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Further developments in PAC method of adjustment of geodetic networks

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Abstract: The paper presents a general concept of geodetic observations adjustment based on application of the Edgeworth' series and the principle of an alternative choice. The Edgeworth' series approximates the empirical distribution of measurement errors and gives an opportunity to modify the empirical characteristics of errors distribution. The method of estimation used is based on the principle of the alternative choice. Its natural robustness for outliers was the basis for newly created method called PAC-E. The paper presents the algorithm and some numerical tests that were carried out to compare the results of the proposed method with the results of the classical LS adjustment. Special attention was paid on the effect of non-zero excess and robustness of the proposed method.

Keywords: Edgeworth' series, robust estimation, the principle of an alternative choice, outliers, excess, asymmetry

1. Introduction

One of the main directions in development of theory of adjustment of geodetic observations is the elaboration of a method that can utilise information on probabilistic properties of measurement errors and also be robust for outliers. Analyses of error properties and their structure were the basis for development of new models of geodetic network. Former research suggest the existence of some anomalies concerning main parameters in empirical distribution of measurement errors. Some researchers (e.g. Szacherska, 1974) point out non-zero values of empirical excess that cannot be neglected and can influence final results of adjustment, especially variance estimators. Thus it is important to analyse observation data and then to apply the most suitable mathematical model of empirical measurement errors.

Some authors (Cymerman, 1991; Dumalski, 1995; Dumalski and Wiśniewski, 1994; Dumalski and Wiśniewski, 1995) proposed to replace the set of possible distributions R^{α} of measurement errors with an approximating function that could represent a density function. The Edgheworth' series, which describes the distribution of empirical measurement errors, could be such a function. This probabilistic model gives an opportunity to consider some anomalies of empirical distribution of geodetic observations. Those distribution anomalies usually concern anomalies of the parameters such as the expected value, empirical asymmetry and the excess. It is a well known fact that if the maximum likelihood method ML (notified earlier as NW) is used then the obtained estimates depend on the assumed distribution of measurement errors. Thus a new method, called ML-E (notified earlier as NW-E), was created by combining the ML method with the Edgeworth' series. It is an alternative for the earlier proposed methods: PD (notified earlier as RP) (Wiśniewski, 1986) and ML-MND (notified earlier as NW-MRN) (Cymerman, 1991), where some Pearson's or modified normal distributions were taken as the probabilistic models of measurement errors.

Robustness for outliers is also a very important problem of adjustment of geodetic observations, especially when data acquisition and measurement processes are highly automated. Methods elaborated so far, can be divided into two groups: the passive methods and the active methods (Kamiński, 1990). Passive methods are based on some statistical tests that can detect and eliminate outliers. Active methods eliminate not the outliers themselves but their influence on adjustment results. Those methods can be divided into three subgroups:

– methods formulated on the basis of probabilistic models of measurement errors,
– methods with modified characteristic functions,
– other robust methods.

Methods like PD or ML-MND should be classified to the first subgroup. Robustness of those methods is the effect of target function modifications and is obtained just "by the way". Huber's, Hampel's or Danish methods are good examples of the methods belonging to the second subgroup. Their robustness is obtained by modification of other characteristic functions, usually a weight function. The third subgroup consists of methods, e.g. PAC or MAD (Minimum Absolute Differences method) that cannot be classified to the first two subgroups.

Most of the presented active methods can also be classified as those belonging to the class called M-estimation (the class especially well known in mathematics). M-estimators are derived from characteristic function modification, usually with strong relationship to influence function (Hampel et al., 1986). The problem of robustness for gross errors related to the M-estimation class was the main task of many papers of the last decades (e.g. Yang, 1991; 1997; Caspary and Hean, 1990). Some authors (Xu, 1989; Yang, 1994) tried also to generalize robust methods to the case of observations-dependent.

The robust method based on the principle of an alternative choice (PAC) is particularly suitable for adjustment of observations when gross errors are present. The specific structure of its target function, that consists of distributions of measurement errors, made possible to create a new method of adjustment called PAC-E (Dumalski and Wiśniewski, 1994). This paper presents new development of the PAC-E method and shows its general practical properties.

2. The principle of an alternative choice (PAC)

The principle of an alternative choice was proposed by Kadaj (1978, 1980). The basis of it was the assumption that likelihood of different values of the parameter vector **X** (when vector **x** is observed) is a sum of conditional probabilities $P(\mathbf{x}/\mathbf{X})$, contrary to the classic method of maximum likelihood where this likelihood is a multiplication of those conditional probabilities. Generally, when measurement errors are mutually independent, the presented idea leads to the following adjustment criterion

$$\sum_{i=1}^{n} f\left(\varepsilon_i\left(\mathbf{X}\right)\right) = \max \tag{1}$$

where $f(\varepsilon_i(\mathbf{X}))$ is a density function of measurement error distribution or any proportional function.

The criterion (1) can be treated as the consequence of a choice of the best sum of random events from the infinite solutions of the **X** vector and the correction vector, which values are of course dependent on **X**. Interpreting the criterion, the $\hat{\mathbf{X}}$ vector is chosen within the interval of largest number of observations, aside from the fact whether there is any outlier or not. This is an advantage of the PAC method as compared to the classic maximum likelihood method, where the "multiplicational" character of the likelihood function disturbs adjustment results when the observational vector includes any outlier. Therefore, the following form of the target function of the PAC method was assumed:

$$L^{PAC}(\mathbf{X}) = \sum_{i=1}^{n} d^* f^*\left(\varepsilon_i; (\mathbf{X})\right) \tag{2}$$

where d^* is a positive coefficient, and $f^*(\varepsilon_i; (\mathbf{X})) = \exp\left[-\dfrac{\varepsilon_i^2}{2\sigma_i^2}\right]$ is a function proportional to the normal density function.

