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On controllability of fractional positive continuous-time linear systems with delay

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In the paper positive fractional continuous-time linear systems are considered. Positive fractional systems without delays and positive fractional systems with a single delay in control are studied. New criteria for approximate and exact controllability of systems without delays as well as a relative controllability criterion of systems with delay are established and proved. Numerical examples are presented for different controllability criteria. A practical application is proposed.

Key words: fractional systems, positive systems, the Caputo derivative, controllability, delay, the Metzler matrix

1. Introduction

Fractional differential calculus is one of the fastest growing branches of mathematics in the 21st century. Katsuyuki Nishimoto [22] predicted this 30 years ago. Especially in the last two decades, the number of publications in the field of fractional differential calculus has shown a significant increase. The reason for the increased interest in this topic is the fact that algorithms based on fractional differential calculus often work better compared to algorithms using classical differential calculus, as shown by theoretical studies [1, 4]. However, practical research shows that it is not only the object of interest of physicists, biologists or economists but it is also an excellent tool for modeling systems, which allows for intensive development in the field of dynamic systems. The reason that for over 250 years fractional differential calculus was dealt with only by mathematicians was several definitions of the fractional derivative, a lack of physical interpretation, and problems with solving fractional differential equations [6].

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The breakthrough came only in the mid-twentieth century, thanks to information technology which, combined with already developed theorems and concepts from the fractional calculus range, enabled analysis and modeling of real phenomena. The cooperation of a chemist Keith Oldham with a mathematician Jerome Spanier, who in 1968 approached the problem of mass and heat exchange using integrals and derivatives of order $\frac{1}{2}$ [19], contributed significantly to this.

Discussions on fractional differential equations and their practical applications can be found, among others, in monographs [17, 20, 21, 23–26].

In many applications, both external and internal variables cannot take negative values. This condition is particularly strictly observed in population ecology. In the matrix model, the non-negative state of x is the age structure of the population, while control u means, for example, pest control [5]. Another example of positive systems are electrical circuits [11, 15, 16].

The growing demand for models of systems described by non-negative quantities caused the development of the theory of positive systems. These systems are defined in the space of cones, not in a linear space, which makes them much more complicated. Octants are called spatial equivalents of the quarters of the plane. Controllability of such systems means that the system can be carried out from any initial state to a final state with positive coordinates by means of a positive control. Positive fractional discrete-time systems are discussed in [33], positive fractional linear systems, both discrete- and continuous-time, are presented in [7, 9, 10] and [14]. However, controllability of positive linear fractional systems is studied only in [10] and [12] for discrete-time system, and [13] for continuous-time linear systems. The controllability problems for linear continuous-time fractional systems with delayed control were analyzed in [2, 3, 27–31, 34]. Controllability of positive continuous-time fractional systems with delayed control has not been studied yet.

The aim of the work is to establish new controllability criteria for positive linear fractional systems both for systems with and without delay.

The work is organized as follows: definitions of the Caputo derivative, the Mittag-Leffler function, a pseudo-transition matrix, the Metzler matrix are given in Section 2 as well as the Cayley-Hamilton method of determining the pseudo-transition matrix. Section 3 contains mathematical models for positive systems with and without delay. Formulas for a solution of the discussed systems are presented. Positivity aspects of the considered systems are examined. Sections 4 and 5 contain main results of the paper: new controllability criteria for both positive systems with and without delays. Section 6 presents several numerical examples that illustrate the theoretical considerations. A practical example is discussed in Section 7. Finally, concluding remarks are included in Section 8.

2. Preliminaries

The integer-order differentiation is the generalization of differentiation of fractional order. The first concepts of fractional differential integral calculus were presented in 1823 by Niels Abel. Currently, one of the most popular definitions of a fractional order derivative is the Riemann–Liouville one, which under certain conditions corresponds to the Grünwald–Letnikov derivative. However, in this paper we use the Caputo fractional derivative due to the fact that in the Caputo approach the initial conditions for fractional differential equations are analogous to the integer-order differential equations case [24]. We recall the definition.

