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MODEL FOR SIMULATION OF VEHICLE DYNAMICS

The paper presents a model of a car with special attention given to the drive system. Two possible drive systems were considered: with standard differential and independent drive of each wheel by means of an electric motor. In both cases, flexibilities of live axle shafts have been taken into consideration. A 3D model of the car was assumed. The model consists of a system of rigid bodies connected one with another by means of elastic-damping elements. The phases of static and kinetic friction were considered in the steering and drive systems. The method of homogenous transformations was used in the mathematical description. The results of computer simulations are presented.

1. Introduction

In most of the literature, one considers vehicle models (2D and 3D) in which motion of a vehicle (body with suspensions and wheels) are treated as independent of vibrations of the steering and drive systems [6], [7], [12], [13]. This results from the fact that frequencies of vibrations of the drive system are significantly greater than frequencies of vibrations of vehicles. It is possible to solve the equations of motion of the drive and steering systems together with the equations of motion for the whole vehicle only in the case when a small step of integration is used. Furthermore, taking into account steering and driving systems means that a greater number of equations is used in the model for description of the vehicle behaviour. This prolongs the time of computer calculations. However, it is necessary in some cases, such as

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comparison of effectiveness of drives or complex consideration of a vehicle and its drive system. In the research program performed for the *'Centro Ricerche FIAT'*, the authors have undertaken an attempt to create mathematical and computer models of a vehicle, together with drive and steering systems. This paper presents the main results of the mentioned investigation. Two types of drives are considered. In the first variant, with standard differential driven by combustion engine, the rear axle shafts are driven. In the second variant, the independent drive of rear wheels (by means of two electric motors) is modelled. In both cases, flexibility of the drive system, especially of axle shafts, is taken into account. The vehicle steering system is considered in the model as well. The steering system is a complex multi-body system which could be modelled in a more complicated way [8] or as a simplified system like in this paper. It was assumed that the angle of steering of the vehicle wheels is determined by: the moment acting on the steering wheel, motion of the steering rack and external load acting on the vehicle wheels. Kinetic and static phases of friction between the steering rack and its housing are taken into account. The authors applied Dugoff-Uffelmann's tire model [2] for description of the contact between the wheels and road surface. The developed computer program makes it possible to use other tire models [1], [4] in order to compare the influence of the assumed tire model on the results of simulation. The numerical results of calculations for some road tests, determined among others by ISO standards, are presented in the paper.

2. Model of the vehicle

It is assumed that in the physical model of the vehicle the following three subsystems are taken into consideration: I – body, suspensions, wheels with tires, II – steering system, III – drive system with differential and flexible axle shafts (Fig. 1).
In general, we can assume that the equations of motion of the subsystems have the following form:

$$\mathbf{M}_r \ddot{\mathbf{q}}_r + \mathbf{D}_{rs}\mathbf{R}_{rs} + \mathbf{D}_{rd}\mathbf{R}_{rd} = \mathbf{F}_r(\mathbf{q}_r, \dot{\mathbf{q}}_r, \mathbf{F}_{re}) \tag{1.1}$$

$$\mathbf{M}_s \ddot{\mathbf{q}}_s + \mathbf{D}_{sr}\mathbf{R}_{rs} = \mathbf{F}_s(\mathbf{q}_s, \dot{\mathbf{q}}_s, M_e) \tag{1.2}$$

$$\mathbf{M}_d \ddot{\mathbf{q}}_d + \mathbf{D}_{dr}\mathbf{R}_{rd} = \mathbf{F}_d(\mathbf{q}_d, \dot{\mathbf{q}}_d, M_d) \tag{1.3}$$

where:

$\mathbf{q}_r = [q_1^{(r)}, ..., q_{n_r}^{(r)}]^T$ – vector of generalized coordinates of subsystem I (body, suspensions, wheels),

$\mathbf{q}_s = [q_1^{(s)}, \ldots, q_{n_s}^{(s)}]^T$ – vector of generalized coordinates of subsystem II (steering system),

$\mathbf{q}_d = [q_1^{(d)}, \ldots, q_{n_d}^{(d)}]^T$ – vector of generalized coordinates of subsystem III (driving system),

$\mathbf{R}_{rs} = [R_1^{(rs)}, \ldots, R_{m_{rs}}^{(rs)}]^T$ – vector of reaction forces acting between subsystems I and II,

$\mathbf{R}_{rd} = [R_1^{(rd)}, \ldots, R_{m_{rd}}^{(rd)}]^T$ – vector of reaction forces acting between subsystems I and III,

$\mathbf{M}_r(\mathbf{q}_r), \mathbf{M}_s(\mathbf{q}_s), \mathbf{M}_d(\mathbf{q}_d)$ – inertial matrices,