The basic properties of the PAC method, including robustness to outliers, issue from properties of the characteristic functions, i.e. influence, weight and rigour functions (Cymerman, 1991). According to (2) and taking the form of $f^*(\varepsilon)$ one can express as follows the component of the target function

$$l^{PAC}(\varepsilon) = d^* \exp\left[-\frac{\varepsilon^2}{2\sigma^2}\right] \tag{3}$$

the influence function

$$\varphi^{PAC}(\varepsilon) = \frac{dl^{PAC}(\varepsilon)}{d\varepsilon} = \frac{\varepsilon}{\sigma^2} \exp\left[-\frac{\varepsilon^2}{2\sigma^2}\right] \tag{4}$$

the weight function

$$w^{PAC}(\varepsilon) = \frac{dl^{PAC}(\varepsilon)}{d\varepsilon^2} = \frac{1}{2\sigma^2}\exp\left[-\frac{\varepsilon^2}{2\sigma^2}\right] \tag{5}$$

and the rigour function

$$u^{PAC}(\varepsilon) = \frac{d^2 l^{PAC}(\varepsilon)}{d\varepsilon^2} = \frac{\sigma^2 - \varepsilon^2}{\sigma^4}\exp\left[-\frac{\varepsilon^2}{2\sigma^2}\right] \tag{6}$$

The rough graphs of the listed functions are shown in Figure 1.

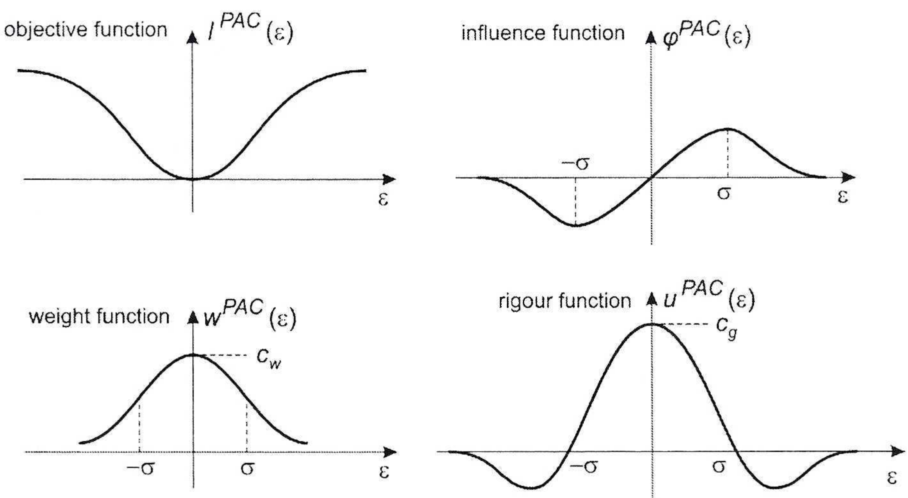


Fig. 1. The characteristic functions of the PAC method

Analysing graphs in Figure 1, according to Kadaj's classification (Kadaj, 1988) it can be written that

(i) $w^{PAC}(\varepsilon)\langle c_w \Rightarrow l^{PAC}(\varepsilon) \in K_-$,

(ii) $u^{PAC}(\varepsilon)\langle 0$ when $\varepsilon \in (-\infty; -\sigma\rangle \cup \langle\sigma; \infty) \Rightarrow l^{PAC}(\varepsilon) \in K_-^{(2)}$.

It is worth to point out that for each $\varepsilon \neq 0$ stands $w^{PAC}(\varepsilon) < \left(w^{PAC}(0) = c_w\right)$. It means that the PAC method is a robust estimation for the whole domain of ε.

3. Edgeworth' series

The Edgeworth' series is usually given in the following form

$$\begin{aligned} f^E(\varepsilon) = f^{ND}(\varepsilon) - \frac{1}{3!}\gamma_1\frac{d^3 f^{ND}(\varepsilon)}{d\varepsilon^3} + \frac{1}{4!}\gamma_2\frac{d^4 f^{ND}(\varepsilon)}{d\varepsilon^4} + \frac{10}{6!}\gamma_1^2\frac{d^6 f^{ND}(\varepsilon)}{d\varepsilon^6} + \\ + \frac{10}{5!}\gamma_1\frac{d^5 f^{ND}(\varepsilon)}{d\varepsilon^5} - \frac{35}{7!}\gamma_1\gamma_2\frac{d^7 f^{ND}(\varepsilon)}{d\varepsilon^7} - \frac{280}{9!}\gamma_1^3\frac{d^9 f^{ND}(\varepsilon)}{d\varepsilon^9} \end{aligned} \tag{7}$$

where $f^E(\varepsilon)$ is a function that represents the Edgeworth' series, $f^N D(\varepsilon)$ is the normal density function, and γ_1, γ_2 are the coefficients of asymmetry and the excess, respectively. The series (7) can also be presented in the form

$$f^E(\varepsilon) = f^{ND}(\varepsilon)\left[1 + \sum_1^6 h_i\right] \tag{8}$$

where

$$h_1 = -\frac{1}{6}\gamma_1 g_3, \quad h_2 = \frac{1}{24}\gamma_2 g_4, \quad h_3 = \frac{1}{72}\gamma_1^2 g_5,$$
$$h_4 = \frac{1}{12}\gamma_1 g_6, \quad h_5 = -\frac{1}{144}\gamma_1\gamma_2 g_7, \quad h_6 = -\frac{1}{1296}\gamma_1^3 g_9,$$

and

$$\begin{gathered} g_3 = 3\varepsilon - \varepsilon^3, \quad g_4 = 3 - 6\varepsilon^3 + \varepsilon^4, \quad g_5 = -15\varepsilon + 10\varepsilon^2 - \varepsilon^5, \\ g_6 = -15 + 45\varepsilon^2 - 15\varepsilon^5 + \varepsilon^6, \quad g_7 = 105\varepsilon - 105\varepsilon^3 + 21\varepsilon^5 - \varepsilon^7, \\ g_9 = -45\varepsilon + 1260\varepsilon^3 - 378\varepsilon^5 + 36\varepsilon^6 - \varepsilon^9. \end{gathered} \tag{9}$$

