GEODEZJA I KARTOGRAFIA
GEODESY AND CARTOGRAPHY
Vol. 56, No 2, 2007, pp. 59-71

Comparison of algorithms of two-stage adjustment and multi-group adjustment of levelling network

Idzi Gajderowicz

Institute of Geodesy, University of Warmia and Mazury
1 Oczapowskiego St., 10-957 Olsztyn, Poland
e-mail: idzigajd@uwm.edu.pl

Received: 18 July 2007/Accepted: 8 October 2007

Abstract: In the research it has been assumed that an observation corresponds to a measured height difference of a levelling section while a pseudo-observation corresponds to a sum of observations for consecutive levelling sections which make up a levelling line. Relations between observations and pseudo-observations are shown. It has also been assumed that observations are not correlated.

The study compares Helmert – Pranis-Praniewicz algorithm of parametric, multi-group (parallel) least squares adjustment of observations with the algorithm of two-stage least squares adjustment of levelling network. The two-stage adjustment consists of least squares adjustment of pseudo-observations and then the adjustment of observations, which is carried out separately for each levelling line.

It was shown that normal equations concerning heights of nodal points, created on the basis of pseudo-observations, are identical to the reduced normal equations formed on the basis of observations in multi-group adjustment. So, adjusted heights of nodal points and their variance-covariance matrix are the same in the case of adjustment of observations and in the case of adjustment of pseudo-observations.

Following a brief presentation of known algorithm of height computation for intermediate benchmarks of levelling lines there is shown the proof that the value of a square root of the *a posteriori* variance of unit weight m_0, known also as mean square error of a typical observation/pseudo-observation, is the same in the case of adjustment of observations and in the case of adjustment of pseudo-observations.

The conclusion states that the results of two-stage adjustment and rigorous least squares adjustment of observations are identical.

Keywords: Levelling network, Helmert – Pranis-Praniewicz algorithm, pseudo-observations, least squares adjustment

1. Introduction

The structure of a levelling network is simple. The network consists of levelling lines which converge in nodal points (junction points), and each levelling line consists of levelling sections which connect the adjacent benchmarks of the line. It has been assumed that the result of measurement of the height difference of a given levelling section constitutes an observation. In this study there is also used the term "pseudo-observation" which means measured height difference of a levelling line.

By performing the rigorous (least squares) adjustment of observations carried out at all levelling sections, the adjusted heights of all benchmarks can be computed. If the parametric method of adjustment is applied, the heights of all determined benchmarks of the network will be used as parameters.

In the 1st order precise levelling network in Poland there are over 16 000 benchmarks including 245 nodal points. In the United European Levelling Network UELN 95/98 (EUREF Report, 2007) there is more than 3 600 nodal points and the number of all benchmarks reaches about 100 000.

In order to avoid serious numerical problems that may arise in the process of simultaneous determination of such large number of unknown parameters, a two-stage adjustment method can be applied. The two-stage adjustment method consists of

- the adjustment of pseudo-observations (summed height differences of levelling sections within levelling lines), which results in adjusted heights of all nodal points,
- the adjustment of observations at levelling lines based on adjusted heights of nodal points.

An algorithm for the two-stage adjustment method is known from literature (e.g. Vaniček and Krakiwsky, 1982; Baran and Gajderowicz, 1993). In both publications the formulae related to the second stage of adjustment were developed under the assumption that observations are not correlated.

The important assumption used in the study was that observations of neighbouring sections are not correlated. The existence of such correlation has, however, been proven (Remmer, 1975; Lucht, 1972, 1983). In the algorithm of levelling network adjustment with the use of *a priori* covariance matrix of observations (Vaniček and Krakiwsky, 1982) one possible family of exponential covariance functions (e.g. Lucht, 1972) has been postulated. Unfortunately, the following sentence of (Vaniček and Krakiwsky, 1982) is still valid: "Research is needed into finding the best kind of covariance function for a given region". There are two problems there. The first – how to divide the network into regions taking into consideration conditions such as slope of lines, atmospheric parameters, vegetation etc, and the second – how to select proper family of covariance functions and then how to determine parameters of those functions. Different assumptions/solutions of the above problems lead to different results of adjustment. That is why, looking for unambiguous solution, practical adjustments are usually carried out under the assumption that observations are not correlated.

