic strain energy for low-cycle fatigue, and c) criteria based on the sum of plastic and elastic strain energies for both low- and high-cycle fatigue. The criterion for random loading is based on the new definition of energy parameter which distinguishes plus and minus signs in history of specific work of stress on strain along chosen directions in the critical fracture plane. The criteria which take into account strain energy density in the critical plane dominate in the energy description of multiaxial fatigue. Parameters dependent on loading and factors dependent on a kind of material and influencing selection of the critical plane have been given. The author presented the mathematical models of the criteria and next distinguished those including influence of mean stresses and stress gradients as well as proportional and non-proportional loading. It has been emphasised that the generalized criterion of maximum shear and normal strain energy density in the critical plane seems to be the most efficient in practice and it should be developed and verified in a future.

1. Introduction

Multiaxial fatigue has been investigated for about 110 years and many different mathematical models of the limit state of strength have been formulated. At present we know more than 50 criteria of fatigue strength for multiaxial loading [1], [2], [3], [4], [5], [6], [7], [8], [9]. There are stress-, strain- and energy-based fatigue failure criteria but there is no one universal criterion

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for different loading conditions. Therefore, new mathematical models are sought. Recently, special attention is being paid to energy-based criteria [10-30]. These criteria can be divided into three groups, depending on the kind of strain energy density per cycle which is assumed as the damage parameter. They are: (i) criteria based on the elastic energy, (ii) criteria based on the plastic energy, and (iii) criteria based on the sum of elastic and plastic energies. In each group, we can distinguish criteria based on the critical fracture plane. A new definition of the energy parameter has been introduced and the mentioned criteria have been generalized to multiaxial random loading.

This paper contains a review of energy criteria of multiaxial fatigue and specification of a kind of energy assumed as the parameter controlling the fatigue process. The author used the previous review [24] and introduced the latest models of the criteria.

2. Criteria based on elastic strain energy

The first criteria were formulated by adaptation of the known static strength hypothesis proposed by Huber-Mises-Hencky and the Beltrami hypothesis to high-cycle fatigue (HCF) under proportional loading for ductile materials.

2.1. Criterion of volume and shear strain energy density (generalized Beltrami hypothesis)

This criterion is based on the assumption that fatigue fracture is influenced by total strain energy [31], [32], [33]

$$\phi = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}, \quad (i,j = x,y,z) \tag{1}$$

where σ_{ij} and ε_{ij} are the amplitudes of stress and strain state components, respectively. Thus, the amplitude of equivalent stress is

$$\sigma_{eq} = [(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})^2 + 2(1+\nu)(\sigma_{xy}^2+\sigma_{xz}^2+\sigma_{yz}^2-\sigma_{xx}\sigma_{yy}-\sigma_{xx}\sigma_{zz}-\sigma_{yy}\sigma_{zz})]^{1/2} \tag{2}$$

where ν is the Poisson's ratio.

2.2. Criterion of shear strain energy density (generalized Huber-Mises-Hencky hypothesis)

In this criterion it is assumed that not the total strain energy ϕ (equal to the sum of energies due to volume change, ϕ_v, and due to distortion change, ϕ_f), i.e.

$$\phi = \phi_v + \phi_f = \frac{1}{6}\sigma_{kk}\varepsilon_{kk} + \frac{1}{2}s_{ij}e_{ij}, \tag{3}$$

but only ϕ_f influences the fatigue fracture [31], [34], [35]. Here

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}, \tag{4}$$

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{ij}\delta_{ij}, \qquad (5)$$

are the deviatoric stress and strain tensors, respectively. In such a case, the equivalent stress is

$$\sigma_{eq} = [(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 - \sigma_{xx}\sigma_{yy} - \sigma_{xx}\sigma_{zz} - \sigma_{yy}\sigma_{zz} + 3(\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2)]^{1/2}. \qquad (6)$$

2.3. Criterion of shear strain energy density in the plane of maximum shear stress

According to this criterion, presented in [22] at the first time, the energy density, ϕ_{sf}, is equal to work of shear stress, $\tau_{\eta s}$, in direction $\bar{s}$ of maximum shear strain, $\varepsilon_{\eta s}$, on the plane with normal $\bar{\eta}$

$$\phi_{sf} = \frac{1}{2}\tau_{\eta s}\varepsilon_{\eta s} \qquad (7)$$

It has been proved in [24] that under proportional loading the shear strain energy density in the plane of maximum shear stress, ϕ_{sf}, when applied as a damage parameter leads to a mathematical form of the maximum shear stress criterion, i.e.

$$\sigma_{eq} = \sigma_1 - \sigma_3, \qquad (8)$$

where σ_1 and σ_3 are the maximum and minimum amplitudes of normal stresses, respectively.

2.4. Criterion of the average strain energy

Palin-Luc and Lasserre [36] proposed a criterion based on energy density averaged over a cyclic period T and within the material volume V* for the case where a stress gradient occurs. Taking the mean volumetric strain energy per cycle

$$\hat{W}_v = \frac{1}{T}\int_0^T \phi_v(t)dt, \qquad (9)$$

and the mean shear strain energy

$$\hat{W}_f = \frac{1}{T}\int_0^T \phi_f(t)dt, \qquad (10)$$

we have the average strain energy in a cycle

$$\hat{W} = \hat{W}_v + \hat{W}_f, \qquad (11)$$

and the energy degree of triaxiality given by

$$T_{ED} = \frac{\hat{W}_v}{\hat{W}}, \qquad (12)$$

F, the ratio of the average strain energy under multiaxial loading $\hat{W}$ to the average strain energy under pure torsion $\hat{W}_t$ is also defined

$$F = \frac{\hat{W}}{W_t} \tag{13}$$

Under an arbitrary but known distribution of energy $\hat{W}$, the strain energy average in a volume at the critical point of a material is determined from the following relation

$$\hat{\varpi} = \frac{1}{V^*} \iiint_{V^*} (\hat{W} - \hat{W}^*) dV^* \tag{14}$$

where

$$\hat{W}^* = \hat{W}_u^* \frac{F}{F_u} \tag{15}$$

$$\hat{W}_u^* = \frac{2\sigma_{af}^2 - \sigma_{afrb}^2}{4E} \tag{16}$$

The integration limit of V* represents the volume around the considered material point at which $\hat{W} \geq \hat{W}^*$. The ratio $F_u = F$ defined for uniaxial load, σ_{af} = fatigue limit under uniaxial load, σ_{afrb} = fatigue limit under rotary bending and E = Young's modulus.
The average strain energy at the fatigue limit

$$\hat{\varpi}_f = \hat{\varpi}_{fu} \frac{F}{F_u}, \tag{17}$$

where

$$\hat{\varpi}_{fu} = \frac{\sigma_{afrb}^2 - \sigma_{af}^2}{4E}. \tag{18}$$

Finally, the criterion proposed by Palin-Luc and Lasserre takes the following form

$$\hat{\varpi} = \hat{\varpi}_f. \tag{19}$$

3. Criteria based on plastic strain energy

These criteria describe fatigue strength using the plastic strain energy dissipated in the material during a loading cycle.

3.1. Criterion of effective strain energy

In this criterion, proposed by Lefebvre et al. [20], it is assumed that the energy density of the plastic effective strain in a loading cycle is the parameter influencing the initiation of a fatigue crack (detected as a small crack which is usually indicated by a distortion of the hysteresis loop) after N_f number of cycles that occurs under uniaxial and multiaxial stress states. It is given by

$$\Delta\bar{\sigma}\Delta\bar{\varepsilon}^p = KN_f^c \tag{20}$$

The effective stress range, $\Delta\bar{\sigma}$, and the range of the effective plastic strain, $\Delta\bar{\varepsilon}^{p}$, are calculated from

$$\Delta\bar{\sigma} = \left(\frac{3}{2}\Delta s_{ij}\Delta s_{ij}\right)^{1/2} \tag{21}$$

$$\Delta\bar{\varepsilon}^{p} = \left(\frac{2}{3}\Delta\varepsilon_{ij}^{p}\Delta\varepsilon_{ij}^{p}\right)^{1/2} \tag{22}$$

$$\Delta s_{ij} = \Delta\sigma_{ij} - \frac{1}{3}\Delta\sigma_{kk}\delta_{ij} \tag{23}$$

$$\Delta\varepsilon_{ij}^{p} = \Delta\varepsilon_{ij} - \Delta\varepsilon_{ij}^{e}, \tag{24}$$

where superscripts e and p mean elastic and plastic strain, respectively. The parameters K and c depend on mechanical properties of the material, and are functions of the stress ratio. Since the range of effective stress $\Delta\bar{\sigma}$ is proportional to the range of octahedral shear stress, $\Delta\tau_{oct}$, and the range of effective plastic strain, $\Delta\bar{\varepsilon}^{p}$, is proportional to the plastic range of the octahedral shear strain, $\Delta\varepsilon_{oct}^{p}$, hence Eq. (20) can be written as

$$\Delta\bar{\sigma}\Delta\bar{\varepsilon}^{p} = \frac{3}{\sqrt{2}}\Delta\tau_{oct}\sqrt{2}\Delta\varepsilon_{oct}^{p} = 3\Delta\tau_{oct}\Delta\varepsilon_{oct}^{p} = KN_{f}^{c}, \tag{25}$$

or

$$\Delta\tau_{oct}\Delta\varepsilon_{oct}^{p} = \frac{1}{3}KN_{f}^{c}, \tag{26}$$

From Eq. (26) it results that the criterion of plastic energy of the effective strain is equivalent to the statement that plastic shear strain energy in the critical plane (one of the octahedral planes) is a parameter influencing failure at the initiation stage under uniaxial and multiaxial proportional stress states [24].