4. The PAC-E method

The original form of the target function of the PAC-E method is as follows

$$L^{PAC}(\mathbf{X}) = \sum_{i=1}^{n} f^{ND}(\varepsilon_i; (\mathbf{X})) \tag{10}$$

The normal density f^{ND} can be replaced by another density function. If additionally one assumes $R \xrightarrow{A} R^E$ then

$$L^{PAC-E}(\mathbf{X}) = \sum_{i=1}^{n} f^E(\varepsilon_i; (\mathbf{X})) \tag{11}$$

where $\mathbf{R}^\alpha[\mathbf{\Theta}^\alpha, \mathbf{X}] \in R$ means a distribution belonging to the set $R = \{\mathbf{R}^\alpha : \alpha \in T^\alpha\}$ of potential probabilistic distributions of measurement errors ($\mathbf{\Theta}^\alpha, \mathbf{X}$ are their parameters) (Wiśniewski, 1986). The distribution is represented by the density function and the relation $f^\alpha(\varepsilon) \xrightarrow{A} f^E(\varepsilon)$.

The properties of the presented method can be derived from the properties of its characteristic functions. Detailed graphs of those functions were presented in (Dumalski, 1995).

Now, let us recall some general conclusions. The graphs of all characteristic functions are generally similar to the graphs shown in Figure 1 (the original PAC method). Thus the properties of the presented method should be identical with the ones of the

PAC method, at least in terms of robustness. The effect of excess is more significant when the influence of observations in the closest vicinity of the centre of distribution on adjustment results is larger. That effect decreases when the error gets bigger. In turn, non-zero values of asymmetry make distribution asymmetric but it does not affect the robustness of the method.

5. Adjustment procedure

The adjustment procedure that applies distributions different than normal can generally be presented in the following way (Kadaj, 1980)

$\mathbf{x} = \mathbf{AX} + \varepsilon$ – the functional model of a geodetic network,

$\mathbf{x} \sim \mathbf{R}^{\alpha}\left[\Theta^{\alpha}, \mathbf{X}\right]$ – the probabilistic model, (12)

$\max\limits_{\mathbf{X}} L^{PAC}(\mathbf{X}) = \max\limits_{\mathbf{X}} \sum\limits_{i=1}^{n} f_x^{\alpha}(\mathbf{X}; x_i)$ – the adjustment criterion stemmed from the PAC method application, where

$\mathbf{x} \in \mathrm{M}_{(n,1)}$ – the vector of measurements or residuals,
$\mathbf{A} \in \mathrm{M}_{(n,u)}$ – the design matrix,
$\mathbf{X} \in \mathrm{M}_{(u,1)}$ – the vector of parameters of the functional model of a network,
$\varepsilon \in \mathrm{M}_{(n,1)}$ – the vector of measurement errors,
$\mathrm{M}_{(a,b)}$ – the set of matrices of dimensions (a, b),

Assuming that

$$\left(R = \{\mathbf{R}^{\alpha} : \alpha \in \mathbf{T}_{\alpha}\} \xrightarrow{A} R^{E}\right) \Rightarrow \left(\forall i : f_x^{\alpha}(\mathbf{X}; x_i)_{\alpha \in T_{\alpha}} \stackrel{A}{=} f_x^{E}(\mathbf{X}; x_i)\right) \tag{13}$$

the adjustment algorithm changes to the form

$$\begin{aligned} &\mathbf{x} = \mathbf{AX} + \varepsilon \\ &\max_{\mathbf{X}} L^{PAC-E}(\mathbf{X}) = \max_{\mathbf{X}} \sum_{i=1}^{n} f_x^{E}(\mathbf{X}; x_i) = \max_{\mathbf{X}} f^{*}(\varepsilon)\left[1 + \sum_{1}^{6} h_i\right] \end{aligned} \tag{14}$$

Any optimisation method, e.g. Newton's method, can be used to find such $\hat{\mathbf{X}} \in X_0$ (X_0 is a set of possible solutions) that satisfies $\max\limits_{\mathbf{X}} L^{PAC-E}(\mathbf{X}) = L^{PAC-E}\left(\hat{\mathbf{X}}\right)$. The following scheme shows the procedure for Newton's optimisation method

$$\begin{aligned} &\mathbf{X}^{j+1} = \mathbf{X}^{j} + \tau\left[\mathbf{Q}\left(\mathbf{X}^{j}\right)\right]^{-1} \mathbf{g}(\mathbf{X})^{j} \\ &\varepsilon^{(j+1)} = \mathbf{x} - \mathbf{AX}^{(j+1)} \end{aligned} \tag{15}$$

where τ is a coefficient that improves the convergence of the method, $\mathbf{g}(\mathbf{X})$ is the gradient of the target function, i.e.

$$\mathbf{g}(\mathbf{X}) = \left[\frac{\partial}{\partial \mathbf{X}} L^{PAC-E}(\mathbf{X})\right]^T = \left[\frac{\partial}{\partial \varepsilon} L^{PAC-E}(\mathbf{X}) \frac{\partial}{\partial \mathbf{X}} \varepsilon(\mathbf{X})\right]^T = \mathbf{A}^{\mathrm{T}}\mathbf{B}(\mathbf{X}) \tag{16}$$

where $[\mathbf{B}(\mathbf{X})]_i = f^*(\varepsilon_i)\left[\varepsilon_i\left(1 + \sum_1^6 h_i\right) + \sum_1^6 h_i'\right]$