The two-stage adjustment algorithm for uncorrelated observations was applied in the adjustment of many national levelling networks, e.g. the 1st order precise levelling network in Poland (Gajderowicz, 2005), the levelling network covering Lithuania and neighbouring countries (Paršeliunas et al., 2000), as well as in the adjustment of the UELN 95/98 (EUREF Report, 2007). Commonly the second stage of the two-stage adjustment was not even mentioned in publications; it is obvious, however, that computation of heights of all benchmarks is a goal of a network adjustment.

Two stages of the solution, creating and using pseudo-observations, and a simple method of observation adjustment on levelling lines, may raise suspicions that the two-stage method is not equivalent to the rigorous least squares adjustment of all the observations in a network. The study aims at showing that the two-stage method of adjustment of a levelling network is fully equivalent to the rigorous least squares adjustment of all the observations. Comparison of the algorithm of two-stage adjustment method with Helmert – Pranis-Praniewicz algorithm of least squares, parametric, multi-group (parallel) adjustment of all observations carried out in a network is an important part of this study.

There may also be mentioned another algorithm for levelling network adjustment (Beluch, 1991) which is similar to Helmert – Pranis-Praniewicz algorithm with simplifications due to specific structure of levelling networks. That algorithm was not applied in the study.

2. Relations between observations and pseudo-observations

An observation dh_i is the result of measurement of i-th levelling section height difference (between two consecutive benchmarks). The variance σ_i^2 of dh_i is expressed as

$$\sigma_i^2 = \sigma_0^2 R_i \tag{1}$$

where σ_0^2 is a unit variance (the variance of a measurement of a levelling section of 1 km length), and R_i is the length of i-th levelling section, expressed in km.

The weight p_i of observed i-th levelling section can be calculated from the formula

$$p_i = \frac{C}{\sigma_i^2} \tag{2}$$

where C is any positive constant. With $C = \sigma_0^2$, one obtains

$$p_i = \frac{\sigma_0^2}{\sigma_0^2 R_i} = \frac{1}{R_i} \tag{3}$$

The observation equation for the i-th observation has the following form

$$dh_i + v_i = (H^0_{B_e,i} + dH_{B_e,i}) - (H^0_{B_s,i} + dH_{B_s,i}) \tag{4}$$

where $H^0_{B_s,i}$, $H^0_{B_e,i}$ are approximate heights of the starting point (B_s) and the end-point (B_e) of the i-th section, and $dH_{B_s,i}$, $dH_{B_e,i}$ are the corrections to the approximate heights (parameters to be determined). Denoting

$$l_i = -H^0_{B_s,i} + H^0_{B_e,i} - dh_i \tag{5}$$

the observation equation (4) can be written as

$$v_i = -dH_{B_s,i} + dH_{B_e,i} + l_i \tag{6}$$

Now let us consider pseudo-observations h. The following relationship is valid for the JK line connecting the nodal points J and K

$$h_{JK} = \sum_{i=1}^{n_{JK}} dh_i \tag{7}$$

where n_{JK} is the number of levelling sections in the line JK.

Usually, in practice of processing levelling data, it is assumed that observations are not correlated; thus the variance of a pseudo-observation can be expressed as follows

$$\sigma^2_{h_{JK}} = \sum_{i=1}^{n_{JK}} \sigma_i^2 = \sigma_0^2 \sum_{i=1}^{n_{JK}} R_i = \sigma_0^2 D_{JK} \tag{8}$$

where D_{JK} is the sum of section lengths in JK line, i.e. the length of the JK line.

