GEODEZJA I KARTOGRAFIA
t. XLIX, z. 1, s. 3 – 6 (2000)

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A Proof of the Chebyshev theorem

The article presents a proof of one of the elementary theorems of mathematical cartography, pertaining to the issues of optimization of distribution of deformations in cartographic projections of a limited area. The Chebyshev theorem provides a basis for prospecting and determination of projecting formulas enabling minimization of oscillation of local length scale in a conformal projection of a given area.

Let S be a regular area of C^3 class determined by equation $\vec{r}=\vec{r}(u, v)$. Let us assume that parametric grid u=const and v=conts is an orthogonal one and let us present the first square form of the area in the following way:

$$ds^2=Edu^2+Gdv^2$$

In differential geometry [1] it has been proved that if so, the Gauss curve might be defined by the following formula:

$$K=-\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)+\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right] \tag{1}$$

After an isothermic grid has been formed upon area S, the first square form of the area may be presented in the following form:

$$ds^2=\mu^2(u, v)(du^2+dv^2) \tag{2}$$

Once quantities $E=G=\mu^2$ have been substituted for (1), the following will be obtained:

$$K=-\frac{1}{2\mu^2}\left[\frac{\partial}{\partial v}\left(2\frac{\mu_v}{\mu}\right)+\frac{\partial}{\partial u}\left(2\frac{\mu_u}{\mu}\right)\right]$$

or

$$K=-\frac{1}{\mu^2}\left[\frac{\partial^2 \ln\mu}{\partial v^2}+\frac{\partial^2 \ln\mu}{\partial u^2}\right] \tag{3}$$

Let us mark

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \tag{4}$$

The following will be obtained:

$$\Delta \ln \mu = -K\mu^2 \tag{5}$$

If $K > 0$ for each $(u,v) \in S$, then $\Delta \ln \mu < 0$, which means that $\ln \mu$ is a superharmonic function [2].

Let $x + iy = f(u + iv)$ define any conformal projection of area S upon plane XOY. If so, local scale m of such a projection is known to be given by means of the following formula:

$$m = \frac{|f'|}{\mu} \tag{6}$$

Let area D located upon plane XOY be the image of surface S in this projection. Let us assume that on edge L of area D scale $m = 1$. Thus dependence (6) implies that edge L of area D the following equality comes about:

$$\ln \mu = \ln |f'| \tag{7}$$

It has been proved before that if $K > 0$, $\ln \mu$ is a superharmonic function. This implies that — in accordance with definition of superharmonic function — within entire area D including the edges the following inequality comes about:

$$\ln \mu \geqslant \ln |f'| \tag{8}$$

As it is, $\ln |f'|$ is known to be a (—) harmonic function. This means that within area D the following inequality comes about:

$$\ln m \leqslant 0 \tag{9}$$

The conclusion thus implied might be formulated in the form of a theorem.

Theorem. If upon surface S the Gauss curve is positive, and if on edge L of area D scale $m = 1$, inside area D the following inequality comes about:

$$m \leqslant 1 \tag{10}$$

Now let us proceed to proving the Chebyshev theorem, which is one of elementary ones in mathematical cartography.

The Chebyshev theorem (Chebyshev criterion).

Within a set of conformal projections upon the plane of surface S whose Gauss curve is of a constant sign ($K>0$ or $K<0$), the smallest oscillation of scale logarithm within given area D is characteristic of the projection whose scale upon edge L of area D is constant.

Proof

Generality of discussion not having been decreased, it might be taken for granted that over entire surface S is $K>0$. Analogously it might be assumed that upon edge L of area D scale $m=1$.

Let conformal projection $F: S \rightarrow D$ be determined by relationship $x+iy=f(u+iv)$. Let us assume that in such a projection scale m upon edge L of area D equals to 1. Let $g: S \rightarrow D$ any conformal projection. The scale m^* is of the following form:

$$m^*=\frac{|g'|}{\mu} \tag{11}$$

Let us mark by ρ the biggest value of $\ln m^*$ within area D, i.e.:

$$\rho=\sup_D \ln m^* \tag{12}$$

Note that functions $\ln m^*$ and $\chi=\ln m^*-\rho$ have the same oscillation within area D. Also note that function $\ln m$ inside area D assumes negative values, while on edge L area D it equals to zero. Function χ inside area D assumes negative values, while it equalts to zero on the subset of edge L.

Let us mark by ω oscillation of a function determined in set $D \cup L$. The following will be obtained:

$$\omega(\ln m)=-\inf_D \ln m \tag{13}$$

and

$$\omega(\ln m^*)=\omega(\chi)=-\left[\inf_D (\ln m^*-\rho)\right\} \tag{14}$$

Now let us consider function

$$\delta=\chi-\ln m \tag{15}$$

Then the following will be obtained:

$$\delta=(\ln|g'|-\ln\mu-\rho)-(\ln|f'|-\ln\mu)=\ln|g'|-\ln|f'|-\rho$$

Function χ upon edge L of area D assumes non-positive values. Whereas upon edge L of area D we have $\ln m=0$. Thus function δ upon edge L of area D assumes non-positive values. Function δ is a harmonic one, therefore — on the basis of the maximum rule for harmonic functions — it reaches in set $D \cup L$ its boundaries (upper and lower) on the edge of area D. This means that over entire area D the following inequality comes about $\delta \leqslant 0$ i.e.

$$\chi \leqslant \ln m \tag{17}$$

One the basis of (13) and (14) it might be inferred that the following inequality comes about:

$$\omega(\ln m^*) \geqslant \omega(\ln m)$$

What should be proved.

REFERENCES

[1] Biernacki M.: *Geometria różniczkowa*, part II, PWN, Warszawa 1955
[2] Leja F.: *Teoria funkcji analitycznych*, PWN, Warszawa 1957.

Received December 17.1999
Accepted March 21.2000

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Dowód twierdzenia Czebyszewa

S t r e s z c z e n i e

W przedstawionej pracy przeprowadzono rozumowanie wykazujące, że w odwzorowaniu konforemnym ograniczonego obszaru wartość oscylacji skali jest najmniejsza gdy skala na brzegu obszaru jest stała. W dowodzie przyjęto, że lokalna skala długości w odwzorowaniu konforemnym na brzegu obszaru jest równa jedności. W praktycznych poszukiwaniach formuł aproksymujących odwzorowanie Czebyszewa wskazane jest, dla minimalizacji odchyleń modułu skali od jedności, przyjąć na brzegu skalę równą połowie wyznaczonej maksymalnej oscylacji, jaka wystąpi w okolicach środka obszaru.

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Доказательство теоремы Чебышева

Р е з ю м е

В представленной роботе проведено рассуждение показывающее, что в конформном отображении ограниченной области величина колебаний масштаба является самой малой, когда масштаб на грамице области постоянный. В доказательстве принято, что местный масштаб длины в конформном отображении на границе области является равным единицы. В практических поисках формул апроксимирующих отображение Чебышева рекомендуется, для минимализирования отклонений модула масштаба от единицы, принять на границе масштаб равный половине определенного максимального колебания, которое происходит в районе середины области.