The Hessian of the same function is as follows

$$\mathbf{Q}(\mathbf{X}) = \frac{\partial^2}{\partial \mathbf{X} \partial \mathbf{X}^T} L^{PAC-E}(\mathbf{X}) = -\mathbf{A}^{\mathrm{T}}\mathbf{D}(\mathbf{X})\mathbf{A} \tag{17}$$

where $\mathbf{D}(\mathbf{X})$ is a diagonal matrix with the following elements on the diagonal

$$[\mathbf{D}(\mathbf{X})]_{ii} = f^*(\varepsilon_i)\left[(\varepsilon_i^2 - 1)\left(1 + \sum_1^6 h_i\right) + 2\varepsilon_i \sum_1^6 h_i' + \sum_1^6 h_i''\right]$$

The sufficient condition for $L^{PAC-E}(\hat{\mathbf{X}}) = \max_{\mathbf{X}} L^{PAC-E}(\mathbf{X})$ is that the Hessian $\mathbf{Q}(\mathbf{X})$ is negative definite. This is true when the matrix $\mathbf{D}(\mathbf{X})$ is positive definite. Taking into consideration the rigour function of the PAC-E method, it is easy to find out, that not every element of that matrix is positive. Thus the Hessian must be checked whether it is negative or positive definite during the iterative process. That test can be done automatically when applying the Banachiewicz-Cholesky scheme. When many outliers occur in the intervals where the rigour function has negative values, sometimes the Hessian will become positive definite in the very first iterative steps. For that reason Kadaj (1980) proposed to enlarge the interval of positive values of the matrix $\mathbf{D}(\mathbf{X})$ by overstating the standard deviation. It can be done especially in the first iterative steps, while later on, the proper value of σ should be restored.

6. Numerical tests

The PAC-E method was tested using the following, simulated level network (Fig. 2).

The set of observations was divided into two parts to test how distribution anomalies affect adjustment results. Each part contained four fixed and five new points (being under adjustment); one fixed and two new points were common for both subsets. The same value of the standard deviation σ but different excess γ_2 were assumed for the subsets. Zero value of γ_2 was assumed for the first subset while for the second – observations were simulated in such a way that $\gamma_2 \neq 0$. Four variants of observations were generated with the parameters shown in Table1. For all variants it was assumed that $\gamma_1 = 0$ (no asymmetry) and that all theoretical heights of the points were equal to zero.

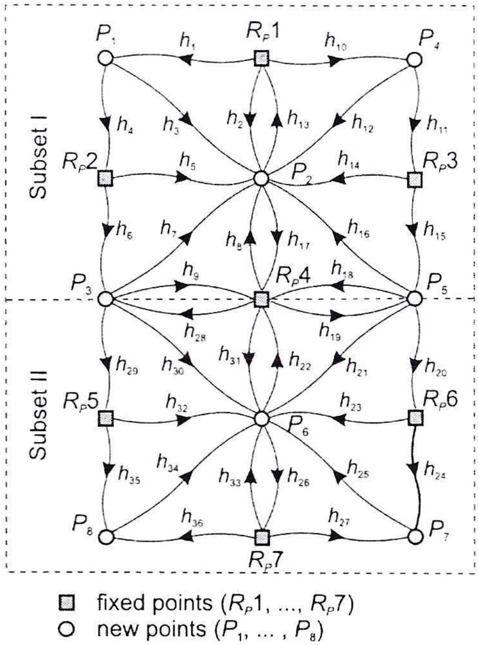


Fig. 2. The tested network

Table 1.

Variant	1	2	3	4
Subset I	$\sigma = 3.06$ $\gamma_2 = 0.00$	$\sigma = 3.06$ $\gamma_2 = 0.00$	$\sigma = 3.06$ $\gamma_2 = 0.00$	$\sigma = 3.06$ $\gamma_2 = 0.00$
Subset II	$\sigma = 3.06$ $\gamma_2 = 0.50$	$\sigma = 3.06$ $\gamma_2 = 1.01$	$\sigma = 3.06$ $\gamma_2 = 1.51$	$\sigma = 3.06$ $\gamma_2 = 0.29$

The tests were aimed to compare the results of the least squares adjustment (LS) with the results of the proposed PAC-E method and to find how the non-zero excess affected the results. The analyses were based on the comparison of the following norms: $\|\hat{\mathbf{X}} - \mathbf{X}^t\|_2 = \|\hat{\mathbf{X}}\|_2$ (taking $\mathbf{X}^t = \mathbf{0}$) – including all the network points; $\|\hat{\mathbf{X}}_I\|_2$ – including the points from the subset I; $\|\hat{\mathbf{X}}_{II}\|_2$ – including the points of the subset II; and $\|\hat{\mathbf{X}}_p\|_2$ – including the points belonging to both subsets. The PAC-E method seemed very robust to outliers, thus an additional test was carried out to confirm this property. Each subset was contaminated firstly with one and later with two outliers. Those errors were placed in observations h_{36} (subset II). The test results are shown in Table 2.

The values of the norms $\|\hat{\mathbf{X}}\|_2$ and $\|\hat{\mathbf{X}}_{II}\|_2$ are shown in Figures 3, 4, 5, 6, 7 and 8 to complete the analyses.

Robustness of the method is especially visible in Figures 9, 10, 11, 12, 13, 14, 15, and 16.

Table 2.