Definition 1 *The Caputo fractional derivative of order α ($n-1<\alpha<n$, $n\in\mathbb{N}$) for a function $f:\mathbb{R}^+\to\mathbb{R}$ is defined as follows*

$$
{}^{C}D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\,\mathrm{d}\tau, \tag{1}
$$

where Γ is the gamma function.

It is obvious that for $\alpha\to n$ the Caputo derivative tends to n-th order conventional derivative of the function f, eg. $\lim_{\alpha\to n}{}^{C}D^{\alpha}f(t)=f^{(n)}(t)$.

A function of a complex variable z given by the formula

$$
E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\quad \alpha>0 \tag{2}
$$

is called the one-parameter Mittag-Leffler function. For $\alpha=1$ we obtain the classical exponential function $E_1(z)=e^z$. Moreover, a function of a complex variable z given by the formula

$$
E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad \alpha>0,\ \ \beta>0 \tag{3}
$$

is called the two-parameter Mittag-Leffler function. For $\beta=1$ we obtain the one-parameter Mittag-Leffler function $E_{\alpha,1}(z)=E_{\alpha}(z)$.

Based on the above definitions, for $\alpha>0$ and an arbitrary n-th order square matrix A we can give the formula for a *pseudo-transition matrix* $\Phi_0(t)$ of the linear fractional system ${}^{C}D^{\alpha}(t)=A(t)x(t)$ [10, 21]:

$$
\Phi_0(t)=E_{\alpha}(At^{\alpha})=\sum_{k=0}^{\infty}\frac{A^kt^{\alpha k}}{\Gamma(k\alpha+1)}
$$

and

$$\Phi(t) = t^{\alpha-1} E_{\alpha,\alpha}(At^{\alpha}) = t^{\alpha-1} \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k}}{\Gamma((k+1)\alpha)}.$$

This means that for $\alpha \in (0,1)$,

$$\Phi_0(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \Phi(t).$$

For $\alpha = 1$ we obtain the classical transition matrix of ordinary differential equations

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = e^{At}.$$

There are several methods of computing the functions $\Phi_0(t)$ and $\Phi(t)$ for fractional order systems. In the paper, we apply the method that is based on the following Cayley-Hamilton theorem which states that a matrix A satisfies its own characteristic equation [21].

Theorem 1 *If $\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_n = 0$ is a characteristic equation for the $n \times n$ matrix A, then*

$$A^n + a_{n-1}A^{n-1} + \ldots + a_1 A + a_0 I = 0,$$

where I denotes the $n \times n$ identity matrix.

Let be given a function f which can be expanded into the Taylor series

$$f(A,t) = \sum_{k=0}^{\infty} a_k(t) A^k.$$

Using Theorem 1, from the above series we obtain the followig finite sum

$$f(A,t) = \sum_{k=0}^{n-1} a_k(t) A^k.$$

Eigenvalues λ_i are roots of the characteristic equation, so $f(\lambda_k, t)$ can also be represented as a finite sum with the same coefficients

$$f(\lambda_k, t) = \sum_{k=0}^{n-1} a_k(t) \lambda^k, \quad k = 1, \ldots n.$$

The matrix form of the above equality is as follows

$$\begin{bmatrix} f(\lambda_1, t) \\ f(\lambda_2, t) \\ \vdots \\ f(\lambda_n, t) \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0(t) \\ a_1(t) \\ \vdots \\ a_{n-1}(t) \end{bmatrix},$$

which makes it possible to determine the coefficients $a_k(t)$ for $k = 0, 1 \ldots, n-1$. Taking

$$f(A, t) = a_0(t)I + a_1(t)A + \ldots + a_{n-1}(t)A^{n-1},$$

the pseudo-transition matrix can be expressed as

$$\Phi_0(t) = E_\alpha(At^\alpha) = f(A, t) = \sum_{k=0}^{n-1} a_k(t)A^k.$$

In positive systems, inputs, outputs and state variables take only non-negative values. Examples of positive systems are models of water and air pollution or interval models in epidemiology. This chapter will provide sufficient and necessary conditions for the external and internal positivity of fractional order systems.