The weight P_{JK} of a pseudo-observation

$$P_{JK} = \frac{C}{\sigma^2_{h_{JK}}} = \frac{C}{\sigma_0^2 D_{JK}} = \frac{\sigma_0^2}{\sigma_0^2 D_{JK}} = \frac{1}{D_{JK}} \tag{9}$$

is related to the weights p_i of observations as follows

$$\frac{1}{P_{JK}} = D_{JK} = \sum_{i=1}^{n_{JK}} R_i = \sum_{i=1}^{n_{JK}} \frac{1}{p_i} \tag{10}$$

The observation equation for a pseudo-observation h_{JK} results from the relationship

$$h_{JK} + V_{JK} = (H_K^0 + dH_K) - (H_J^0 + dH_J) \tag{11}$$

Therefore

$$V_{JK} = -dH_J + dH_K + l_{JK} \tag{12}$$

where H_J^0, H_K^0 are approximate heights, and dH_J, dH_K are the corrections to the approximate heights of nodal points J and K, respectively, and

$$l_{JK} = -H_J^0 + H_K^0 - h_{JK} \tag{13}$$

3. Multi-group adjustment of observations

All observations dh in the network should be adjusted using the least squares method. Let us apply Helmert – Pranis-Praniewicz's method, which is multi-group, parallel, parametric adjustment (e.g. Wolf, 1975; Baran, 1983).

A levelling network can easily be divided into groups. Let each levelling line of a network constitutes a separate group. Heights of all points of a given levelling line, except for the nodal points, will be considered the internal unknowns of the group, while the heights of the nodal points – the external unknowns.

Thanks to such a natural division of the network into groups

- the groups are linked with external unknowns (heights of the nodal points),
- each observation belongs to one specific group,
- there is no such observation in the network which would link internal unknowns belonging to two groups.

For each group (levelling line) the following processing actions should be performed:

- forming the observation equations (calculating the elements of the design matrix),
- forming the normal equations,
- reducing such normal equations to 2 equations which bind two external unknowns of the group.

Those actions can be demonstrated, taking as an example the levelling line (the j-th group) which connects the nodal points J and K, and which has two intermediate points 1 and 2 (Fig. 1).

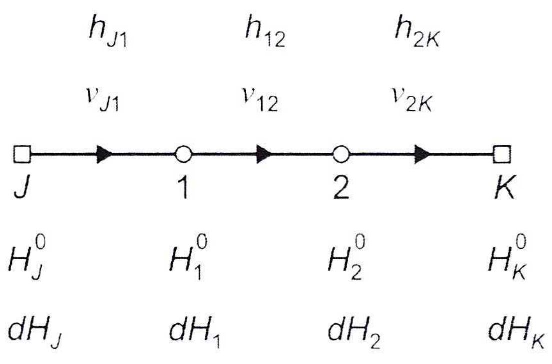

Fig. 1. The levelling line JK

The internal unknowns of the group will make up the matrix X^j_{int}, whereas the external ones – the matrix X^j_{ext}

$$\mathbf{X}^j_{\mathrm{int}} = \begin{bmatrix} dH_1 \\ dH_2 \end{bmatrix} \tag{14}$$

$$\mathbf{X}^j_{\mathrm{ext}} = \begin{bmatrix} dH_J \\ dH_K \end{bmatrix} \tag{15}$$

The observation equations for the j-th group will have the following form

$$\mathbf{V}^j = \mathbf{A}^j_{\mathrm{int}}\mathbf{X}^j_{\mathrm{int}} + \mathbf{A}^j_{\mathrm{ext}}\mathbf{X}^j_{\mathrm{ext}} + \mathbf{l}^j \tag{16}$$

where

$$\mathbf{V}^j = \begin{bmatrix} v_{J1} \\ v_{12} \\ v_{2K} \end{bmatrix} \tag{17}$$

$$\mathbf{l}^j = \begin{bmatrix} l_{J1} \\ l_{12} \\ l_{2K} \end{bmatrix} = \begin{bmatrix} H_1^0 - H_J^0 - h_{J1} \\ H_2^0 - H_1^0 - h_{12} \\ H_K^0 - H_2^0 - h_{2K} \end{bmatrix} \tag{18}$$

$$\mathbf{A}_{\text{int}}^j = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \tag{19}$$

$$\mathbf{A}_{\text{ext}}^j = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{20}$$

while the weight matrix $\mathbf{P}^j$ will be

$$\mathbf{P}^j = diag\begin{bmatrix} 1/D_{J1} & 1/D_{12} & 1/D_{2K} \end{bmatrix} \tag{21}$$

It is obvious that the external unknowns of the j-th group, found in $\mathbf{X}_{\text{ext}}^j$, belong to the column vector $\mathbf{X}_{\text{ext}}$ of all the external unknowns of the network.