$\mathbf{D}_{rs}(\mathbf{q}_r, \mathbf{q}_s)$, $\mathbf{D}_{sr}(\mathbf{q}_r, \mathbf{q}_s)$, $\mathbf{D}_{rd}(\mathbf{q}_r, \mathbf{q}_d)$, $\mathbf{D}_{dr}(\mathbf{q}_r, \mathbf{q}_d)$ – matrices of coefficients,

$\mathbf{F}_r$, $\mathbf{F}_s$, $\mathbf{F}_d$ – vectors,

$\mathbf{F}_{re}$ – vector of reaction forces of the road surface on the wheels,

M_e – external torque acting on the steering wheel,

M_d – engine torque.

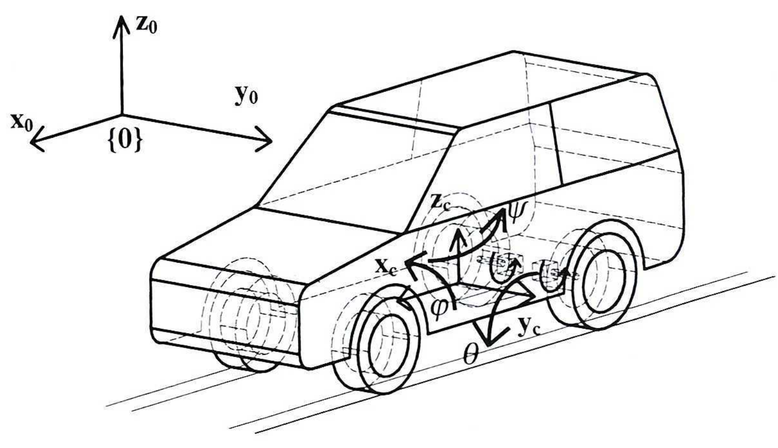

Fig. 1. General scheme of the vehicle

These equations have to be completed with constraint equations, which can be written as:

$$h_i^{(rs)}(\mathbf{q}_r, \mathbf{q}_s) = \mathbf{0}, \; i = 1, \ldots, m_{rs} \tag{2.1}$$

$$h_i^{(rd)}(\mathbf{q}_r, \mathbf{q}_d) = \mathbf{0}, \; i = 1, \ldots, m_{rd} \tag{2.2}$$

In literature we can find many models [6], [7], [9] which take into account equations (1.1), (1.2) and (2.1), basing on the assumption that there is no influence of drive system structure on vehicle motion (except engine torque). In our consideration we concentrate on the problem how the drive system structure influences the vehicle motion. Thus, we will describe the models of drive systems considered.

The internal forces arising from suspension flexibility and damping are usually considered as nonlinear [5], [10]. They depend on kinematic structure and stiffness and damping characteristics of suspension and steering system applied. In our consideration, we assumed that the vector of generalized coordinates of subsystem I has the following components:

$$\mathbf{q}_r = [\mathbf{q}_n \ \mathbf{q}_1^{(z)} \ \varphi_1 \ \mathbf{q}_2^{(z)} \ \varphi_2 \ \mathbf{q}_3^{(z)} \ \varphi_3 \ \mathbf{q}_4^{(z)} \ \varphi_4]^T, \tag{3}$$

where:

$\mathbf{q}_n = [x_c \ y_c \ z_c \ \psi \ \theta \ \varphi]^T$ – generalized coordinates of vehicle body,

$\mathbf{q}_k^{(z)} = [x_k^{(z)} \ y_k^{(z)} \ z_k^{(z)} \ \delta_k]^T$ – generalized coordinates of k-th suspension (Fig. 2),

φ_k – angle of rotation of k-th wheel.

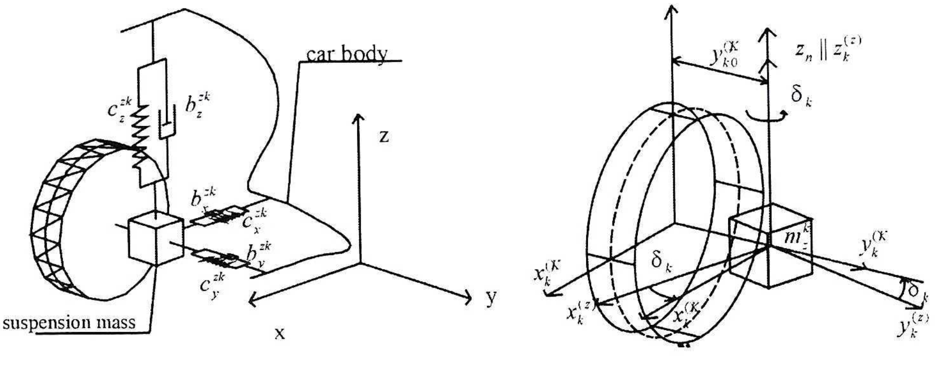

Fig. 2. Model of a flexible suspension and co-ordinate systems of suspension and wheel