3.2. Criterion of normal and shear strain energy

Garud [16] assumed that plastic strain energy (equal to the sum of energies from all the stress state components in a cycle) is the damage parameter influencing the initiation fatigue life, i.e. up to the moment when visible cracks can be observed. The incremental theory of plasticity was used to describe relations between cyclic stresses and strains under multiaxial non-proportional loading (with phase displacements). The theory includes the path of plastic strains. The Garud's criterion can be written as

$$\Delta W_{ij}^{p} = \int_{cycle}\sigma_{ij}d\varepsilon_{ij}^{p} = AN_{f}^{-\beta}, \tag{27}$$

where ΔW_{ij}^{p} is the sum of hysteresis loop areas from the nine stress state components, while A and β are the material constants.

Under combined tension and torsion Eq. (27) can be written as

$$\Delta W_{ij}^{p} = \Delta W_{xx}^{p} + 2\Delta W_{xy}^{p} = \int_{cycle} [\sigma_{xx} d\varepsilon_{xx}^{p} + 2\sigma_{xy} d\varepsilon_{xy}^{p}] = AN_f^{-\beta}. \quad (28)$$

From the experimental results it appears, however, that under combined tension and torsion Eq. (28) should be modified to the following form

$$\Delta W_{ij}^{p} = \Delta W_{xx}^{p} + H_G \Delta W_{xy}^{p} = \int_{cycle} [\sigma_{xx} d\varepsilon_{xx}^{p} + H_G \sigma_{xy} d\varepsilon_{xy}^{p}] = AN_f^{-\beta}, \quad (29)$$

where the weight coefficient H_G at plastic shear strain energy, ΔW_{xy}^{p}, is equal 2 [37], [38], 1 [16], [39], [40], 0.6 [10] or 0 [41], depending on the material type. When $H_G=1$, the plane x-y can be treated as the critical plane, and the sum $\Delta W_{xx}^{p} + \Delta W_{xy}^{p}$ is not the energy from all the stress state components in a cycle, but only the energy associated with the critical plane.

3.3. Criterion of normal and shear strain energy in the critical plane

Lately, a combined plastic energy density and critical plane concept has been proposed by Chen et al. [26] for low-cycle fatigue life under non-proportional loading. They consider different failure mechanisms for a shear-type failure and a tensile-type failure, and from which different damage parameters for the critical plane-strain energy density are proposed. The critical plane energy density criterion for a tensile-type failure is given as follows

$$\Delta\varepsilon_1^{max} \Delta\sigma_1 + \Delta\gamma_1 \Delta\tau_1 = 4\frac{(\sigma_f')^2}{E}(2N_f)^{2b} + 4\sigma_f'\varepsilon_f'(2N_f)^{b+c} \quad (30)$$

where $\Delta\varepsilon_1^{max}$ is the maximum normal strain range, $\Delta\sigma_1$, $\Delta\gamma_1$ and $\Delta\tau_1$ are the normal stress, shear strain and shear stress ranges that occur on the maximum normal strain plane, respectively, and b, c, $\sigma_f', \varepsilon_f', E$ are parameters of the Coffin-Manson axial fatigue relationship.

For a shear-type failure, the critical plane energy density criterion is

$$\Delta\gamma_{max} \Delta\tau + \Delta\varepsilon_n \Delta\sigma_n = 4\frac{(\tau_f')^2}{G}(2N_f)^{2b_t} + 4\tau_f'\gamma_f'(2N_f)^{b_t+c_t} \quad (31)$$

where $\Delta\gamma_{max}$ is the maximum shear strain range, $\Delta\tau$, $\Delta\varepsilon_n$ and $\Delta\sigma_n$ are the shear stress, normal strain and normal stress ranges on the maximum shear strain plane, respectively, and b_t, c_t, τ_f, γ_f, G are parameters of Coffin-Manson torsional fatigue relationship.

4. Criteria based on the sum of elastic and plastic strain energy

It is known that elastic strain energy alone cannot influence fatigue damage accumulation in the low cycles to failure range; also the plastic strain energy is minimal in high-cycle fatigue. Thus, a third group of criteria have been proposed in which authors take into account both forms of energy.

4.1. Criterion of the sum of elastic and plastic shear strain energy

This criterion was proposed by Ellyin [12] in 1974. He assumed that the cyclic shear strain energy, expressed as

$$W_f = \int_{cycle} s_{ij} de_{ij}\,, \tag{32}$$

was a fatigue damage parameter under proportional loading.
Since components of the strain state deviator e_{ij} in Eq.(32) are the sum of elastic and plastic strains, energy W_f is also the sum of elastic and plastic energies of the shear strains. Reference [12] contains calculations of the integral in Eq. (32) using the cyclic stress-strain curve and the deformation theory of plasticity. Hence

$$W_f = \xi \Delta\bar{\varepsilon}\Delta\bar{\sigma} + \Delta\bar{\varepsilon}\bar{\sigma}_m\,, \tag{33}$$

where ξ = coefficient dependent on the stress level when the material is in the non-elastic state, and $\bar{\sigma}_m = (\bar{\sigma}_U + \bar{\sigma}_L)/2$ is the equivalent mean stress calculated from the upper, $\bar{\sigma}_U$, and the bottom, $\bar{\sigma}_L$, limits of the effective stress (for symmetric uniaxial loading $\bar{\sigma}_m = \Delta\sigma/2$).
In [42] Ellyin and Kujawski developed the ideas contained in the criterion expressed in Eq.(32) and assumed the following form of energy

$$\Psi = \frac{\xi \Delta\bar{\varepsilon}\Delta\bar{\sigma}\left(1 + \alpha \frac{\sigma_m}{\bar{\sigma}}\right)}{f(\bar{\rho})} \tag{34}$$

where $\sigma_m = \sigma_{mkk}$ is the sum of mean values of normal stresses,
α = coefficient characterizing the material sensitivity to mean stresses σ_m,
$\bar{\sigma} = \frac{\Delta\bar{\sigma}}{2}$ is the amplitude of effective stress, and
$f(\bar{\rho})$ = function of the multiaxial constraints, where $\bar{\rho}$ is a measure of the multiaxial constraints as expressed by the ratio of the maximum normal and shear stresses and the effective Poisson ratio.
For experimental verification of Eq. (34) the following power function was assumed

$$f(\bar{\rho}) = \bar{\rho}^n \tag{35}$$

where $n \geq 0$ is a parameter (in some cases n = 0; 0.5; 1).
For the uniaxial loading state and for $\alpha = 0.5\,\xi$ and $f(\bar{\rho}) = 1$ then Eq. (34), becomes Eq. (33), i.e. $\Psi = W_f$.
As in the plastic effective strain energy criterion, Eq.(20), we may treat Eqs. (32) and (34) as energy (elastic and plastic) of the shear strain in the critical plane which is one of the octahedral planes.

4.2. Criterion of the sum of elastic and plastic strain energies

Leis [13] assumed that under multiaxial proportional fatigue loading and fatigue combined with creep, the total internal strain energy

$$U_T = \int_{cycle} \sigma_{ij} d\varepsilon_{ij} \tag{36}$$

is the parameter influencing failure up to the crack initiation stage.
The integral of Eq. (36) has been evaluated, and the following simple general form of damage parameter has been proposed

$$U_T = \bar{s}_m \Delta\bar{e}_t + \Delta\bar{s}\Delta\bar{e}_t \tag{37}$$

where $\bar{s}_m$ = equivalent mean stress, $\Delta\bar{e}_t$ = range of total equivalent strain and $\Delta\bar{s}$ = range of equivalent stress are defined as

$$\bar{s}_m = \left(\frac{3}{2}\sigma_{mij}\sigma_{mij}\right)^{1/2} \tag{38}$$

$$\Delta\bar{e}_t = \left(\frac{2}{3}\Delta\varepsilon_{ij}\Delta\varepsilon_{ij}\right)^{1/2} \tag{39}$$

$$\Delta\bar{s} = \left(\frac{3}{2}\Delta\sigma_{ij}\Delta\sigma_{ij}\right)^{1/2} \tag{40}$$

The energy term U_T consists of volume and shear strain energies, and contains elastic and plastic components. The criterion also includes components of mean stresses σ_{mij}.
Under uniaxial fatigue with the mean stress σ_m the Leis damage parameter agrees with the experimental damage parameter proposed by Smith-Watson-Topper [43], i.e.

$$U_T = \sigma_{max}\frac{\Delta\varepsilon}{2}. \tag{41}$$

4.3. Criterion of the sum of elastic strain energy under tension and plastic energy of effective strain

This criterion is a developed form of the Ellyin [12] approach. It was assumed by Ellyin and Gołoś [14], [15], [18] that the damage parameter is a special form of energy ΔW^t, equal to the sum of elastic strain energy in the interval of positive stresses in a cycle ΔW^{e+} and plastic strain energy ΔW^p which can be written as

$$\Delta W^t = \Delta W^{e+} + \Delta W^p \tag{42}$$

where

$$\Delta W^{e+} = \int_{cycle} \sigma_{ij} d\varepsilon_{ij}^{e} = \Delta W_D + \Delta W_V = \frac{1+\nu}{E}\sigma_{maxij}\varepsilon_{maxij} - \frac{\nu}{2E}\sigma_{maxkk}^2 = \frac{1+\nu}{3E}\bar{\sigma}_{max}^2 + \frac{1-2\nu}{6E}\sigma_{maxkk}^2 \tag{43}$$

$$\Delta W^{p} = \int_{cycle} \sigma_{ij} d\varepsilon_{ij}^{p} = \int_{cycle} s_{ij} d\varepsilon_{ij}^{p} = \frac{1-n'}{1+n'} \Delta\bar{\sigma}\Delta\bar{\varepsilon}^{p} . \tag{44}$$