The normal equations for the j-th group will have the following form

$$\begin{bmatrix} \mathbf{\Theta}_{\text{int,int}}^j & \mathbf{\Theta}_{\text{int,ext}}^j \\ \mathbf{\Theta}_{\text{ext,int}}^j & \mathbf{\Theta}_{\text{ext,ext}}^j \end{bmatrix} \begin{bmatrix} \mathbf{X}_{\text{int}}^j \\ \mathbf{X}_{\text{ext}}^j \end{bmatrix} + \begin{bmatrix} \mathbf{L}_{\text{int}}^j \\ \mathbf{L}_{\text{ext}}^j \end{bmatrix} = 0 \tag{22}$$

where

$$\mathbf{\Theta}_{\text{int,int}}^j = \left(\mathbf{A}_{\text{int}}^j\right)^{\mathrm{T}} \mathbf{P}^j \mathbf{A}_{\text{int}}^j \tag{23}$$

$$\mathbf{\Theta}_{\text{int,ext}}^j = \left(\mathbf{A}_{\text{int}}^j\right)^{\mathrm{T}} \mathbf{P}^j \mathbf{A}_{\text{ext}}^j \tag{24}$$

$$\mathbf{\Theta}_{\text{ext,int}}^j = \left(\mathbf{\Theta}_{\text{int,ext}}^j\right)^{\mathrm{T}} \tag{25}$$

$$\mathbf{\Theta}_{\text{ext,ext}}^j = \left(\mathbf{A}_{\text{ext}}^j\right)^{\mathrm{T}} \mathbf{P}^j \mathbf{A}_{\text{ext}}^j \tag{26}$$

$$\mathbf{L}_{\text{int}}^j = \left(\mathbf{A}_{\text{int}}^j\right)^{\mathrm{T}} \mathbf{P}^j \mathbf{l}^j \tag{27}$$

$$\mathbf{L}^j_{\text{ext}} = \left(\mathbf{A}^j_{\text{ext}}\right)^{\mathrm{T}} \mathbf{P}^j \mathbf{l}^j \tag{28}$$

Having eliminated the internal unknowns $\mathbf{X}^j_{\text{int}}$ one obtains the reduced normal equations for the j-th group in the following form

$$\mathbf{B}^j_{\text{ext}} \mathbf{X}^j_{\text{ext}} + \mathbf{C}^j_{\text{ext}} = \mathbf{0} \tag{29}$$

where

$$\mathbf{B}^j_{\text{ext}} = \mathbf{\Theta}^j_{\text{ext,ext}} - \mathbf{\Theta}^j_{\text{ext, int}} \left(\mathbf{\Theta}^j_{\text{int,int}}\right)^{-1} \mathbf{\Theta}^j_{\text{int,ext}} \tag{30}$$

$$\mathbf{C}^j_{\text{ext}} = \mathbf{L}^j_{\text{ext}} - \mathbf{\Theta}^j_{\text{ext, int}} \left(\mathbf{\Theta}^j_{\text{int,int}}\right)^{-1} \mathbf{L}^j_{\text{int}} \tag{31}$$

After calculating and reducing the normal equations for all n_l groups, a collective normal equations can be computed for a given network, with only external unknowns.

$$\left(\sum_{j=1}^{n_l} \mathbf{B}^j_{\text{ext}}\right) \mathbf{X}_{\text{ext}} + \left(\sum_{j=1}^{n_l} \mathbf{C}^j_{\text{ext}}\right) = \mathbf{0} \tag{32}$$

The matrices $\mathbf{B}^j_{\text{ext}}$ and $\mathbf{C}^j_{\text{ext}}$ were calculated from (30) and (31), then expanded and completed with zeros so that they corresponded to the vector of all external unknowns $\mathbf{X}_{\text{ext}}$.