	Variant	Method	Parameter vector $\hat{\mathbf{X}}$								$\lVert\hat{\mathbf{X}}\rVert_2$	$\lVert\hat{\mathbf{X}}_I\rVert_2$	$\lVert\hat{\mathbf{X}}_{II}\rVert_2$	$\lVert\hat{\mathbf{X}}_p\rVert_2$
			$\hat{X}_1$	$\hat{X}_2$	$\hat{X}_3$	$\hat{X}_4$	$\hat{X}_5$	$\hat{X}_6$	$\hat{X}_7$	$\hat{X}_8$				
$\gamma_1 = 0$ $\gamma_2 = 0.5$	No outliers	LS	–0.26	–0.84	0.25	2.10	0.22	–0.41	0.58	1.17	2.68	2.28	1.37	0.33
		PAC-E	–1.02	–0.86	–0.34	1.54	0.12	–0.37	0.50	0.60	2.24	2.04	0.86	0.36
	1 outlier	LS	–0.26	–0.83	0.29	2.11	0.26	–0.17	0.66	3.38	4.15	2.28	3.45	0.39
		PAC-E	–1.02	–0.86	–0.34	1.54	0.12	–0.36	0.50	0.77	2.29	2.04	0.99	0.36
	2 outliers	LS	–0.26	–0.82	0.33	2.11	0.30	0.04	0.73	5.39	5.91	2.28	5.44	0.44
		PAC-E	–1.02	–0.86	–0.34	1.54	0.12	–0.36	0.50	0.89	2.33	2.04	1.08	0.36
$\gamma_1 = 0$ $\gamma_2 = 1.01$	No outliers	LS	0.48	–0.02	–0.33	1.35	–0.35	0.55	0.17	1.11	1.96	1.44	1.25	0.48
		PAC-E	0.43	0.30	0.13	1.14	–0.11	0.22	0.08	0.43	1.36	1.26	0.49	0.17
	1 outlier	LS	0.48	–0.01	–0.30	1.36	–0.31	0.78	0.25	3.18	3.61	1.44	3.28	0.43
		PAC-E	0.43	0.30	0.13	1.14	–0.11	0.23	0.09	0.50	1.39	1.26	0.56	0.17
	2 outliers	LS	0.49	0.00	–0.26	1.36	–0.27	1.00	0.32	5.26	5.56	1.44	5.36	0.37
		PAC-E	0.43	0.30	0.13	1.15	–0.11	0.24	0.09	0.59	1.43	1.26	0.64	0.17
$\gamma_1 = 0$ $\gamma_2 = 1.51$	No outliers	LS	–0.31	–0.99	–0.42	2.06	–0.48	0.39	1.64	–0.64	2.99	2.30	1.80	0.64
		PAC-E	–1.07	1.04	–0.74	1.45	–0.92	0.28	0.99	–0.40	2.63	2.08	1.10	1.18
	1 outlier	LS	–0.31	–0.98	–0.38	2.06	–0.44	0.61	1.71	1.43	3.32	2.30	2.31	0.59
		PAC-E	–1.07	–1.04	–0.74	1.45	–0.92	0.27	0.99	–0.44	2.64	2.08	1.11	1.18
	2 outliers	LS	–0.30	–0.97	–0.35	2.06	–0.40	0.83	1.79	3.50	4.66	2.30	4.02	0.53
		PAC-E	–1.07	–1.04	–0.74	1.45	–0.92	0.26	0.98	–0.49	2.65	2.08	1.13	1.18
$\gamma_1 = 0$ $\gamma_2 = 0.29$	No outliers	LS	–0.26	–0.83	0.31	2.11	0.26	–0.23	0.68	08 7	2.58	2.28	1.13	0.41
		PAC-E	–1.01	–0.83	–0.10	1.55	0.21	–0.35	0.65	0.37	2.20	2.03	0.83	0.10
	1 outlier	LS	–0.26	–0.82	0.35	2.11	0.30	–0.01	0.75	2.95	3.83	2.28	3.04	0.46
		PAC-E	–1.01	–0.83	–0.10	1.55	0.02	–0.35	0.65	0.46	2.21	2.03	0.87	0.10
	2 outliers	LS	–0.25	–0.81	0.39	2.11	0.34	0.21	0.82	5.02	5.60	2.28	5.09	0.52
		PAC-E	–1.02	–0.86	–0.99	1.53	–0.12	–0.27	0.66	0.04	2.16	2.03	0.71	0.15

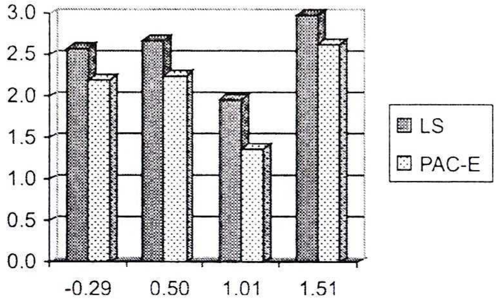


Fig. 3. Values of $\|\hat{\mathbf{X}}\|_2$
(variant $\gamma_1 = 0$; $\gamma_2 \neq 0$; no outliers)

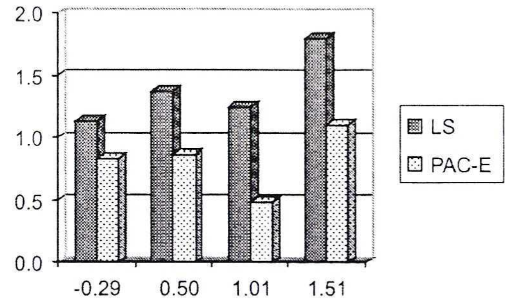


Fig. 4. Values of $\|\hat{\mathbf{X}}_{II}\|_2$
(Subset II, variant $\gamma_1 = 0$; $\gamma_2 \neq 0$; no outliers)

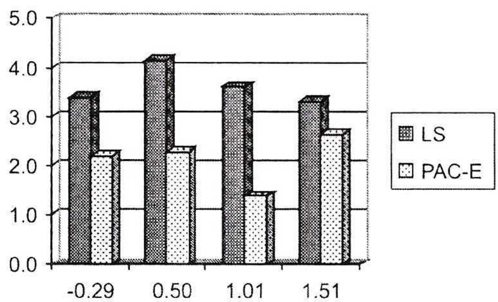


Fig. 5. Values of $\|\hat{\mathbf{X}}\|_2$
(variant $\gamma_1 = 0$; $\gamma_2 \neq 0$; 1 outlier)