The assumed simplified model of the steering system is presented in Fig. 3. The vector of the generalized coordinates of the steering system takes the form:

$$\mathbf{q}_s = \begin{bmatrix} \varphi_w \\ x_L \end{bmatrix} \tag{4}$$

where:

φ_w – rotation angle of steering wheel,

x_L – translation of steering rack.

The vector of reactions between subsystem I and II has the following components:

$$R_{rs} = \begin{bmatrix} F_1 \\ F_2 \\ F_t \end{bmatrix} \tag{5}$$

where:

F_1, F_2 – forces acting on the rack,

F_t – friction force between rack and mounting.

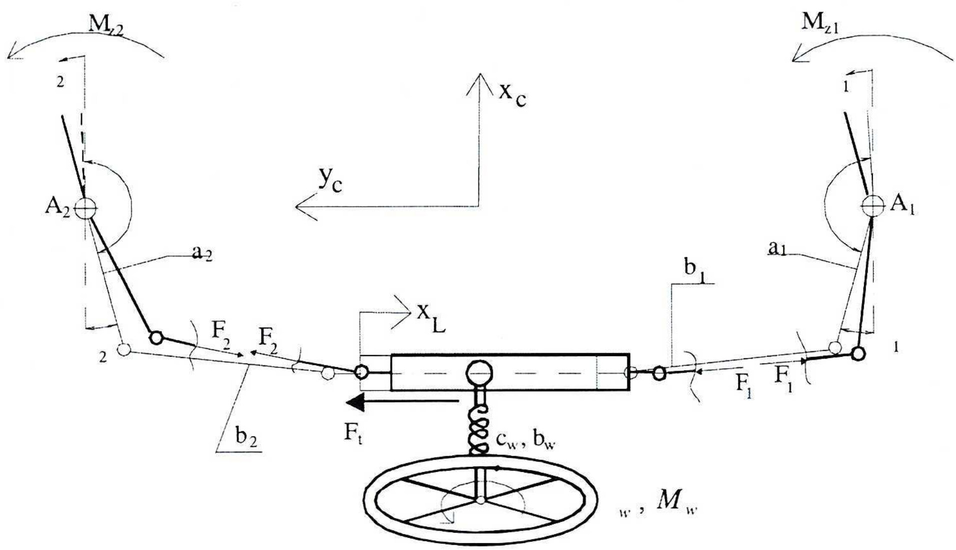

Fig. 3. Model of steering system

The external forces of reactions of road surface on wheel tires can be obtained using different models [2], [3], [11]. In our calculations, the Dugoff-Uffelmann model has been applied. This model takes into account tire slip as well as tires radial and circumferential flexibility.

The drive system of the vehicle rear wheels may have one of the forms presented in Fig. 4. Both models differ essentially in their functional principles, and therefore their mathematical models are different. It is assumed that a drive system is a series of rotating masses which are connected by means of elastic-damping elements. It could be a substitutional model of any system in which elements are subjected to torsional vibrations. In this way, different subsystems can be modeled, e.g. both independent vehicle drive systems and shafts of torsional flexibility. Interpretation of masses, transmission ratios, damping and stiffness parameters depends on the approach to the object modeled [12].

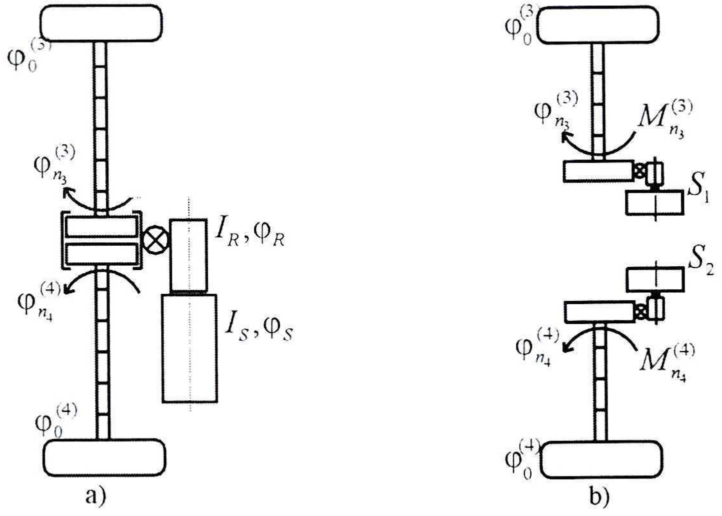

Fig. 4. Drives of vehicle rear wheels: a) common differential, b) independent

Kinetic and potential energy of elastic deformations and function of energy dissipation of the k-th driven axle shaft are expressed, respectively, by the following formula:

$$T^{(k)} = \frac{1}{2}\dot{\mathbf{q}}^{(k)^T}\mathbf{M}^{(k)}\dot{\mathbf{q}}^{(k)} \tag{6}$$

$$V^{(k)} = \frac{1}{2}\mathbf{q}^{(k)^T}\mathbf{C}^{(k)}\mathbf{q}^{(k)} \tag{7}$$

$$D^{(k)} = \frac{1}{2}\dot{\mathbf{q}}^{(k)^T}\mathbf{B}^{(k)}\dot{\mathbf{q}}^{(k)} \tag{8}$$

where: $\mathbf{q}^{(k)} = [\varphi_0^{(k)}, ..., \varphi_{nk}^{(k)}]^T$,

$\mathbf{M}^{(k)}$ – diagonal inertial matrix,

$\mathbf{C}^{(k)}, \mathbf{B}^{(k)}$ – three-diagonal matrices of stiffness and damping.

The vectors of generalized co-ordinates for both types of drive systems are described below.

A. Independent drive system

The drive system pertain to this case is presented in Fig. 5. In this case, the drive system has $n_A^{(d)} = 1 + n_3 + 1 + n_4$ degrees of freedom (i.e. angles $\varphi_0^{(3)}, ..., \varphi_{n_3}^{(3)}$ and $\varphi_0^{(4)}, ..., \varphi_{n_4}^{(4)}$), which are the components of the vector:

$$\mathbf{q}_d^{(A)} = [\mathbf{q}^{(3)} \; \mathbf{q}^{(4)})^T \tag{9}$$

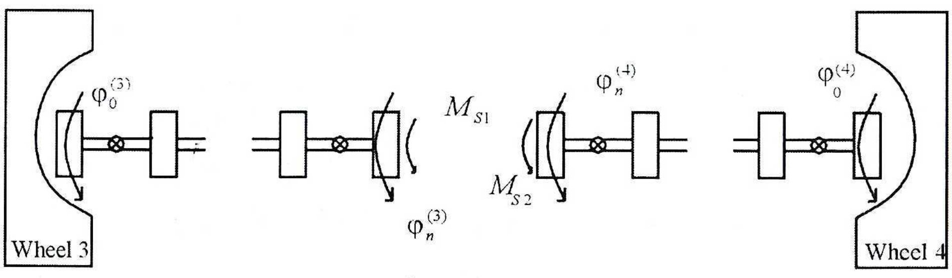

Fig. 5. Independent drive

B. Common drive with the differential

The system is presented in Fig. 6. The denotation is as follows:
M_R – reduced moment acting on the differential, φ_R – rotational angle of the differential, I_R – mass inertial moment of the differential, M_S – engine moment, φ_S – engine rotational angle, I_S – mass inertial moment of the engine, j_S – transmission ratio.

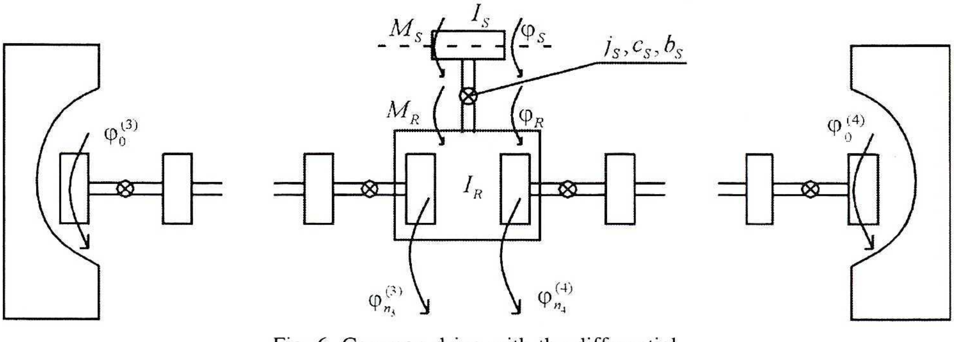

Fig. 6. Common drive with the differential

In this case, the drive system has $n_B^{(d)} = 1 + n_3 + 1 + n_4 + 2$ degrees of freedom and motion of the elements is described by means of the following vector:

$$\mathbf{q}_d^{(B)} = [\mathbf{q}^{(3)}\ \mathbf{q}^{(4)}\ \mathbf{q}^{(RS)}]^T \tag{10}$$