Here $\sigma_{maxij} = \sigma_{aij} + \sigma_{mij}$,
n' = the cyclic strain hardening exponent, and
$\bar{\sigma}_{max}$ = the maximum value of the effective stress, calculated similarly to $\Delta\bar{\sigma}$ of Eq. (21).
For symmetric cycles, when stress mean values are equal to zero, energy $\Delta W^{e+} = \phi$, $\Delta W_D = \phi_f$ and $\Delta W_V = \phi_V$, according to Eq. (3).
The fatigue criterion formulated by Ellyin et al. is now being improved [19], [44], [45]; its one form is

$$\Delta W^{t} = \chi(\rho) N_f^{\alpha} + c(\rho) \tag{45}$$

where $\chi(\rho)$ and $c(\rho)$ are functions of the axial/tangential strain ratio ρ [19], [45], i.e.

$$\chi(\rho) = a\rho + b , \tag{46}$$

$$c(\rho) = e\rho + f . \tag{47}$$

where a,b,e and f are material constants.
In [19] we can find versions of Eqs. (46) and (47) which include non-zero mean values of strains, χ (ρ, ε_m) and c (ρ, ε_m). In [46], [47] Gołoś and Eshtewi have presented another version of functions χ (ρ, σ_m) and c (ρ, σ_m) where non-zero mean values of stresses are taken into account.

4.4. Criterion of the geometric mean of normal and shear strain energies

Itoh et al. [28] have proposed the energy parameter for correlating the non-proportional fatigue lives. The energy parameter is represented by

$$\bar{E} = \sqrt{(\Delta\sigma_{max} \cdot \Delta\varepsilon_{max})^2 + (\Delta\tau_{max} \cdot \Delta\gamma_{max})^2} . \tag{48}$$

In this equation, $\Delta\sigma_{max}$ and $\Delta\varepsilon_{max}$ are the maximum normal stress and strain ranges, and $\Delta\tau_{max}$ and $\Delta\gamma_{max}$ are the maximum shear stress and strain ranges.

4.5. Criterion of the effective total strain energy

Under cyclic loading, Park and Nelson [30] distinguish four components of strain energy density

$$W = \Delta W_p^d + \Delta W_e^d + \Delta W^h + W_m^h \tag{49}$$

where ΔW_p^d and ΔW_e^d are variable plastic and elastic deviatoric strain energies, respectively, and ΔW^h and ΔW_m^h are variable and static (or mean) volumetric strain energies, respectively.

The first term of Eq. (49) is defined as

$$\Delta W_p^d = \int_{cycle} s_{ij} de_{ij}^p, \tag{50}$$

and the second one as

$$\Delta W_e^d = \frac{\Delta s_{ij}}{2} \cdot \frac{\Delta e_{ij}^e}{2} = \frac{1+\nu}{4E} \Delta s_{ij} \Delta s_{ij}. \tag{51}$$

The volumetric strain energy ΔW^h is used to correct plastic energy term ΔW_p^d by means of a triaxial factor for stress state TF_s, and an effective plastic distortion strain energy parameter W_p^* is defined as

$$W_p^* = 2^{k_1(TF_s - 1)} \cdot \Delta W_p^d, \tag{52}$$

where the constant k_1 may be determined from two sets of test data with different stress states, for instance uniaxial and torsional data or uniaxial and equibiaxial data.

The triaxial factor, TF_s, characterizes different stress states by the following expression

$$TF_s = \frac{(\sigma_1 + \sigma_2 + \sigma_3)_a}{s_{eq}}, \tag{53}$$

where s_{eq} is the equivalent deviatoric stress amplitude defined as

$$s_{eq} = \left(\frac{3}{8} \Delta s_{ij} \Delta s_{ij} \right)^{1/2}, \tag{54}$$

and the subscript a refers to amplitude of stress.

To introduce mean stresses, the elastic energy term ΔW_e^p is modified by the static strain energy W_m^h, and an effective elastic distortion strain energy parameter W_e^* is defined as follows

$$W_e^* = 2^{k_2 TF_m} \cdot \Delta W_e^d, \tag{55}$$

where TF_m represents a triaxial factor for mean stresses and k_2 is a constant.

The triaxial factor, TF_m, is represented by the ratio of sum of mean principal stresses and the equivalent deviatoric stress amplitude s_{eq} from (54) as

$$TF_m = \frac{(\sigma_1 + \sigma_2 + \sigma_3)_m}{s_{eq}} \tag{56}$$

where the subscript m refers to mean values.

The constant, k_2 in Eq. (55) may be determined from fully-reversed ($TF_m = 0$) and zero-to-maximum ($TF_m = 1$) uniaxial fatigue data. Finally, the relation for the effective total strain energy parameter W_t^* vs fatigue life N_f can be written as

$$W_t^* = W_e^* + W_p^* = A^* N_f^{\alpha^*} + B^* N_f^{\beta^*} \tag{57}$$

where the constants A^*, B^*, α^* and β^* can be expressed as follows using uniaxial fatigue properties and cyclic stress-strain curve data (see Eq. (30))

$$A^{*} = \frac{2^{2b+1}(1+\nu)(\sigma_f')^2}{3E} \tag{58}$$

$$\alpha^{*} = 2b \tag{59}$$

$$B^{*} = 2^{b+c+2}\sigma_f'\varepsilon_f'\left(\frac{c-b}{c+b}\right) \tag{60}$$

$$\beta^{*} = b+c \tag{61}$$

4.6. Strain energy criteria in the critical plane

Under different combinations of proportional and non-proportional tension-compression Socie [48] described multiaxial fatigue, and he used the uniaxial Smith-Watson-Topper [43] model written in the form

$$\sigma_{max}\frac{\Delta\varepsilon}{2} = \frac{(\sigma_f')^2}{E}(2N_f)^{2b} + \sigma_f'\varepsilon_f'(2N_f)^{b+c} \tag{62}$$

This means that the energy parameter influences only the fatigue fracture plane, and that energy on other planes may be omitted. The Smith-Watson-Topper (SWT) parameter includes elastic and plastic energies of the normal strains on the critical plane which is the plane with the maximum range of normal strains; it also includes the mean stresses of a cycle, σ_m ($\sigma_{max} = \sigma_a + \sigma_m$). More detailed observations of crack initiation and growth under cyclic tension-compression and torsion [48], [49] give additional arguments for the importance of the SWT parameter in the description of multiaxial fatigue of materials which fail according to Mode I.

Nitta, Ogata and Kuwabara [23] used energy fatigue parameters connected with the fracture plane for approximating the results obtained from fatigue tests under proportional and non-proportional torsion with tension. The tests were carried out for seven ratios of the controlled strain ranges $\Delta\gamma/\Delta\varepsilon$ and five phase displacements between those strains.

Under proportional loading, for the strain ratio $\Delta\gamma / \Delta\varepsilon \leq 1.7$, the fracture planes agreed with the plane suffering the maximum range of normal strain $\Delta\varepsilon_1$ (Mode I), and the energy term

$$\Delta E_1 = \frac{1}{2}\Delta\sigma_1\Delta\varepsilon_1 = AN_f^{-\alpha} \tag{63}$$

was a suitable fatigue parameter. When the strain ratio was $\Delta\gamma / \Delta\varepsilon > 1.7$, the fracture planes agreed with the plane of maximum range of shear strain, $\Delta\gamma_{max}$ (Mode II), and energy in that plane was calculated from

$$\Delta E_2 = \Delta\tau_{max}\Delta\gamma_{max} = BN_f^{-\alpha} \tag{64}$$

Under non-proportional loading, the fatigue damage was influenced by both crack types. The following equation for calculating the number of cycles to fracture seemed to be useful

$$\frac{1}{N_{fou}} = \frac{1}{N_{f1}} + \frac{1}{N_{f2}} \tag{65}$$

where N_{f1}, N_{f2}, N_{fou} are the numbers of cycles to fracture for the first (Mode I – Eq. (63), second (Mode II – Eq. (64)) and mixed types of cracking, respectively. Under a γ and ε phase shift of $\pi/2$ the fracture planes were perpendicular to the specimen axis and the energy terms were calculated from

$$\Delta E_1 = \frac{1}{2}\Delta\sigma\Delta\varepsilon \tag{66}$$

$$\Delta E_2 = \Delta\tau\Delta\gamma \tag{67}$$

For the other phase shifts ($\pi/6$, $\pi/4$ and $\pi/3$) the fracture planes coincided with the planes of the maximum range of shear strain $\Delta\gamma_{max}$ and for this condition the energy was calculated as

$$\Delta E_1 = \frac{1}{2}\Delta\sigma_n\Delta\varepsilon_n \tag{68}$$

$$\Delta E_2 = \Delta\tau_{max}\Delta\gamma_{max} \tag{69}$$

where $\Delta\sigma_n$ and $\Delta\varepsilon_n$ are the ranges of normal stress and strain in the plane of $\Delta\gamma_{max}$, respectively.