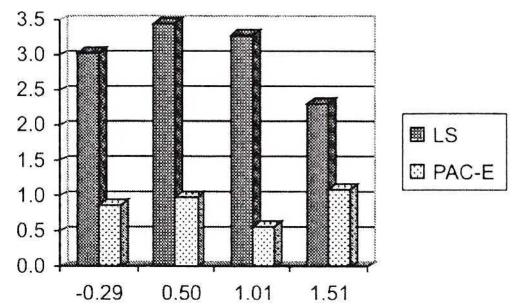


Fig. 6. Values of $\|\hat{\mathbf{X}}_{II}\|_2$
(Subset II, variant $\gamma_1 = 0$; $\gamma_2 \neq 0$; 1 outlier)

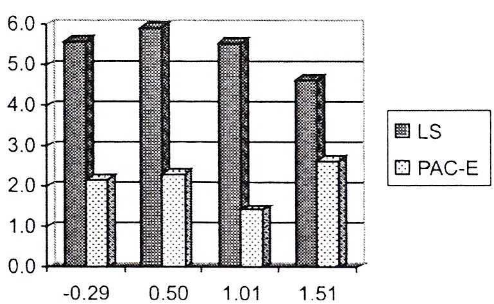


Fig. 7. Values of $\|\hat{\mathbf{X}}\|_2$
(variant $\gamma_1 = 0$; $\gamma_2 \neq 0$; 2 outliers)

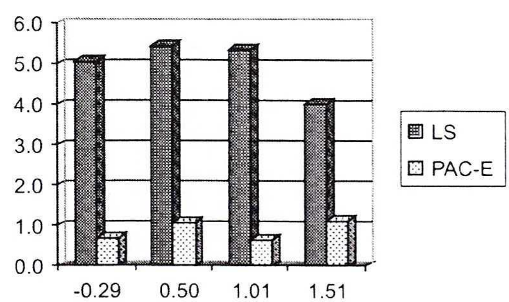


Fig. 8. Values of $\|\hat{\mathbf{X}}_{II}\|_2$
(Subset II, variant $\gamma_1 = 0$; $\gamma_2 \neq 0$; 2 outliers)

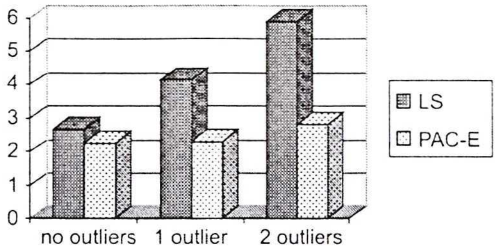


Fig. 9. Values of $\|\hat{\mathbf{X}}\|_2$
(variant $\gamma_1 = 0$; $\gamma_2 = 0.5$)

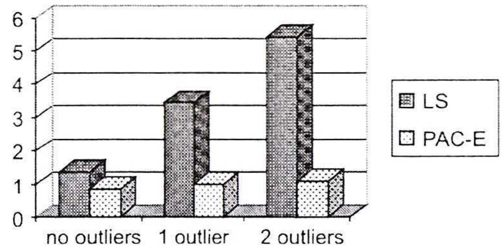


Fig. 10. Values of $\|\hat{\mathbf{X}}_{II}\|_2$
(Subset II, variant $\gamma_1 = 0$; $\gamma_2 = 0.5$)

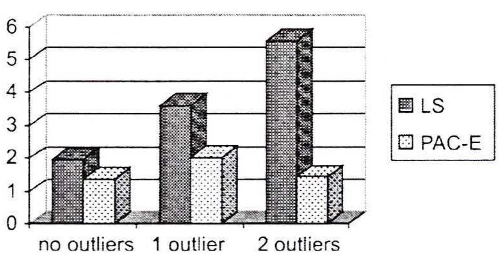


Fig. 11. Values of $\|\hat{\mathbf{X}}\|_2$
(variant $\gamma_1 = 0$; $\gamma_2 = 1.01$)

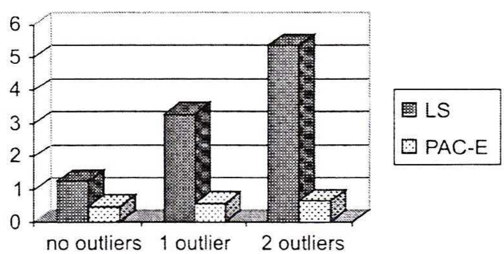


Fig. 12. Values of $\|\hat{\mathbf{X}}_{II}\|_2$
(Subset II, variant $\gamma_1 = 0$; $\gamma_2 = 1.01$)

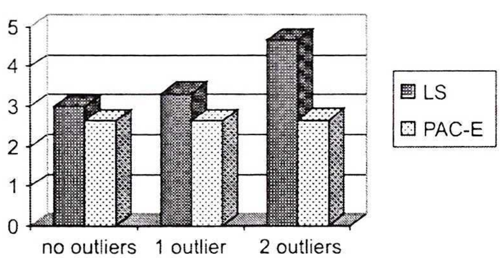


Fig. 13. Values of $\|\hat{\mathbf{X}}\|_2$
(variant $\gamma_1 = 0$; $\gamma_2 = 1.51$)

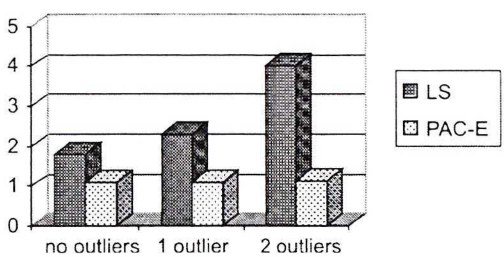


Fig. 14. Values of $\|\hat{\mathbf{X}}_{II}\|_2$
(Subset II, variant $\gamma_1 = 0$; $\gamma_2 = 1.51$)