Equations of motion of the wheels and drives, regardless of the method of drive (in both cases A and B) are coupled by the moments M_3 and M_4 acting on the driven wheels (Fig. 7):

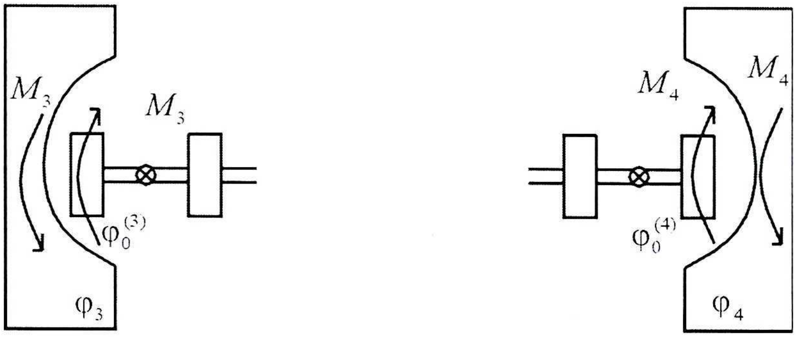

Fig. 7. Interaction of vehicle wheels and drive system

Moments M_3 and M_4 are unknown. The constraint equations have the following forms, respectively:

$$\dot{\varphi}_3 = \dot{\varphi}_0^{(3)}, \tag{11}$$

$$\dot{\varphi}_4 = \dot{\varphi}_0^{(4)}. \tag{12}$$

The problem is even more complicated in the case of drive mechanism B. In this case, the additional (unknown) variable may be the moment M_{TR} (Fig. 8) when friction is static:

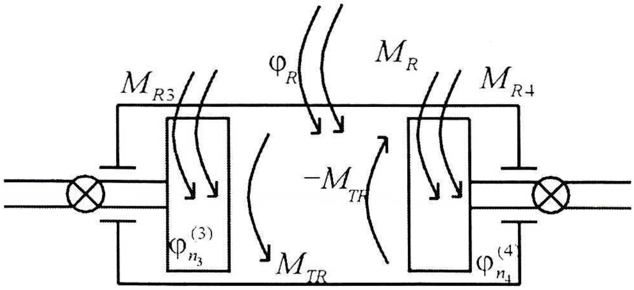

Fig. 8. Model of differential

The constraint equation in this case has the following form:

$$\dot{\varphi}_R = \frac{1}{2}\left[\dot{\varphi}^{(3)}_{n_3} + \dot{\varphi}^{(4)}_{n_4}\right] \tag{13}$$

Assuming that the differential is symmetric, the relationships can be written as:

$$M_{R3} = \frac{1}{2}M_R \tag{14}$$

$$M_{R4} = \frac{1}{2}M_R \tag{15}$$

If dry friction appears in the differential, then two cases should be considered:
1°. $w_R = 0$, static friction. The additional variable is the moment M_{TR} and the appropriate constraint equation has the form:

$$\dot{\varphi}^{(3)}_{n_3} = \dot{\varphi}^{(4)}_{n_4} \tag{16}$$

2°. $w_R = 1$, kinetic friction. In this case, the moment M_{TR} can be determined by the formula:

$$M_{TR} = \operatorname{sgn}\left[\dot{\varphi}^{(4)}_{n_4} - \dot{\varphi}^{(3)}_{n_3}\right] \cdot k \cdot |M_R| \tag{17}$$

where: k – coefficient of blocking of the differential.

The equations of motion for particular drive subsystems can be written as below. Equations of motion for the drive system are considered according to its type.

1°. <u>Type A</u>
Using the formula (9), we can obtain the equation:

$$\mathbf{M}^{(A)}_d \ddot{\mathbf{q}}^{(A)}_d + \mathbf{D}^{(A)}_d \mathbf{R}^{(A)}_d = \mathbf{F}^{(A)}_d, \tag{18}$$

where: $\mathbf{D}^{(A)}_d$ – coefficient matrix,
$\mathbf{R}^{(A)}_d$ – vector of reactions.

2°. Type B

$$\mathbf{M}_d^{(B)} \ddot{\mathbf{q}}_d^{(B)} + \mathbf{D}_d^{(B)} \mathbf{R}_d + \mathbf{D}_d^{(MR)} \mathbf{M}_{MR} = \mathbf{F}_d^{(B)}, \tag{19}$$

where $\mathbf{D}_d^{(B)}$, $\mathbf{D}_d^{(MR)}$ – coefficient matrices.
The equations (18) and (19) should be completed by the corresponding constraint equations. Depending on the type of the drive system, the equations have different forms:

- In case A of drive system – equations (11) and (12),
- In case B of drive system – equations (11)÷(15) and when static friction in differential occurs, equation (16).