After computation of the unknowns $\mathbf{X}_{\text{ext}}$ using (32), one can compute, separately for each group, the unknowns $\mathbf{X}^j_{\text{int}}$ solving (22), and then the corrections $\mathbf{V}^j$ using (16).

In the case of the levelling line shown in Figure 1, the matrices found in normal equations (22), computed according to (23)–(28) with the use of (19)–(21), have the following forms

$$\mathbf{\Theta}^j_{\text{int,int}} = \begin{bmatrix} 1/D_{J1} + 1/D_{12} & -1/D_{12} \\ -1/D_{12} & 1/D_{12} + 1/D_{2K} \end{bmatrix} \tag{33}$$

$$\mathbf{\Theta}^j_{\text{int,ext}} = \begin{bmatrix} -1/D_{J1} & 0 \\ 0 & -1/D_{2K} \end{bmatrix} \tag{34}$$

$$\mathbf{\Theta}^j_{\text{ext,ext}} = \begin{bmatrix} 1/D_{J1} & 0 \\ 0 & 1/D_{2K} \end{bmatrix} \tag{35}$$

$$\mathbf{L}^j_{\text{int}} = \begin{bmatrix} l_{J1}/D_{J1} - l_{12}/D_{12} \\ l_{12}/D_{12} - l_{2K}/D_{2K} \end{bmatrix} \tag{36}$$

$$\mathbf{L}^j_{\text{ext}} = \begin{bmatrix} -l_{J1}/D_{J1} \\ l_{2K}/D_{2K} \end{bmatrix} \tag{37}$$

Subsequent matrices, which are necessary to get the reduced normal equations of the j-th group have, according to (30) and (31), the following forms

$$\left(\mathbf{\Theta}^{j}_{\mathrm{int.int}}\right)^{-1} = \frac{1}{D_{J1}+D_{12}+D_{2K}}\begin{bmatrix} D_{J1}(D_{12}+D_{2K}) & D_{J1}+D_{2K} \\ D_{J1}+D_{2K} & (D_{J1}+D_{12})D_{2K} \end{bmatrix} \tag{38}$$

$$\mathbf{B}^{j}_{\mathrm{ext}} = \frac{1}{D_{J1}+D_{12}+D_{2K}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{39}$$

$$\mathbf{C}^{j}_{\mathrm{ext}} = \frac{1}{D_{J1}+D_{12}+D_{2K}}\begin{bmatrix} -(l_{J1}+l_{12}+l_{2K}) \\ +(l_{J1}+l_{12}+l_{2K}) \end{bmatrix} \tag{40}$$

The formulae (39) and (40) concern the levelling line with 2 intermediate points. Similar formulae have been derived for lines containing 1, 3 or 4 intermediate points, yielding the following results

$$\mathbf{B}^{j}_{\mathrm{ext}} = \frac{1}{D_{JK}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{41}$$

$$\mathbf{C}^{j}_{\mathrm{ext}} = \frac{1}{D_{JK}}\begin{bmatrix} -l_{JK} \\ +l_{JK} \end{bmatrix} \tag{42}$$

where D_{JK} is the sum of the lengths of all levelling sections which make up the levelling line, i.e. the length of the levelling line JK, and l_{JK} is obtained from (13).

Deriving the formulae for $\mathbf{B}^{j}_{\mathrm{ext}}$, $\mathbf{C}^{j}_{\mathrm{ext}}$ for levelling lines containing more than 3 intermediate points is a tedious task. However, numerical experiments indicate that the formulae (41) and (42) are correct for each line, regardless of the number of intermediate points.