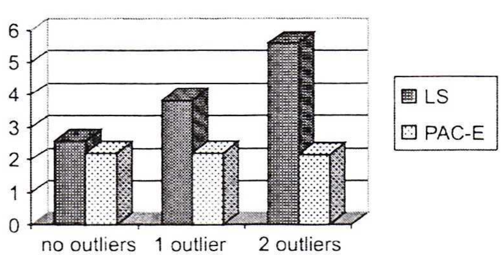


Fig. 15. Values of $\|\hat{\mathbf{X}}\|_2$
(variant $\gamma_1 = 0$; $\gamma_2 = -0.29$)

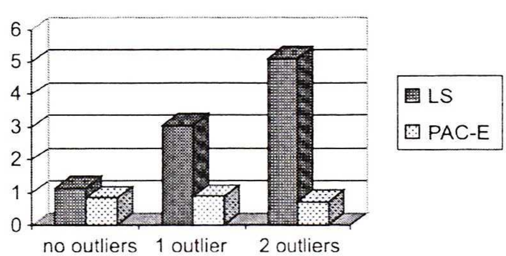
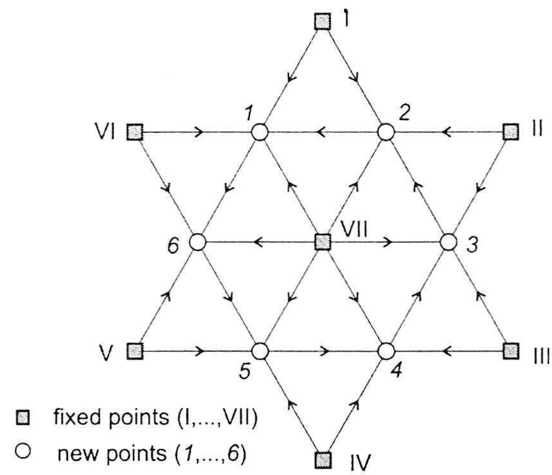


Fig. 16. Values of $\|\hat{\mathbf{X}}_{II}\|_2$
(Subset II, variant $\gamma_1 = 0$; $\gamma_2 = -0.29$)

The next test was carried out using the simulated network shown in Figure 17. The set of observations was generated in such a way that the asymmetry was non-zero (so was the excess).



Rys. 17. The test network of geometrical levelling

Table 3.

	Variant	Method	Parameter vector $\hat{X}$						$\Vert\hat{X}\Vert_2$
			$\hat{x}_1$	$\hat{x}_2$	$\hat{x}_3$	$\hat{x}_4$	$\hat{x}_5$	$\hat{x}_6$	
$\gamma_1 = 0.45$ $\gamma_2 = 1.00$ $\sigma = 3.00$	No outliers	LS	0.54	0.53	0.57	0.01	0.31	0.82	1.29
		PAC-E	–0.20	–0.26	–0.10	–0.56	–0.53	–0.30	0.89
	1 outlier	LS	0.89	0.61	0.60	0.08	0.65	2.45	2.82
		PAC-E	–0.18	–0.26	–0.10	–0.55	–0.51	–0.39	0.91
	2 outliers	LS	1.16	0.65	0.55	–0.18	–0.64	3.75	4.07
		PAC-E	–0.21	–0.26	–0.09	–0.53	–0.39	–0.19	0.77

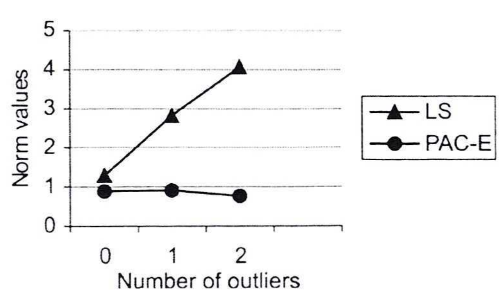


Fig. 18. Influence of outliers on adjustment results (LS and PAC-E methods)

Table 4.