If under uniaxial fatigue the equivalent energy is defined as product of normal stress and strain ranges

$$\Delta E_{eq} = \frac{1}{2}\Delta\sigma_{eq}\Delta\varepsilon_{eq} = AN_{fou}^{-\alpha} \tag{70}$$

then Eq. (65) can be transformed and we obtain

$$\Delta E_{eq} = \left[(\Delta E_1)^{1/\alpha} + \left(\frac{A}{B}\right)^{1/\alpha} (\Delta E_2)^{1/\alpha} \right]^{\alpha} \tag{71}$$

From Eq. (71) it appears that the equivalent energy (elastic and plastic) in the fracture plane is a nonlinear function of the energy associated with the normal and shear strains.

For multiaxial proportional loading, Hoffman and Seeger [49] suggest the application of the SWT parameter written as

$$P = \sqrt{E\sigma\varepsilon_{1a}} \tag{72}$$

where $\sigma = \sigma_a + \sigma_m$.

whereas fatigue damage of a material is better described by the maximum amplitude of strain ε_{1a}. The modified form of the parameter

$$P = \sqrt{G\tau\gamma_{max,a}} \tag{73}$$

should be used when fatigue damage of the material is better described by the maximum amplitude of shear strain $\gamma_{max,a}$. Thus, the authors suggest that only the energy (elastic and plastic) associated with the normal strain (Eq. (72)) in the plane of maximum normal strain, or only the shear strain energy (Eq. (73)) in the plane of maximum shear strain, should be assumed as the fatigue damage parameter.

Chu et al. [50] proposed another damage parameter, namely the specific work of shear and normal stresses in the critical plane

$$C_{SWT} = 2\tau_{max}\gamma_a + (\sigma_n)_{max}(\varepsilon_n)_a \tag{74}$$

This can be treated as a development of the SWT approach [43]. The factor of 2 was assumed in order to include the same participation of work under simple shearing and simple tension. The plane in which the parameter C_{SWT} reaches its maximum value, is the critical plane.

Liu [21] proposes the virtual strain energy (VSE) in the critical fracture plane as a parameter to describe multiaxial fatigue under proportional and nonproportional cyclic loading

$$\Delta W = \Delta W_n + \Delta W_s = \Delta\sigma_n\Delta\varepsilon_n + \frac{\Delta\tau_n\Delta\gamma_n}{2} \tag{75}$$

where:

$\Delta W_n = \Delta\sigma_n\Delta\varepsilon_n$ (normal strain energy in the critical plane) and,

$\Delta W_s = \frac{\Delta\tau_n\Delta\gamma_n}{2}$ (shear strain energy in the same critical plane).

Each of the above energies include elastic, ΔW_e and plastic, ΔW_p energies. The position of the critical fracture plane depends on the material, temperature, strain range and loading history. Under uniaxial loading the energy, term ΔW agrees with the energy formulated by the SWT parameter and can be expressed with the Manson-Coffin-Basquin characteristic of fatigue failure behaviour as follows

$$\Delta W = \Delta W_e + \Delta W_p = \Delta\sigma\Delta\varepsilon = 4\frac{(\sigma_f')^2}{E}(2N_f)^{2b} + 4\sigma_f'\varepsilon_f'(2N_f)^{b+c}. \tag{76}$$

Liu [21] considered thin-walled cylindrical specimens under a combination of proportional torsion and tension cracked according to Modes I and II. For Mode II, the crack driven along the surface of the specimen is defined as type A, and the crack acting through the thickness of the specimen is defined as type B. For materials cracking according Mode I we have

$$\Delta W = \Delta W_I = (\Delta W_n)_{max} + \Delta W_s = (\Delta\sigma_n\Delta\varepsilon_n)_{max} + \frac{\Delta\tau_n\Delta\gamma_n}{2}. \tag{77}$$

For Mode II and type A we obtain

$$\Delta W = \Delta W_{IIA} = \Delta W_n + (\Delta W_s)_{max} = \Delta\sigma_n\Delta\varepsilon_n + \left(\frac{\Delta\tau_n\Delta\gamma_n}{2}\right)_{max}, \tag{78}$$

and for Mode II and type B we have

$$\Delta W = \Delta W_{IIB} = \Delta W_n + (\Delta W_s)_{max} = \Delta\sigma_n\Delta\varepsilon_n + \left(\frac{\Delta\tau_n\Delta\gamma_n}{2}\right)_{max}. \tag{79}$$

If the critical plane of cracking is defined according to Mode I, then in Eq. (77) the maximum value of the normal strain energy $(\Delta W_n)_{max}$ should be assumed. In the case of Mode II, we assume the maximum values of shear strain energy $(\Delta W_s)_{max}$ in Eqs. (78) and (79). We should remember that in spite of the same symbols in Eqs. (78) and (79) the strain ranges $\Delta\varepsilon_n$, $\Delta\gamma_n$ and the stress ranges $\Delta\sigma_n$ and $\Delta\tau_n$ are determined in different ways. For calculation of strain and stress ranges we use ε_1, ε_3, σ_1, σ_3 in Eq. (78) and ε_1, ε_2, σ_1, σ_2 in Eq. (79), where $\varepsilon_1 \geq \varepsilon_2 \geq \varepsilon_3$ and $\sigma_1 \geq \sigma_2 \geq \sigma_3$. Maximization of one term of strain energy in Eqs. (77) to (79) allows for a precise selection of the critical fracture plane when directions of strains and stresses with maximum ranges do not coincide.

For uniaxial loading we have

$$\Delta W = \Delta W_I = \Delta W_n \tag{80}$$

For pure torsion of thin-walled cylindrical specimens we obtain

$$\Delta W = \Delta W_{IA} = \Delta W_{IIA} \tag{81}$$

Liu also proposes a way of determining the VSE parameters under non-proportional histories of strains (i.e. out-of-phase) and noted as $\Delta\hat{W}_I$ and $\Delta\hat{W}_{II}$.

Glinka et al. [17] used energy (elastic and plastic) associated with the normal and shear strains in the critical plane for description of multiaxial symmetric proportional loading. Their energy parameter

$$W^* = \frac{\Delta\sigma}{2}\cdot\frac{\Delta\varepsilon}{2} + \frac{\Delta\tau}{2}\cdot\frac{\Delta\gamma}{2} \tag{82}$$

is the sum of normal and shear strain energies in the critical plane, which is the plane of the maximum shear strain.

Assuming the notation used in [17], we can write the criterion presented as Eq.(82) as

$$\begin{aligned} W^* &= \frac{\Delta\sigma_{22}}{2}\cdot\frac{\Delta\varepsilon_{22}}{2} + \frac{\Delta\tau_{21}}{2}\cdot\frac{\Delta\gamma_{21}}{2} = \\ &= \frac{\Delta\sigma_{22}}{2}\cdot\frac{\Delta\varepsilon_{22}}{2} + 2\frac{\Delta\sigma_{21}}{2}\cdot\frac{\Delta\varepsilon_{21}}{2} = W_{22} + 2W_{21} \end{aligned} \tag{83}$$

Thus, shear strain energy

$$W_{21} = \frac{\Delta\sigma_{21}}{2}\cdot\frac{\Delta\varepsilon_{21}}{2} \tag{84}$$

in the critical plane 2–1 is summed with coefficient 2 (see Eq. (83)).

To consider the occurrence of both mean, normal and shear stresses in the cycles σ_{m22} and σ_{m21}, Glinka et al. [51], [52] have modified the Eq. (83) as follows

$$W^* = \frac{\Delta\sigma_{21}}{2}\frac{\Delta\gamma_{21}}{2}\left[\frac{1}{1-\frac{\sigma_{max21}}{\tau'_f}} + \frac{1}{1-\frac{\sigma_{max22}}{\sigma'_f}}\right] = 2W_{21}\left[\frac{1}{1-\frac{\sigma_{max21}}{\tau'_f}} + \frac{1}{1-\frac{\sigma_{max22}}{\sigma'_f}}\right] \tag{85}$$

The modification consists in resignation in damage parameter (85) from the normal strain energy density W_{22} in the critical plane (with the maximum amplitude of shear strain), whereas two material parameters σ'_f and τ_f' and the maximum normal stress $\sigma_{max\,22} = \sigma_{m22} + \Delta\sigma_{22}/2$ and shear stress $\sigma_{max\,21} = |\sigma_{m21} + \Delta\sigma_{21}/2|$ or $|\sigma_{m21} - \Delta\sigma_{21}/2|$ are introduced from which the larger σ_{max21} should be used.

Recently, Pan et al. [27], to improve correlation of some experimental data, have made next modification of Eq. (83) by including two weight constants $H_1 = \sigma'_f/\tau'_f$ and $H_2 = \gamma'_f/\varepsilon'_f$ for stress and strain amplitudes, respectively. The modified parameter is

$$W^* = \frac{\Delta\sigma_{21}}{2}\cdot\frac{\Delta\gamma_{21}}{2} + H_1H_2\frac{\Delta\sigma_{22}}{2}\cdot\frac{\Delta\varepsilon_{22}}{2} \tag{86}$$

To simulate the stress-strain relationship for multiaxial loading, Pan et al. used the endochronic constitutive equations.