Summing up the considerations which have resulted in formulae (41) and (42), the reduced normal equations for each group can be presented as

$$\begin{bmatrix} +1/D_{JK} & -1/D_{JK} \\ -1/D_{JK} & +1/D_{JK} \end{bmatrix}\begin{bmatrix} dH_J \\ dH_K \end{bmatrix} + \begin{bmatrix} -l_{JK}/D_{JK} \\ +l_{JK}/D_{JK} \end{bmatrix} = \mathbf{0} \tag{43}$$

4. Two-stage adjustment

4.1. Adjustment of pseudo-observations

In the levelling network divided into levelling lines, the procedure of obtaining reduced normal equations for a given levelling line can be much more simple than it was shown in the previous section. Note that for each levelling line (i.e. each group considered previously) there is one pseudo-observation described with (7)–(9).

The observation equation (12) for the pseudo-observation can be written as

$$V_{JK} = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} dH_J \\ dH_K \end{bmatrix} + l_{JK} \tag{44}$$

Having taken into account the weight of pseudo-observation $P_{JK} = 1/D_{JK}$, two normal equations, binding the nodal points J, and K of a given levelling line, will have the form

$$\begin{bmatrix} +1/D_{JK} & -1/D_{JK} \\ -1/D_{JK} & +1/D_{JK} \end{bmatrix} \begin{bmatrix} dH_J \\ dH_K \end{bmatrix} + \begin{bmatrix} -l_{JK}/D_{JK} \\ +l_{JK}/D_{JK} \end{bmatrix} = \mathbf{0} \tag{45}$$

Comparison of (45) with (43) leads to an important conclusion: *two normal equations, obtained directly from the observation equation for a given pseudo-observation, are identical to the reduced normal equations, obtained in the process of elaboration of the given levelling line with the multi-group method.*

The *a priori* covariance matrix for all pseudo-observations in a levelling network, is a diagonal matrix as those pseudo-observations are not correlated. Therefore, the normal equations can sequentially be processed for the whole network. For each consecutive pseudo-observation, partial normal equations are formed according to (45); elements of those equations are then added to the appropriate elements in collective normal equations (for the whole network).

After applying all the observation equations for pseudo-observations, collective normal equations will be obtained, binding all heights of the levelling network's nodal points. Such collective normal equations are identical to the normal equations (32), obtained with the use of the multi-group method. Therefore, *the adjusted heights of nodal points and their variance-covariance matrix (see also the Section 4.3 concerning m_0) are the same, regardless of whether adjustment of observations or adjustment of pseudo-observations is performed.*

4.2. Heights of intermediate benchmarks

The adjustment of pseudo-observations provides the adjusted heights of the nodal points as well as corrections to pseudo-observations. The adjusted heights of intermediate benchmarks on the levelling lines can now be calculated.

For each levelling line, there is one condition equation of the form

$$dh_1 + v_1 + dh_2 + v_2 + \ldots + dh_{n_{JK}} + v_{n_{JK}} = h_{JK} + V_{JK} \tag{46}$$

where v_i is the correction for the i-th observation (dh_i), and V_{JK} is the correction for the pseudo-observation (h_{JK}).

The right-hand side of (46) is known. According to the definition of pseudo-observation (7) h_{JK} is equal to the sum of all (n_{JK}) observations on a levelling line. Therefore,

$$v_1 + v_2 + \ldots + v_{n_{JK}} = V_{JK} \tag{47}$$

In the case of one condition equation, the solution is simple: corrections $v_1, v_2, \ldots, v_{n_{JK}}$ are inversely proportional to the weights $p_i = \dfrac{1}{R_i}$ and the sum of corrections must be equal to V_{JK} (Baran and Gajderowicz, 1993). Therefore

$$v_i = V_{JK}\frac{R_i}{D_{JK}} \tag{48}$$

Conclusion: *corrections v to the observations carried out on the levelling line JK have been calculated correctly, as this was done with a correctly calculated correction V_{JK} to pseudo-observation h_{JK} and with the only condition equation binding the corrections.*

Mean square errors of the adjusted heights of intermediate points of the line JK are calculated according to the formula (Vaniček and Krakiwsky, 1982)

$$m_i^2 = (1-q)^2 Q_{JJ} + 2q(1-q)Q_{JK} + q^2 Q_{KK} + q(1-q)D_{JK}m_0^2 \tag{49}$$

where $q = D_{Ji}/D_{JK}$, and Q_{JJ}, Q_{JK}, Q_{KK} are the elements of the variance-covariance matrix, which corresponds to points J, and K.