In the computer program we applied Dugoff-Uffelmann's model of tire contact with road surface. In this the model, the following relations are assumed:

$$F_x^{'(k)} = \alpha_x^{(k)} F_z^{'(k)} \tag{20.1}$$

$$F_y^{'(k)} = \alpha_y^{(k)} F_z^{'(k)} \tag{20.2}$$

$$M_S^{(k)} = \alpha_S^{(k)} \left(F_x^{'(k)}, F_y^{'(k)}\right) \tag{20.3}$$

where coefficients $\alpha_x^{(k)}$, $\alpha_y^{(k)}$, $\alpha_S^{(k)}$ depend on wheel centre of mass velocity and wheel angular velocities (it allows us to take into account tire slipping) as well as wheel geometry, and tire radial and circumferential flexibilities. Detailed description of above dependencies is given in [7].

3. Results of numerical simulations

Based upon the mathematical models presented in section 2, a computer program was written. The program allows for numerical simulations of vehicle behaviour during different road tests to be performed. In the figures, the following denotations are used:

MR – vehicle which is driven by the classic drive system (differential),
NE A/B – vehicle which is driven by independent drives, ratio of moments in the motors is equal to A/B,
V – initial velocity of vehicle [km/h].

All of the tests were performed for the parameters (dimensions, masses) characteristic for a small passenger car. The most important parameters are listed in the table below.

Table 1.

Parameters of vehicle

	Parameter	Value
1	*Gross vehicle weight*	1250 [kg]
2	*Inertial moments of car body (Ix, Iy, Iz)*	304/886/882 [kgm^2]
3	*Rate of the suspension – front (linear characteristics)*	18000 [N/m]
4	*Rate of the suspension – rear (linear characteristics)*	20000 [N/m]
5	*Lateral/longitudinal rate of the suspensions (linear characteristics)*	5 10^6/5 10^6 [N/m]
6	*Wheel base*	~2,4 [m]
7	*Wheel track*	~1,2 [m]
8	*Circumferential stiffness of tyre*	100000 [N/m]
9	*Radial stiffness of tyre*	150000 [N/m]
10	*Differential blocking coefficient*	k = 0,08
11	*Coefficient of stiffness of steering column (linear characteristics)*	1500 [Nm/rd]

I. Stepwise input of steering wheel

The test consists in an impulse turn of the steering wheel by 80 [deg] in 0.2 [sec]. The courses of the input angle of steering wheel are presented in Fig. 9. The values of $\dot{\psi}$ (presented in the figure above) obtained for the velocity 40, 60, 80 and 100 [km/h] are different. However, the unstable/transient state fades, and the vehicle reaches a good stability of motion.

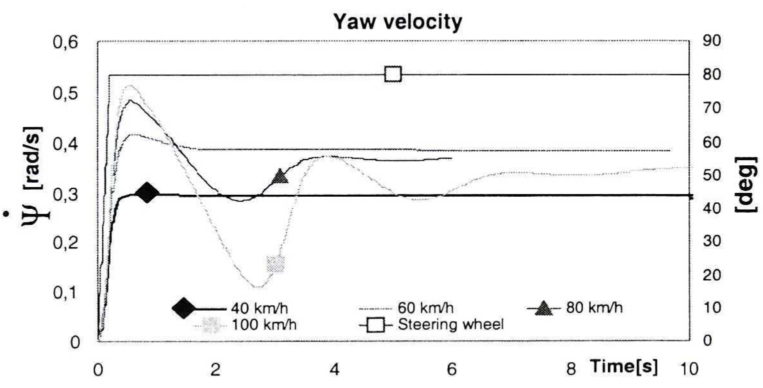

Fig. 9. Steering wheel input and angular velocity of the vehicle body for different velocities

II. Passing through an area of low surface adhesion

The results of simulation of vehicle motion over an area of low surface adhesion μ (Fig. 10) are presented below. The tests were performed for two values of velocity for both drive systems: independent and with differential (k – 0.08). Drive moments in the first case were symmetrical and constant, 200 [Nm] for each motor. In the second case, the drive moment was constant, too, but equal to 400 [Nm].