Lately, Varvani-Farahani and Topper [29] have proposed the energy parameter in the critical plane for non-proportional and mean loading. This parameter versus the fatigue life $f(N_f)$ is defined as

$$\frac{1}{\sigma'_f\varepsilon'_f}\Delta\sigma_n\Delta\varepsilon_n + \frac{1+\dfrac{\sigma_n^m}{\sigma'_f}}{\tau'_f\gamma'_f}\cdot\Delta\tau_{max}\Delta\left(\frac{\gamma_{max}}{2}\right) = f(N_f) \tag{87}$$

This parameter contains the sum of the normal strain energy range, $\Delta\sigma_n\Delta\varepsilon_n$, and the shear strain energy range, $\Delta\tau_{max}\Delta(\gamma_{max}/2)$, calculated for the critical plane on which both strain and stress Mohr's circles are the largest while loading (at the angle θ_1) and unloading (at the angle θ_2) of a cycle. The normal and shear strain energies in this parameter have been weighted by the tensile $(\sigma'_f, \varepsilon'_f)$ and shear (τ'_f, γ'_f) fatigue properties, respectively.

The correction for mean loading is based on the mean normal stress applied to the critical plane.

$$\sigma_n^m = \frac{\sigma_n^{max} + \sigma_n^{min}}{2} \tag{88}$$

The range of the maximum shear strain and the corresponding normal strain range on the critical plane are calculated as

$$\Delta\left(\frac{\gamma_{max}}{2}\right) = \left(\frac{\varepsilon_1 - \varepsilon_3}{2}\right)_{\theta 1} + \left(\frac{\varepsilon_1 - \varepsilon_3}{2}\right)_{\theta 2} \tag{89}$$

$$\Delta\varepsilon_n = \left(\frac{\varepsilon_1 + \varepsilon_3}{2}\right)_{\theta 1} + \left(\frac{\varepsilon_1 + \varepsilon_3}{2}\right)_{\theta 2} \tag{90}$$

where $\varepsilon_1 \geq \varepsilon_2 \geq \varepsilon_3$ are the principal strains. Similarly, the range of the maximum shear stress and the corresponding normal stress range are calculated as

$$\Delta\tau_{max} = \left(\frac{\sigma_1 - \sigma_3}{2}\right)_{\theta 1} + \left(\frac{\sigma_1 - \sigma_3}{2}\right)_{\theta 2} \tag{91}$$

$$\Delta\sigma_n = \left(\frac{\sigma_1 + \sigma_3}{2}\right)_{\theta_1} + \left(\frac{\sigma_1 + \sigma_3}{2}\right)_{\theta_2} \tag{92}$$

where $\sigma_1 \geq \sigma_2 \geq \sigma_3$ are the principal stresses,

Some general form of energy based critical plane damage parameter for both proportional and non-proportional multiaxial cyclic loading with mean values has been also proposed by Rolovic and Tipton [25]. Here, failure is defined as the development of an engineering size crack (approximately 1 mm in surface length) after N_f cycles. Their criterion can be expressed in a general form as

$$[\tau_a + f_1(\sigma_n)]\gamma_a + [\sigma_{n,a} + f_2(\sigma_n)]\varepsilon_{n,a} = f_3(N_f) \tag{93}$$

where τ_a and γ_a are the maximum shear stress and strain amplitudes on the critical plane, respectively, $\sigma_{n,a}$ and $\varepsilon_{n,a}$ are the maximum normal stress and strain amplitudes on the critical plane, respectively and σ_n is the normal (static and cyclic) stress on the critical plane.

The first term of the left side of Eq. (93) represents Mode II crack loading modified by a function $f_1(\sigma_n)$ to account for the crack closure effects. The second term of the left side of Eq. (93) accounts for Mode I crack loading modified by another function $f_2(\sigma_n)$. The right side of Eq. (93) is a function $f_3(N_f)$ of the uniaxial strain life relation.

The critical plane can be determined on the basis of the maximum damage parameter or the observed material cracking behaviour.

For experimental verification, the following specific form of general criterion has been used

$$(\tau_a + 0.3\sigma_n)\gamma_a + \sigma_{n,max}\varepsilon_{n,a} = \frac{(\sigma_f')^2}{E}(2N_f)^{2b} + \sigma_f'\varepsilon_f'(2N_f)^{b+c} \tag{94}$$

where σ_n is the normal stress on the critical plane at the same moment as γ_a, and $\sigma_{n,max}$ is the maximum normal stress on the critical plane during a cycle. Both normal (Mode I) and shear (Mode II) crack loading are incorporated in the model.

5. Energy parameter in the multiaxial random stress state

The random strain and stress tensors have been described as six-dimensional stationary and ergodic Gaussian processes with the wide-band frequency spectra and zero-expected values. Łagoda et al.[53] have defined the energy parameter for uniaxial random loading which distinguishes the strain energy density for tension (positive) and the strain energy density for compression (negative) as follows

$$W(t) = \frac{1}{2}\sigma(t)\cdot\varepsilon(t)\,\mathrm{sgn}[\sigma(t),\varepsilon(t)] \tag{95}$$

where $\sigma(t)$ and $\varepsilon(t)$ are random stress and strain, respectively and sgn(x,y) is the two-argument logical function, sensitive to signs of variables x and y. The logical function is defined as

$$\operatorname{sgn}(x,y)=\begin{cases} 1 & \text{when } \operatorname{sgn}(x)=\operatorname{sgn}(y)=1 \\ 0.5 & \text{when } x=0 \text{ and } \operatorname{sgn}(y)=1 \text{ or } y=0 \text{ and } \operatorname{sgn}(x)=1 \\ 0 & \text{when } \operatorname{sgn}(x)=-\operatorname{sgn}(y) \\ -0.5 & \text{when } x=0 \text{ and } \operatorname{sgn}(y)=-1 \text{ or } y=0 \text{ and } \operatorname{sgn}(x)=-1 \\ -1 & \text{when } \operatorname{sgn}(x)=\operatorname{sgn}(y)=-1 \end{cases} \tag{96}$$

Later Lagoda and Macha [54] generalized some known energy criteria of multiaxial cyclic fatigue to the random loading. The proposed generalized energy criterion is based on the selected components of specific work of stress on the strains in the critical plane.

5.1. Generalized criterion of maximum shear and normal strain energy density on the critical plane

It is assumed that

(1) Fatigue fracture is caused by that part of strain energy density which corresponds to the specific work of normal stress $\sigma_\eta(t)$ on normal strain $\varepsilon_\eta(t)$ i.e. $W_\eta(t)$ and specific work of shear stress $\tau_{\eta s}(t)$ on shear strain $\varepsilon_{\eta s}(t)$ acting in the $\bar{s}$ direction, on the plane with a normal $\bar{\eta}$, i.e. $W_{\eta s}(t)$.

(2) The direction $\bar{s}$ in the critical plane (the expected fracture plane) coincides with the mean direction of the maximum shear strain energy density $W_{\eta s max}(t)$

(3) In the limit state that conforms to the fatigue strength, the maximum value of combined $W_{\eta s}(t)$ and $W_\eta(t)$ energies under multiaxial random loading satisfies the following equation

$$\max_t\left\{\beta W_{\eta s}(t)+\kappa W_\eta(t)\right\}=Q \tag{97}$$

where β = constant for a particular form of Eq. (97), κ and Q = material constants determined from sinusoidal fatigue tests.

The left side of Eq. (97) can be written as $\max_t\{W(t)\}$, and should be interpreted as the 100% quantile of the random variable W. If the maximum value of W(t) exceeds the value of Q, then damage will accumulate resulting in fracture. The random process W(t) can be interpreted as a stochastic process of the fatigue strength of a material. The positions of the unit vectors $\bar{\eta}$ and $\bar{s}$ are determined with use of one of the following procedures: weight functions method, variance method or damage accumulation method [55].

A choice of constants β, κ and Q in Eq. (97), together with the assumed position of the critical plane, leads to particular cases of the generalized criterion. Three special cases are considered here.

5.2. Criterion of maximum normal strain energy density in the critical plane

If $\beta = 0$, $\kappa = 1$ and $Q = W_{af}$ (fatigue limit under tension-compression expressed by normal strain energy density) and if we assume that the unit vector $\bar{\eta}$ coincides with the mean direction along which the maximum normal strain energy density $W_{\eta max}(t)$ occurs, i.e.

$$\bar{\eta} = \hat{l}_\eta \bar{i} + \hat{m}_\eta \bar{j} + \hat{n}_\eta \bar{k} \tag{98}$$

then criterion (97) becomes

$$\max_t \{W_\eta(t)\} = W_{af} \tag{99}$$

where

$\hat{l}_\eta, \hat{m}_\eta, \hat{n}_\eta$ = mean direction cosines of $\bar{\eta}$ in relation to the constant system of axes Oxyz,

$\bar{i}, \bar{j}, \bar{k}$ = versors of the axes 0x,y,z.