4.3. Mean square error m_0 of typical measurement

Pseudo-observation adjustment also involves the computation of the mean square error m_0 of a typical measurement ($P = 1$) of a section/line (square root of the *a posteriori* variance of unit weight) with the following formula:

$$m_0 = \sqrt{\frac{1}{n_p}\sum_{i=1}^{n_l} P_i V_i V_i} \tag{50}$$

where P is a weight of a pseudo-observation, V is a correction to the pseudo-observation, n_l is a number of levelling lines, and n_p is a number of redundant pseudo-observations.

Another determination of the mean square error m_0 can be based on corrections v to the observations

$$m_0 = \sqrt{\frac{1}{n_o}\sum_{i=1}^{n_k} p_i v_i v_i} \tag{51}$$

where p is a weight of an observation, v is a correction to the observation, n_k is a number of levelling sections (number of observations), and n_o is a number of redundant observations.

Let us compare the value of m_0 determined with the use of (50) with the one determined using (51). For each levelling line JK

$$P_{JK}V_{JK}V_{JK} = \sum_{i=1}^{n_{JK}} p_i v_i v_i \tag{52}$$

what can easily be proven using (48). Therefore, in the whole network, the sum of the products PVV is equal to the sum of the products pvv

$$\sum_{i=1}^{n_l} P_i V_i V_i = \sum_{i=1}^{n_k} p_i v_i v_i \tag{53}$$

The number of redundant pseudo-observations n_p is

$$n_p = n_l - w + s \tag{54}$$

where n_l is a number of levelling lines (number of pseudo-observations), w is a number of all nodes, and s is a number of nodes of known heights.

In the case of adjustment of observations, the number of redundant observations n_o is

$$n_o = n_k - r + s \tag{55}$$

where n_k is a number of all observations, r is a number of all points, and s is a number of points (nodes) of known heights.

The number of all observations is

$$n_k = (n_{w1} + 1) + (n_{w2} + 1) + \ldots + (n_{wn_l} + 1) = \sum_{i=1}^{n_l} (n_{wi} + 1) = \sum_{i=1}^{n_l} n_{wi} + n_l \tag{56}$$

where $n_{w1}, n_{w2}, \ldots, n_{wn_l}$ are the numbers of intermediate points of consecutive levelling lines.

The number of all points of the network can be computed as follows

$$r = n_{w1} + n_{w2} + \ldots + n_{wn_l} + w = \sum_{i=1}^{n_l} n_{wi} + w \tag{57}$$

Now, applying (56) and (57), the number of redundant observations is

$$n_o = \left(\sum_{i=1}^{n_l} n_{wi} + n_l\right) - \left(\sum_{i=1}^{n_l} n_{wi} + w\right) + s = n_l - w + s \tag{58}$$

and thus $n_o = n_p$. Considering also (53), the following statement can be formulated: *The mean error m_0 of a typical observation/pseudo-observation has the same value, regardless of whether its calculation is based on corrections v to the observations or on corrections V to the pseudo-observations.*

5. Conclusion

The two-stage adjustment of a levelling network, consisting of

- rigorous least squares adjustment of pseudo-observations,
- calculation of heights of intermediate points, based on the condition (46),

yields the same results which would have been obtained in the process of rigorous least squares adjustment of observations. The two-stage adjustment of a levelling network is a rigorous adjustment of all the observations which make up a network.

Acknowledgements

The research was supported by the University of Warmia and Mazury in Olsztyn and the Head Office of Geodesy and Cartography in Poland. The author expresses his gratitude to Prof. Lubomir W. Baran for the remarks which initiated the research presented in the section 4.3 and to Prof. Jan Kryński for his support in preparing the final version of the manuscript.

References

Baran W., (1983): *Theoretical bases of elaboration of geodetic survey results* (in Polish), PWN, Warszawa (Chapter 3.7.7).