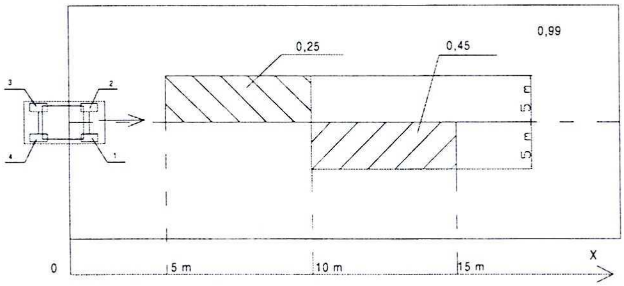

Fig. 10. Areas of different surface adhesion

The graphs presented below show particular parameters obtained for the moment of loss of adhesion (the wheels do not lose adhesion while travelling through the area with a coefficient $\mu = 0.45$).

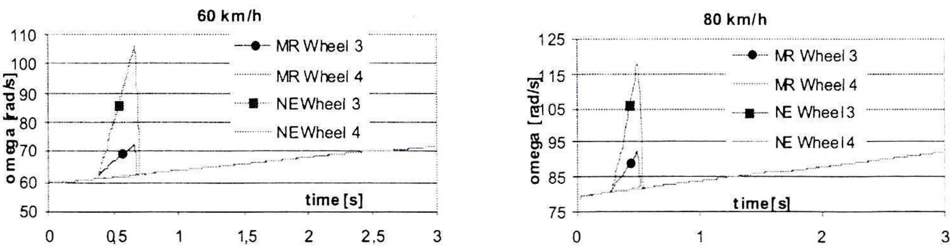

Fig. 11. Velocities of wheels driven during motion over surface with low adhesion

III. Vibrations of the drive system

Fig. 12 presents the time function of drive moments applied to electric motors (symmetric drive), which trigger the force impulse in the drive system. In the case of the vehicle with traditional drive, the drive moment has the same time function, but the value of the moment is twice as large as that presented in Fig. 12.

Numerical tests were performed for the straight motion of the vehicle for the model in which flexible axle shafts were divided into 4 rigid finite elements. The springs, whose stiffness coefficients were equal to 10^5 [Nm/rd], were placed between the elements. The inertia of elements was 0.01 [kgm^2]. All of the courses were determined for initial velocity 60km/h and for dry ground ($\mu = 0.9$).

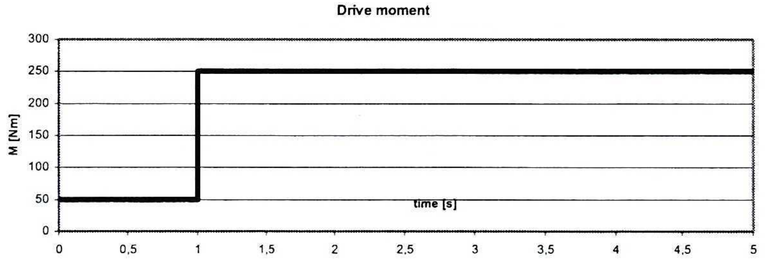

Fig. 12. Course of drive moment

Fig. 13 depicts the courses obtained for the following conditions: the vehicle with independent drive, different stiffness of axle shafts (left – stiffness 5^5 [Nm/rad], right – stiffness 10^5 [Nm/rad], respectively). In the figure, the symbol "L" denotes the ratio of angular velocity of the first finite element (from the drive side) to angular velocity of the last (fourth) finite element (from the wheel side) for the left axle shaft.

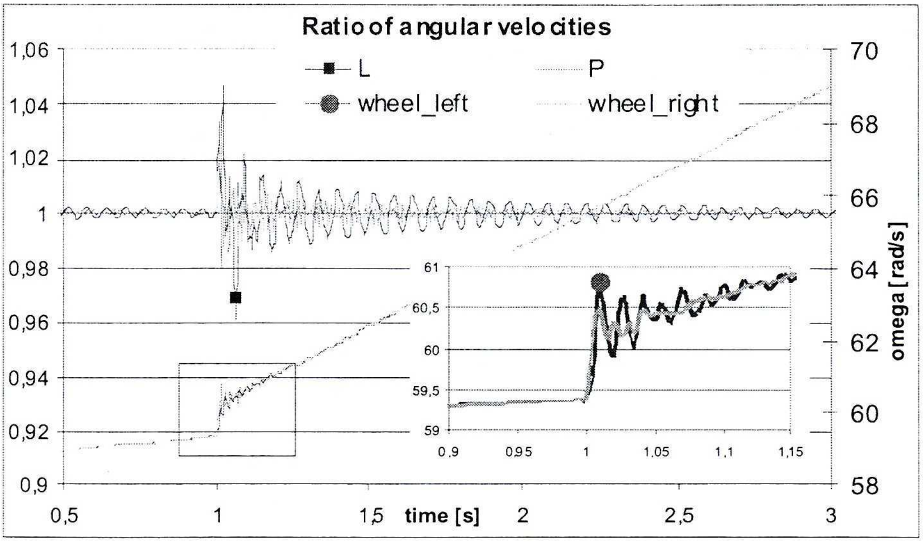

Fig. 13. Angular velocities in the drive system with different stiffness of axle shafts