The equivalent strain energy derived from criterion (99) is as follows

$$W_{eq}(t) = W_\eta(t) = \tfrac{1}{2}\sigma_\eta(t)\varepsilon_\eta(t)\,\mathrm{sgn}[\sigma_\eta(t), \varepsilon_\eta(t)] \tag{100}$$

where $\sigma_\eta(t)$ and $\varepsilon_\eta(t)$ are normal stress and strain in the critical plane respectively, i.e.

$$\begin{aligned}\sigma_\eta(t) = {} & \hat{l}_\eta^2 \sigma_{xx}(t) + \hat{m}_\eta^2 \sigma_{yy}(t) + \hat{n}_\eta^2 \sigma_{zz}(t) + 2\hat{l}_\eta \hat{m}_\eta \sigma_{xy}(t) + \\ & + 2\hat{l}_\eta \hat{n}_\eta \sigma_{xz}(t) + 2\hat{m}_\eta \hat{n}_\eta \sigma_{yz}(t),\end{aligned} \tag{101}$$

$$\begin{aligned}\varepsilon_\eta(t) = {} & \hat{l}_\eta^2 \varepsilon_{xx}(t) + \hat{m}_\eta^2 \varepsilon_{yy}(t) + \hat{n}_\eta^2 \varepsilon_{zz}(t) + 2\hat{l}_\eta \hat{m}_\eta \varepsilon_{xy}(t) + \\ & + 2\hat{l}_\eta \hat{n}_\eta \varepsilon_{xz}(t) + 2\hat{m}_\eta \hat{n}_\eta \varepsilon_{yz}(t)\end{aligned} \tag{102}$$

If under proportional multiaxial sinusoidal loading we assume that normal stress and normal strain having the maximum amplitudes act along the axis x, i.e.

$$\sigma_{xx}(t) = \sigma_{axx} \sin \omega t ; \qquad \varepsilon_{xx}(t) = \varepsilon_{axx} \sin \omega t \tag{103}$$

and further if we assume $\hat{l}_\eta = 1$, then according to Eqs. (99) – (102), we obtain

$$\max_t \left\{ \frac{1}{2} \sigma_{xx}(t) \varepsilon_{xx}(t)\,\mathrm{sgn}\left[\sigma_{xx}(t), \varepsilon_{xx}(t)\right] \right\} = \frac{1}{2} \sigma_{axx} \varepsilon_{axx} = W_{af} \tag{104}$$

This result leads to the energy parameter used by Socie, Eq. (62), based on the idea of Smith-Watson-Topper [43] and to the criterion proposed by Nitta, Ogata and Kuwabara, Eq. (63) for Mode I. Thus, criterion (99) is a generalization of the mentioned energy criteria, applied under cyclic loading.

5.3. Criterion of maximum shear strain energy density on the critical plane

For $\beta=4/(1+\nu)$, (ν - Poisson's ratio), $\kappa = 0$ and $Q = W_{af}$ we assume that the critical plane with normal $\overline{\eta}$ is determined by the mean position of one of two planes on which the maximum shear strain energy acts. On this plane we choose a direction $\overline{s}$ coincident with the mean position along which the energy $W_{\eta s\max}(t)$ occurs, i.e.

$$\overline{s} = \hat{l}_s\overline{i} + \hat{m}_s\overline{j} + \hat{n}_s\overline{k} \tag{105}$$

where $\hat{l}_s, \hat{m}_s, \hat{n}_s$ are mean direction cosines of $\overline{s}$ in relation to the axes Oxyz. Under the above assumptions, the criterion (97) becomes

$$\max_t\left\{\frac{4}{1+\nu}W_{\eta s}(t)\right\} = W_{af} \tag{106}$$

The equivalent strain energy density derived from criterion (106) is as follows

$$W_{eq}(t) = \frac{4}{1+\nu}W_{\eta s}(t) = \frac{2}{1+\nu}\tau_{\eta s}(t)\varepsilon_{\eta s}(t)\,\mathrm{sgn}[\tau_{\eta s}(t),\varepsilon_{\eta s}(t)] \tag{107}$$

where

$$\begin{aligned}\tau_{\eta s}(t) = {} & \hat{l}_\eta\hat{l}_s\sigma_{xx}(t) + \hat{m}_\eta\hat{m}_s\sigma_{yy}(t) + \hat{n}_\eta\hat{n}_s\sigma_{zz}(t) + 2\hat{l}_\eta\hat{m}_s\sigma_{xy}(t) + \\ & + 2\hat{l}_\eta\hat{n}_s\sigma_{xz}(t) + 2\hat{m}_\eta\hat{n}_s\sigma_{yz}(t)\end{aligned} \tag{108}$$

$$\begin{aligned}\varepsilon_{\eta s}(t) = {} & \hat{l}_\eta\hat{l}_s\varepsilon_{xx}(t) + \hat{m}_\eta\hat{m}_s\varepsilon_{yy}(t) + \hat{n}_\eta\hat{n}_s\varepsilon_{zz}(t) + 2\hat{l}_\eta\hat{m}_s\varepsilon_{xy}(t) + \\ & + 2\hat{l}_\eta\hat{n}_s\varepsilon_{xz}(t) + 2\hat{m}_\eta\hat{n}_s\varepsilon_{yz}(t)\end{aligned} \tag{109}$$

In the case of proportional multiaxial sinusoidal loading when the normal stress and strain having the maximum amplitudes act along the x axis and the normal stress and strain with minimum amplitudes act along the z axis, i.e.

$$\begin{aligned}\sigma_{xx}(t) &= \sigma_{axx}\sin\omega t; \qquad & \varepsilon_{xx}(t) &= \varepsilon_{axx}\sin\omega t; \\ \sigma_{zz}(t) &= \sigma_{azz}\sin\omega t; & \varepsilon_{zz}(t) &= \varepsilon_{azz}\sin\omega t;\end{aligned} \tag{110}$$

and when

$$\begin{aligned}\hat{l}_\eta &= \frac{1}{\sqrt{2}}, \quad & \hat{m}_\eta &= 0, \quad & \hat{n}_\eta &= \frac{1}{\sqrt{2}}, \\ \hat{l}_s &= \frac{1}{\sqrt{2}}, & \hat{m}_s &= 0, & \hat{n}_s &= -\frac{1}{\sqrt{2}},\end{aligned} \tag{111}$$

then, according to Eqs. (106)—(111) we obtain

$$\begin{aligned}&\max_t\left\{\frac{4}{1+\nu}W_{\eta s}(t)\right\} = \max_t\left\{\frac{2}{1+\nu}\tau_{\eta s}(t)\varepsilon_{\eta s}(t)\,\mathrm{sgn}[\tau_{\eta s}(t),\varepsilon_{\eta s}(t)]\right\} = \\ &= \frac{2}{1+\nu}\frac{\sigma_{axx}-\sigma_{azz}}{2}\frac{\varepsilon_{axx}-\varepsilon_{azz}}{2} = \frac{1}{1+\nu}\tau_{a\max}\gamma_{a\max} = W_{af}\end{aligned} \tag{112}$$

Strain energy density expressed by Eq. (112) is also applied in the criterion proposed by Nitta, Ogata and Kuwabara for Mode II under cyclic loading (Eq. (64)). Thus, criterion (106) is a generalization of the next energy criterion applied under cyclic loading.

5.4. Criterion of the maximum shear and normal strain energy density on the critical plane – case I

For $\beta = 2(1+\nu)$, $\kappa = 2/(1-\nu)$ and $Q = W_{af}$ we assume, as in the above section, that the critical plane with normal $\bar{\eta}$ is determined by the mean position of one of two planes on which the maximum shear strain energy acts. On this plane, the direction $\bar{s}$ is coincident with the mean position along which the energy $W_{\eta s\,max}(t)$ occurs – see Eq. (105).
The general criterion (97) now has the following form

$$\max_t \left\{ \frac{2}{1+\nu} W_{\eta s}(t) + \frac{2}{1-\nu} W_{\eta}(t) \right\} = W_{af} \tag{113}$$

From criterion (113) we can derive the equivalent strain energy density as

$$\begin{aligned} W_{eq}(t) &= \frac{2}{1+\nu} W_{\eta s}(t) + \frac{2}{1-\nu} W_{\eta}(t) = \\ &= \frac{1}{1+\nu} \tau_{\eta s}(t)\varepsilon_{\eta s}(t)\,\mathrm{sgn}[\tau_{\eta s}(t),\varepsilon_{\eta s}(t)] + \frac{1}{1-\nu} \sigma_{\eta}(t)\varepsilon_{\eta}(t)\,\mathrm{sgn}[\sigma_{\eta}(t),\varepsilon_{\eta}(t)] \end{aligned} \tag{114}$$

where $\tau_{\eta s}(t)$, $\varepsilon_{\eta s}(t)$, $\sigma_{\eta}(t)$ and $\varepsilon_{\eta}(t)$ are expressed by Eqs. (108), (109), (101), (102) respectively.
Under multiaxial sinusoidal in-phase loading and on the assumption as in the previous section – see Eqs. (110) and (111) – from Eqs. (113) and (114) it follows that

$$\begin{aligned} &\max_t \left\{ \frac{2}{1+\nu} W_{\eta s}(t) + \frac{2}{1-\nu} W_{\eta}(t) \right\} = \\ &= \max_t \left\{ \frac{1}{1+\nu} \tau_{\eta s}(t)\varepsilon_{\eta s}(t)\,\mathrm{sgn}[\tau_{\eta s}(t),\varepsilon_{\eta s}(t)] + \frac{1}{1-\nu} \sigma_{\eta}(t)\varepsilon_{\eta}(t)\,\mathrm{sgn}[\sigma_{\eta}(t)\varepsilon_{\eta}(t)) \right\} = \\ &= \frac{1}{2(1+\nu)} \tau_{a\,max}\gamma_{a\,max} + \frac{1}{1-\nu} \sigma_{a\eta}\varepsilon_{a\eta} = W_{af} \end{aligned} \tag{115}$$

where $\sigma_{a\eta}$ and $\varepsilon_{a\varepsilon}$ are amplitudes of normal stress and strain on the plane of the maximum shear stress and strain amplitudes τ_{amax} and γ_{amax}, respectively.
The strain energy density in Eq. (115) is also assumed by Liu, Eq. (78) or (79), in his virtual strain-energy parameter (VSE) under in-phase cyclic loading for Mode II fracture (for $\beta = \kappa = 1$).