	Variant	Method	Parameter vector $\hat{\mathbf{X}}$								$\lVert\hat{\mathbf{X}}\rVert_2$	$\lVert\hat{\mathbf{X}}_I\rVert_2$	$\lVert\hat{\mathbf{X}}_{II}\rVert_2$	$\lVert\hat{\mathbf{X}}_P\rVert_2$
			$\hat{X}_1$	$\hat{X}_2$	$\hat{X}_3$	$\hat{X}_4$	$\hat{X}_5$	$\hat{X}_6$	$\hat{X}_7$	$\hat{X}_8$				
$\gamma_1 = 0$ $\gamma_2 = 0.5$	No outliers	LS	–0.26	–0.84	0.25	2.10	0.22	–0.41	0.58	1.17	2.68	2.28	1.37	0.33
	1 outlier	LS	–0.26	–0.83	0.29	2.11	0.26	–0.17	0.66	3.38	4.15	2.28	3.45	0.39
		LS*	–0.26	–0.84	0.25	2.11	0.22	–0.39	0.59	1.33	2.75	2.28	1.51	0.33
	2 outliers	LS	–0.26	–0.82	0.33	2.11	0.30	0.04	0.73	5.39	5.91	2.28	5.44	0.44
		LS*	–0.26	–0.84	0.26	2.11	0.22	–0.38	0.59	1.44	2.81	2.28	1.60	0.34
$\gamma_1 = 0$ $\gamma_2 = 1.01$	No outliers	LS	0.48	–0.02	-0.33	1.35	–0.35	0.55	0.17	1.11	1.96	1.44	1.25	0.48
	1 outlier	LS	0.48	–0.01	–0.30	1.36	–0.31	0.78	0.25	3.18	3.61	1.44	3.28	0.43
		LS*	0.48	–0.01	–0.33	1.35	–0.35	0.56	0.18	1.19	2.01	1.43	1.33	0.48
	2 outliers	LS	0.49	0.00	–0.26	1.36	–0.27	1.00	0.32	5.26	5.56	1.44	5.36	0.37
		LS*	0.48	–0.01	–0.33	1.35	–0.35	0.57	0.18	1.28	2.37	1.43	1.41	0.48
$\gamma_1 = 0$ $\gamma_2 = 1.51$	No outliers	LS	–0.31	–0.99	–0.42	2.06	–0.48	0.39	1.64	–0.64	2.99	2.30	1.80	0.64
	1 outlier	LS	–0.31	–0.98	–0.38	2.06	–0.44	0.61	1.71	1.43	3.32	2.30	2.31	0.59
		LS*	–0.31	–0.99	–0.42	2.06	–0.48	0.39	1.64	–0.69	3.01	2.31	1.82	0.64
	2 outliers	LS	–0.30	–0.97	–0.35	2.06	–0.40	0.83	1.79	3.50	4.66	2.30	4.02	0.53
		LS*	–0.31	–0.99	–0.42	2.06	–0.48	0.38	1.64	–0.75	3.02	2.31	1.84	0.64
$\gamma_1 = 0$ $\gamma_2 = 0.29$	No outliers	LS	–0.26	–0.83	0.31	2.11	0.26	–0.23	0.68	0.87	2.58	2.28	1.13	0.41
	1 outlier	LS	–0.26	–0.82	0.35	2.11	0.30	–0.01	0.75	2.95	3.83	2.28	3.04	0.46
		LS*	–0.26	–0.83	0.32	2.11	0.26	–0.23	0.68	0.94	2.63	2.28	1.18	0.41
	2 outliers	LS	–0.25	–0.81	0.39	2.11	0.34	0.21	0.82	5.02	5.60	2.28	5.09	0.52
		LS*	–0.26	–0.83	0.31	2.11	0.26	–0.24	0.67	0.78	2.54	2.28	1.06	0.40

(LS* – the iterative way)

The theoretical point heights were assumed equal to zero, so $\mathbf{X}^t = \mathbf{0}$ and $\mathbf{E}\{h_i\} = 0$. Every observable was "measured" four times to generate the set including 96 elements and having the following values of the parameters $\gamma_1 = 0.45$, $\gamma_2 = 1.00$ and $\delta = 3.00$.

The test was similar to the previous one, i.e. it was aimed to compare LS and PAC-E adjustments with one or two gross errors added to the set of observation to confirm robustness of the method. The norms $\lVert\hat{\mathbf{X}}\rVert_2$ were also used to analyse the results (Table 3).

The PAC-E method provided better results; the norm values of the PAC-E method were smaller than the respective ones of the LS method in all variants. The results of the LS adjustment were also worse when some outliers occurred. That confirmed well known property of the LS method. All observations, including outliers, were treated in the same, equivalent way. On the contrary, the PAC-E method exhibited its robustness for gross errors (like the classic PAC method) and ignored outliers during the adjustment process. To facilitate the analyses, some norm values are shown in Figure 18.

Considering the weight function, the LS method can be classified as a neutral estimation (Kadaj, 1984), thus not robust for gross errors. However, applying the iterative way of the LS solution, i.e. eliminating observations that are suspected to be outliers, the final results of the adjustment should not differ from those obtained on the basis of observations free of gross errors. Data in Table 4 containing the results of the LS adjustment without gross errors and the results of iterative LS method (with outliers eliminated) confirms that statement.

7. Conclusions

The presented tests are the basis for the following conclusions. The use of additional information from the probabilistic model of measurement errors improves final results of adjustment (Table 2 proves the statement clearly). Let us remind, that the test network was divided into two parts differing in the value of excess coefficient γ_2. The application of the proposed method improves the results (values of the norms $\|\hat{\mathbf{X}}_{II}\|_2$ are lower), especially for the second subgroup where $\gamma_2 \neq 0$. Excess coefficient influences the results in the similar way like weights do in the classical method of adjustment, i.e. measurements with the smallest errors are of bigger importance.

The proposed method is also robust for outliers (similarly to the original PAC method). The outliers, that occurred in the second subgroup, make the adjustment result of LS method much worse (higher values of $\|\hat{\mathbf{X}}_{II}\|_2$). On the contrary, PAC-E method "ignored" the outliers, thus it proved its robustness. Also values of the norm listed in Table 2 confirm the robustness (they are all close to one another, including $\|\hat{\mathbf{X}}_{II}\|_2$ norm). This property is an effect of weight function modification that imputes lower weights to identified outliers. Thus, if one combines PAC method with another probabilistic model, such a new method should also be robust to outliers.

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Dalszy rozwój metody ZWA wyrównania sieci geodezyjnych

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Streszczenie

W niniejszej pracy ukazano koncepcję metody wyrównania sieci geodezyjnych z zastosowaniem szeregów Edgewortha i Zasady Wyboru Alternatywy. Szereg Edgewortha jest aproksymantą opisującą empiryczne rozkłady błędów pomiarów. Pozwala ona na uwzględnienie w zadaniach wyrównawczych istotnych anomalii dotyczących podstawowych parametrów empirycznego rozkładu prawdopodobieństw błędów obserwacji geodezyjnych. Jako metodę estymacji zastosowano Zasadę Wyboru Alternatywy. Na szczególną uwagę zasługuje jej naturalna odporność na obserwacje odstające, co dało podstawę opracowania metody ZWA-E. W pracy przedstawiono algorytm rozwiązania oraz testy numeryczne. Celem testów było porównanie wyników wyrównania metody ZWA-E z metodą NK, a w szczególności ustalenie wpływu niezerowych wartości ekscesu na wyniki wyrównania oraz wskazanie na naturalną odporność nowej metody na obserwacje odstające.