For the right axle shaft, the similar angular velocity ratio is denoted by "P". The chart of velocities for left ("wheel_left") and right wheel ("wheel_right") is presented, too. It can be clearly seen that there are greater differences of velocities of particular elements in the axle shaft that is more flexible, and therefore greater torsion vibrations are expected. However, the stiffer axle shaft has greater eigen-frequencies of vibrations, which additionally can cause a beat effect. The vibrations of the axle shaft directly influence the velocities of the driven wheels (it is more significant on the side of the more flexible axle shaft).

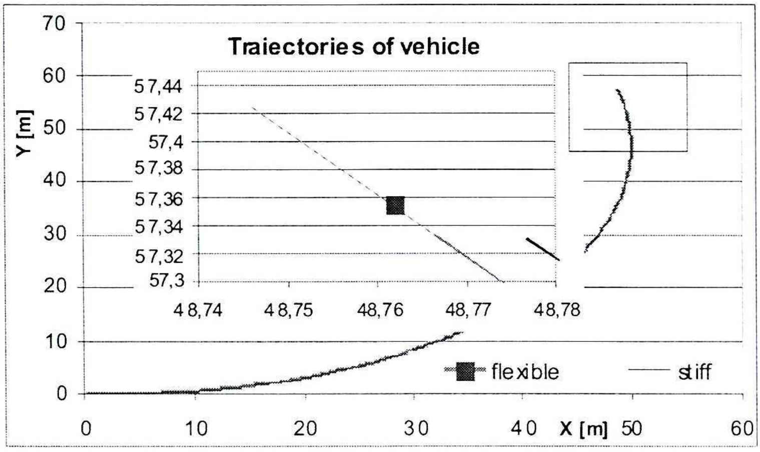

Fig. 14. Trajectories for vehicles with different drive systems

Comparative tests were also performed for the vehicles in which flexibility of the drive system was taken into account (each of the axle shafts were divided into 5 elements) and in the case when the vehicle axle shafts are fully stiff. The results of comparison of motion trajectories for both vehicles are presented in Fig. 14. As it can be seen, the resulting differences in the final position are in the range of few centimetres; therefore it can not be stated that flexibility of the drive system has a significant influence on vehicle motion. It does not have any greater influence on controllability of the vehicle either (test of impact (jerk) movement of the steering wheel, drive change as is in Fig. 12, independent drive system).

4. Final remarks and conclusions

The elaborated models and computer programs can be useful in the analysis of dynamics of vehicles during different road tests. In spite of the fact that, until now, any experimental verification has not been performed, it can be stated that the results obtained show qualitative correctness. They can be applicable in an introductory calculations aiming at the selection of constructional parameters of chassis and drive system. The model presented is complex since it has over thirty degrees of freedom, which has a significant influence on the duration of computer calculations. An additional drawback is that the vibration frequencies of the elements of the drive system are high, which makes it necessary to use a small step of integration. Further research should focus on preparation of special algorithms for the control of independently driven electric motors to ensure the optimal utilisation of vehicle motion properties [6]. Taking into account flexibilities of the drive system, which does not have a significant influence on the vehicle trajectory and other general dynamic characteristics, may have a considerable inpact in the optimization process when the vehicle moves at the limit if adhesion. Small differences arising from introducing the flexible drive system can be crucial for vehicle characteristics. While passing over road unevenness, especially on curves, extensive vibrations of the drive system can occur, this may cause loss of motion stability. The presented mathematical and computer models of the vehicle make it possible to analyse these phenomena, too.

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Model do symulacji dynamiki pojazdu

Streszczenie

W pracy przedstawiono model pojazdu osobowego wraz z układami: przeniesienia napędu oraz układem kierowniczym. Rozważono dwa możliwe układy napędowe: z klasycznym mechanizmem różnicowym oraz niezależny napęd na każde koło poprzez silniki elektryczne. W obu przypadkach uwzględniono podatność półosi napędowych. Model pojazdu jest przestrzenny, stanowi układ ciał sztywnych połączonych ze sobą elementami sprężysto– tłumiącymi. W układzie kierowniczym i napędowym uwzględniono fazy tarcia statycznego i kinetycznego. Do opisu matematycznego wykorzystano metodę przekształceń jednorodnych. Przedstawiono wyniki symulacji komputerowych wykonane przy pomocy programu komputerowego napisanego na podstawie przedstawionych modeli matematycznych.