Thus, the criterion (113) is a generalization of the energy criterion formulated by Liu [21] to the range of random loading. Moreover, it is possible to prove that for $\beta = 2$ and $\kappa = 1$ the criterion (113) is a generalization of the energy criterion proposed by Glinka, Shen and Plumtree, Eq. (82) for cyclic loading. Further cases of this criterion may be defined by the choice of another position of the critical plane. The position is determined by the given values of the direction cosines $\hat{l}_n, \hat{m}_n, \hat{n}_n (n = \eta, s)$ of the unit vectors $\overline{\eta}$ and $\overline{s}$ occuring in the fatigue criteria. The following three methods of determination of the expected critical plane position are proposed:

a) The method of weight function, presented in [55], consists in averaging the random values of angles $\alpha_n(t)$, $\beta_n(t)$, $\gamma_n(t)$, determining instantaneous positions of the principal strain/stress axes position in relation to the constant system of 0xyz axes with use of a special weight functions.

b) The method of damage accumulation, presented in [55]. Here the fatigue damage is accumulated on the all possible planes. Next, the plane on which damage is maximum is selected. Thus, we obtain not only direction of the expected critical plane, but the fatigue life as well.

c) The method of variance maximum. It is the method the most often applied so far. It gives good results when the stress and strain criteria are applied [55], [56]. Here, in the method of variance it is assumed that the planes in which equivalent strain energy density variance according to the chosen criterion reaches its maximum are critical for the material.

6. Some further considerations

The criteria based on elastic strain energy can be used for fatigue under a high number of cycles to fracture (HCF), while the criteria based on plastic strain energy are more appropriate for a low number of cycles to failure (LCF). The criteria based on the sum of plastic and elastic strain energies can be applied for both LCF and HCF.

In the group of criteria based on elastic strain energy, a criterion involving the average strain energy in the period of a cycle and within a material volume for HCF, as proposed by Palin-Luc and Lasserre [36], should be mentioned because it takes into account stress and energy gradients under uniaxial and multiaxial loading.

The advantage of this criterion is that it is sensitive to different stress distributions in circular section bars under tension and under bending as well as to the different energy distributions under plane bending and rotary bending when the stress distributions are the same [36].

The criteria based on plastic strain energy include the models formulated by Lefebvre et al., Eq. (20), Garud, Eq. (27) and Chen et al., Eqs. (30) and (31). In the first two models, there are only plastic strains, and it has been shown that in these criteria, strain energy is not the plastic energy from all the stress state components in a cycle, but it is only the energy associated with the critical plane. That new interpretation of the models supports the energy-based critical plane approach to multiaxial fatigue. The critical plane energy criteria for a tensile-type failure, Eq. (30), and for a shear-type failure, Eq. (31), have been proposed and verified only for LCF [26]. Thus, we may suppose that their authors consider only plastic strains and neglect small elastic ones. Since on the right sides of Eqs. (30) and (31) there are also terms representing the elastic energy of strains, we can expect that the criteria could be useful for both LCF and HCF.

The largest group of criteria is based on the sum of elastic and plastic strain energy densities. The first Ellyin proposals presented in 1974 concentrated on the shear strain energy W_f (see Eq. (32)), and were only theoretical in nature. Later Ellyin and Kujawski modified Eq. (32) and obtained energy Ψ (Eq. (34) which was experimentally confirmed for some materials.

Other modifications by Ellyin and Gołoś assumed a specific form of energy ΔW^t (Eq. (42)) which is the sum of the elastic strain energy in the positive stress range of the cycle ΔW^{e+} and the plastic strain energy ΔW^p. In comparison with Eqs. (32) and (34), the modifications are more complex, because the energy ΔW^{e+} includes the normal and the shear strain energy density, Eq. (43). It should be emphasised that the plastic strain energy ΔW^p is equal to the area of the effective stress-strain hysteresis loop, Eq. (44).

The models by Ellyin and Gołoś, Eq. (42), and Park and Nelson, Eq. (49) are based on the same plastic energy of effective strain $\Delta W^p = \Delta W_p^d$. In the model by Park and Nelson, Eq. (49), there is, however, the double value of elastic energy comparing with the model by Ellyin and Gołoś, Eq. (42), i.e. $\Delta W_e^d = 2\Delta W^{e+}$ [14], [15], [18], [30]. Further differences between these models are connected, among others, with the approach to influence of the mean stresses.

The Leis model given as Eq. (36) includes the normal and shear strain energy U_T with the elastic and plastic parts, and hence contains most of the energy components. However, from a theoretical point of view, the Leis model overestimates damage in relation to those parameters based on the energy in the critical plane. Since the models were verified in some tests, we may expect that in the calculation of the total or only the partial (in the critical plane) plastic strain energy, the assumed form of the constitutive equations relating the amplitudes of cyclic stress and plastic strain (especially under non-proportional loading) plays a very important role.

According to the model by Itoh et al., Eq. (48), the equivalent strain energy reduced to uniaxial state of stress is non-linear function of normal and shear strain energies.

With respect to energy criteria associated with the critical plane, the models proposed by Socie (see Eq. (62)) when $\sigma_m = 0$ and $\sigma_{max} = \sigma_a$ and the energy $\sigma_{max}\varepsilon$, or Nitta et al., Eq. (63) and the energy ΔE_I, or Hoffman and Seeger,

Eq. (72) and the energy P, then these approaches concern the same normal strain energy in the plane corresponding to Mode I cracking, and are based on the concept of the SWT parameter. The proposals by Nitta et al. Eq. (64), and energy ΔE_2, the Hoffman and Seeger proposals, Eq. (73), and energy P, and Glinka et al., Eq. (85), and energy W*, they concern the same shear strain energy in the γ_{max} plane and correspond to Mode II cracking.
The advantage of the models formulated by Chu, Eq. (74) and energy C_{SWT}, Liu, Eq. (75) and energy ΔW, Glinka et al., Eq. (83) and energy W*, Pan et al., Eq. (86), Varvani-Farahani and Topper, Eq. (87), and Rolovic and Tipton, Eq. (93), is that they include two kinds of energy in the damage parameter; the normal strain energy ΔW_{xx} and shear strain energy ΔW_{xy} in the critical plane.
Comparing these models, we notice that the participation of these energies in the damage parameter is different. Under symmetric loading, in Chu's model we have $C_{SWT} = \Delta W_{xx} + 4W_{xy}$, in Liu's model $\Delta W = \Delta W_{xx} + \Delta W_{xy}$, and in the Glinka et al. model there is $W^* = \Delta W_{xx} + 2\Delta W_{xy}$., in Pal et al. model $W^* = H_1 H_2 \Delta W_{xx} + \Delta W_{xy}$, in Varvani-Farahani and Topper's model $\Delta W_{xx} + H_3 \Delta W_{xy}$ (where H_3 =constant) and in the Rolovic and Tipton model $\Delta W_{xx} + 2\Delta W_{xy}$. Therefore, we can see that the individual participation of energies ΔW_{xx} and ΔW_{xy} in the Glinka et al. model is the same as in the Relovic and Tipton model.
If we now consider these interpretations of the damage parameter for the symmetric cycles in which

- the plastic effective strain energy, which is equivalent to the plastic shear strain energy on the octahedral plane, $\Delta\tau_{oct}\Delta\varepsilon_{oct}^{p}$ (see Eq. (26)), or
- the sum of two from nine areas of the closed stress-strain hysteresis loops, which is equal to the sum of the plastic normal and shear strain energies in the x-y plane $\Delta W_{xx}^{p} + \Delta W_{xy}^{p}$ (see Eq. (29) for $H_G=1$), or
- the elastic and plastic shear strain energy, which is equal to the elastic and plastic shear strain energy in the octahedral plane W_f (see Eq. (32) and Ψ in Eq. (34)),

then it will be noticed that those criteria involving the strain energy in the critical plane dominate in the description of multiaxial fatigue and that is why we may accept them as being the most promising criteria. They are shown in Table 1.
Also the generalized criterion of maximum shear and normal strain energy density for multiaxial random loading, Eq. (97), is based on the critical plane. To extend energy approach to random fatigue, it was necessary to introduce a new definition of the energy parameter for distinguishing positive and negative signs in history of specific work of stress on strain along the chosen direction. This new damage parameter has made it possible to generalize some energy criteria of multiaxial cyclic fatigue to the random loading.
The mean stress effects in energy criteria are formulated in a different manner. In the group of criteria based on the elastic strain energy, we can calculate the

mean equivalent stress σ_{meq}, in a manner similar to the amplitude of the equivalent stress σ_{aeq}, from Eqs. (2), and (6) and then calculated the transformed (enlarged) amplitude σ_{aeqT} of the cycle with a zero mean value according to one of the relationships σ_a versus σ_m (for example Goodman [57], Marin [58], Troost and El-Magd [59] applicable to uniaxial loading. The mean shear stress τ_m in the criterion of elastic shear strain energy in the critical plane Φ_{sf} (Eq. (7)) is treated in a similar way. The mean stresses in those energy criteria are included as in the stress based criteria of multiaxial fatigue failure. The criterion proposed by Palin-Luc and Lasserre, Eq. (19), concerns symmetric cycles with zero mean values.