Baran W., Gajderowicz I., (1993): *Adjustment of levelling network – Chapter 8*, in: Precise levelling (in Polish), PPWK, Warszawa – Wroclaw.

Beluch J., (1991): *Parametric two-step adjustment of levelling networks*, Zeszyty Naukowe Akademii Górniczo-Hutniczej, Geodezja 112, Cracow, pp. 69-78.

Report, (2007): *United European Levelling Network (UELN)*, in: European Vertical Reference System (EVRS), http://crs.bkg.bund.de/evrs/.

Gajderowicz I., (2005): *Estimation of accuracy of Polish first order precise levelling network measured in the period 1997-2003* (in Polish), Technical Sciences, Supplement 2, Publisher UWM (University of Warmia and Mazury), Olsztyn, pp. 123-134.

Lucht H., (1972): *Korrelation im Präzisionsnivellement*, Wissenschaftliche Arbeiten der Lehrstuhle für Geodäsie, Photogrammetrie und Kartographie an der Technischen Universität, Nr 48, Hannover.

Lucht H., (1983): *Neighbourhood correlations among observations in levelling networks*, in: H. Pelzer, W. Niemeier (eds.), Precise Levelling, 38 Contribution to the Workshop on Precise Levelling held in Hannover, Fed. Rep. of Germany, 16–18 March, pp. 315-325.

Paršeliunas E., Sacher M., Ihde J., (2000): *Preparations of Lithuanian levelling network data for United European Levelling Network*, Geodezija ir Kartografija, Vol. XXVI, No 4, Vilnius, pp. 171-186.

Remmer O., (1975): *Levelling errors in statu nascendi*, Geodaetisk Institut, Meddelelse No 51, Copenhagen.

Vaniček P., Krakiwsky E., (1982): *Geodesy, the concepts*, North – Holland, Amsterdam (Chapter 19.2).

Wolf H., (1975): *Ausgleichungsrechnung. Formeln zur praktischen Anwendung*, Ferd. Dũmmlers Verlag, Bonn (Chapter A4.2.8).

Porównanie algorytmów wyrównania dwuetapowego i wyrównania wielogrupowego sieci niwelacyjnej

Idzi Gajderowicz

Instytut Geodezji, Uniwersytet Warmińsko-Mazurski
ul. Oczapowskiego 1, 10-957 Olsztyn
e-mail: idzigajd@uwm.edu.pl

Streszczenie

W pracy przyjęto, że obserwacją jest pomierzone przewyższenie odcinka niwelacyjnego, zaś pseudoobserwacją jest suma obserwacji wykonanych dla kolejnych odcinków tworzących linię niwelacyjną. Przyjęto także, że obserwacje nie są wzajemnie skorelowane.

Porównano algorytm Helmerta – Pranis-Praniewicza parametrycznego, wielogrupowego (równoległego) wyrównania obserwacji z algorytmem dwuetapowego wyrównania sieci niwelacyjnej. Dwuetapowe wyrównanie składa się z wyrównania pseudoobserwacji metodą najmniejszych kwadratów i wyrównania obserwacji, które wykonywane jest oddzielnie dla każdej linii niwelacyjnej.

Wykazano, że równania normalne dotyczące wysokości punktów węzłowych, utworzone w oparciu o pseudoobserwacje, są identyczne ze zredukowanymi równaniami normalnymi utworzonymi w oparciu o obserwacje w procesie wyrównania wielogrupowego. A zatem, wyrównane wysokości punktów węzłowych i ich macierz wariancyjno-kowariancyjna są takie same w przypadku wyrównywania obserwacji i w przypadku wyrównywania pseudoobserwacji.

W dalszej kolejności przedstawiono algorytm obliczania wysokości reperów pośrednich linii niwelacyjnych. Wykazano, że wartość błędu średniego m_0 typowej obserwacji/pseudoobserwacji jest taka sama w przypadku wyrównywania obserwacji i w przypadku wyrównywania pseudoobserwacji.

W konkluzji stwierdzono, że wyniki wyrównania dwuetapowego i ścisłego wyrównania obserwacji są identyczne.