In the criteria based on plastic strain energy there is a lack of information as how to involve the mean value of stresses in Eqs. (20), (27), (30) and (31) proposed by respectively Lefebvre et al., Garud and Chen et.al. It is known that mean stresses usually relax in the regime of LCF.

Table 1.
The criteria of multiaxial cyclic fatigue including the strain energy density in the critical plane

No.	The criterion proposed by	Kind of energy	Range of application
1.	Ellyin [12] (1974)	elastic and plastic shear strain energy in the octahedral plane, Eqs. (32) and (34)	HCF and LCF
2.	Macha [22] (1979)	elastic shear strain energy in the plane of maximum shear stress, Eq. (7)	HCF
3.	Garud [16] (a particular case $H_G = 1$) (1981)	sum of plastic energies of normal and shear strains in the critical plane, Eq. (29)	LCF
4.	Smith, Watson and Topper [42], (1970) Socie [47] (1987)	elastic and plastic energies of normal strain in the plane of maximum range of normal strain, Eq. (62)	HCF and LCF
5.	Lefebvre, Neale and Ellyin [20] (1988)	plastic energy of the shear strain in the octahedral plane, Eq. (26)	LCF
6.	Nitta, Ogata and Kuwabara [23] (1989)	i) elastic and plastic energies of normal strain in the plane of maximum range of normal strain Eq. (63), or ii) elastic and plastic energies of shear strain in the plane of maximum range of shear strain Eq. (64)	HCF and LCF
7.	Smith, Watson and Topper [42], (1970) Hoffman and Seeger [48] (1989)	i) elastic and plastic energies of the normal strain in the plane of maximum normal strain Eq. (72) ii) elastic and plastic energies of the shear strain in the plane of maximum shear strain Eq. (73)	HCF and LCF

8.	Chu, Conle and Bonnen [49] (1993)	sum of energies (elastic and plastic) of normal and shear strains in the critical plane (with the maximum sum of both energies), Eq. (74)	HCF and LCF
9.	Liu [21] (1993)	sum of energies (elastic and plastic) of normal and shear strains in the critical plane (with maximum value of one component of energy, according to Mode I or Mode II), Eq. (75)	HCF and LCF
10.	Glinka, Shen and Plumtree [17] (1995)	sum of energies (elastic and plastic) of the normal and shear strains in the plane of maximum shear strains, Eq. (82)	HCF and LCF
11.	Chen, Xu and Huang [26] (1999)	(i) sum of plastic energies of normal and shear strains in the maximum normal strain plane, Eq. (30) (ii) sum of plastic energies of normal and shear strains in the maximum shear strain plane, Eq. (31)	LCF
12.	Pan, Hung and Chen [27] (1999)	sum of energies (elastic and plastic) of shear and weighted normal strains in the plane of maximum shear strains, Eq. (86)	HCF and LCF
13.	Rolovic and Tipton[25] (1999)	sum of energies (elastic and plastic) of normal and shear strains in the critical plane (with maximum sum of both energies or the observed cracking behaviour of the material), Eq. (93)	HCF and LCF
14.	Varvani-Farahani and Topper [29] (2000)	sum of weighted energies (elastic and plastic) of normal and shear strains in the critical plane (on which the stress and strain Moh'r circles are the largest during the loading and unloading parts of a cycle), Eq. (87)	HCF and LCF

As for the criteria based on the sum of elastic and plastic strain energy, the mean value of stresses has been included in the models of Ellyin and Kujawski, Eq. (34), Leis, Eq. (36), Ellyin and Gołoś, Eq. (42), Park and Nelson, Eq. (57), Socie, Eq. (62), Hoffman and Seeger, Eqs. (72) and (73), Glinka et al., Eq. (85), Varvani-Farahani and Topper, Eq. (87) and Rolovic and Tipton, Eq. (93). In the energy models of Itoh et al., Eq. (48), Nitta et al., Eqs. (63), (64) and (71), Chu et al., Eq. (74), Liu, Eq. (75), Glinka et al., Eq. (82), Pan et al., Eq. (86) and in the model for multiaxial random loading, Eq. (97), their authors did not say how to take into account the mean stress effects. From the above specification it appears that the role of mean stresses in the energy criteria requires further experimental evidence, especially under random loading.

Some of the energy criteria discussed in this paper are also effective under non-proportional loading. Here we could mention the parameters proposed by Palin-Luc and Lasserre, Eq. (19), Garud, Eq. (27), Chen et al., Eqs. (30) and (31), Itoh et al., Eq. (48), Park and Nelson, Eq. (57), Socie, Eq. (62), Nitta et al., Eq. (71), Chu et al., Eq. (74), Liu, Eq. (63), Varvani-Farahani and Topper, Eq. (87), Rolovic and Tipton, Eq. (93) and the generalized criterion of maximum shear and normal strain energy on the critical plane, Eq. (97). There is no experimental evidence for the application of other criteria under non-proportional loading.

It is observed that in HCF regime the crack initiation period is dominating and the propagation time is very short in the total fatigue life of typical structural materials. In LCF regime, the initiation and propagation times are comparable. The discussed energy criteria concern only the crack initiation. As for the propagation time, the calculation should be based on a separate group of criteria for multiaxial fatigue, using elements of fracture mechanics [2], [3], [4], [5], [6], [7], [8], [9]. At present, it is not easy to say how to apply the discussed criteria in practice for real structures. First of all, the experimental verification, using complex components stressed under multiaxial random load-time histories, is necessary. We must also pay attention to the material used, because — depending on the nature of the material (ductile, semi-ductile or brittle) — different failure processes and mechanisms must be taken into account. The generalized criterion of maximum shear and normal strain energy density in the critical plane seems to be the most useful in engineering practice. This criterion should be developed, and the mean stresses, stress concentration, non-stationary loading and external factors (for instance, corrosion, high or low temperature) should be taken into account.

7. Conclusions

1. From the review of known energy criteria of multiaxial cyclic fatigue failure it appears that they can be divided into three groups when assuming the strain energy density per cycle as the damage parameter. They are:
   a) criteria based on elastic energy for high-cycle fatigue,
   b) criteria based on plastic energy for low-cycle fatigue,
   c) criteria based on the sum of elestic and plastic energies for high- and low-cycle fatigue.
2. The proposed criterion of multiaxial random fatigue failure is based on the new definition of energy parameter which distinguishes both positive and negative signs in history of specific work of stress on strain along the chosen directions in critical fracture plane.
3. From laboratory tests it appears that the criteria that do not include all the strain energy, but only the components connected with the critical fracture plane dominate in the energy description of multiaxial fatigue, and that is why we may accept them as being the most promising criteria.
4. At present it is not possible to recommend one criterion for application in industry. Therefore, experimental verification using complex-shaped

components subjected to multiaxial random load-time histories and different environments is necessary.

5. The generalized criterion of maximum shear and normal strain energy density in the critical plane seems to be the best for application in practice. It should be developed, and mean stresses, stress concentration and non-stationary loading as well as external factors such as corrosion, high or low temperature should be taken into account. It is also necessary to consider the material used, because its kind (ductile, semi-ductile, brittle) influences different failure processes and mechanisms.

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Przegląd energetycznych kryteriów wieloosiowego zniszczenia zmęczeniowego

Streszczenie

Praca zawiera przegląd energetycznych kryteriów wieloosiowego zniszczenia zmęczeniowego w warunkach cyklicznego i losowego obciążenia. Kryteria odnoszące się do cyklicznego obciążenia podzielono na trzy grupy, zależnie od rodzaju gęstości energii odkształcenia na cykl, którą przyjmuje się jako parametr uszkodzenia. Są to: a) kryteria oparte na energii sprężystej odkształcenia dla zmęczenia wysokocyklowego, b) kryteria oparte na energii plastycznej odkształcenia dla zmęczenia niskocyklowego oraz c) kryteria oparte na sumie energii sprężystej i plastycznej dla zmęczenia wysoko- i niskocyklowego. Kryterium dotyczące losowego obciążenia jest oparte na nowej definicji parametru energetycznego, który odróżnia dodatnie i ujemne znaki w historii pracy właściwej naprężenia na odkształceniu w wybranych kierunkach krytycznej płaszczyzny złomu. Kryteria, które uwzględniają gęstość energii odkształcenia w płaszczyźnie krytycznej, dominują w energetycznym opisie zmęczenia wieloosiowego. Podano parametry zależne od obciążeń oraz czynniki zależne od rodzaju materiału, decydujące o wyborze płaszczyzny krytycznej. Omówiono modele matematyczne kryteriów, a następnie wyróżniono te, które uwzględniają wpływ naprężeń średnich i gradientów naprężeń, oraz obciążenia proporcjonalne i nieproporcjonalne. Zwrócono uwagę, że najbliższe potrzebom obliczeń inżynierskich jest uogólnione kryterium gęstości energii maksymalnego odkształcenia normalnego i stycznego w płaszczyźnie krytycznej i należy je nadal rozwijać i weryfikować.