For simplicity, writing $x(t) \in \mathbb{R}_+^n$, we understand that the values of the vector $x(t)$ are non-negative. Moreover, by $\mathcal{M}_{n\times n}(\mathbb{R}_+)$ we denote the set of all matrices with dimensions $n \times n$ and non-negative elements.

Definition 2 *A square matrix $A = [a_{ij}]$, $i, j = 1, 2, \ldots, n$ with real elements is called the Metzler matrix if its elements lying outside the diagonal are non-negative, i.e. $a_{ij} \geqslant 0$ for $i \neq j$. We will denote the set of $n \times n$ (i.e. of order n) Metzler matrices by M_n.*

Theorem 2 *[10] Let A be a square matrix of order n with real elements. Then*

$$\Phi_0(t) \in \mathcal{M}_{n\times n}(\mathbb{R}_+) \quad \textit{and} \quad \Phi(t) \in \mathcal{M}_{n\times n}(\mathbb{R}_+) \quad \textit{for} \quad t \geqslant 0 \tag{4}$$

if and only if A is a Metzler matrix.

3. Mathematical model

We consider a linear fractional dynamical system with a single delay in control described by the following state equations

$$^{C}D^\alpha x(t) = Ax(t) + B_0u(t) + B_1u(t-h), \tag{5}$$

$$y(t) = Cx(t) + Du(t) \tag{6}$$

for $t \geqslant 0$ and $0 < \alpha < 1$, where $x(t) \in \mathbb{R}^n$ is a state vector, $u(t) \in \mathbb{R}^m$ is an input vector (control), $y(t) \in \mathbb{R}^p$ is an otput vector, A is a $n \times n$ matrix with real elements, B_i are $n \times m$ matrices with real elements (for $i = 0, 1$), C, D are $p \times n$, $p \times m$ matrices with real elements, respectively, and h is a constant delay in control.

Let $L^2_{\text{loc}}([0,\infty), \mathbb{R}^m)$ denote the Hilbert space of locally square integrable functions with values from $\mathbb{R}^m$. For the control function u we assume that $u \in L^2_{\text{loc}}([0,\infty), \mathbb{R}^m)$.

Moreover, let be given the initial conditions $z(0) = \{x(0), u_0\}$ called the initial complete state. For time-delay systems, the complete state $z(t) = (x(t), u_t(s))$, where $u_t(s) = u(s)$ for $s \in [t-h, t)$, completely describes the behavior of the system at time t.

By the Laplace transform it is easy to prove the following theorem about the form of the solution of (5) (see [27]).

Theorem 3 *For given initial conditions $z(0) = \{x(0), u_0\} \in \mathbb{R}^n \times L^2([-h,0], \mathbb{R}^m)$ and a control $u \in L^2_{loc}([0,\infty), \mathbb{R}^m)$, there exists a unique solution $x(t) = x(t, z(0), u)$ of (5), for every $t \geqslant 0$, taking the following form*

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)\left[B_0 u(t) + B_1 u(t-h)\right] \mathrm{d}\tau. \tag{7}$$

Definition 3 *The fractional system with delay in control (5)–(6) is called positive if and only if $x(t, z(0), u) \in \mathbb{R}^n_+$ with any initial conditions $z(0) \in \mathbb{R}^n_+$ and non-negative control values $u(t) \in \mathbb{R}^m_+$ for $t \geqslant 0$.*

The following theorem follows from [8].

Theorem 4 *The fractional system (5)–(6) is (internally) positive if and only if*

$$A \in M_n, \qquad B_0 \in \mathcal{M}_{n\times m}(\mathbb{R}_+), \qquad B_1 \in \mathcal{M}_{n\times m}(\mathbb{R}_+),$$

$$C \in \mathcal{M}_{p\times n}(\mathbb{R}_+), \qquad D \in \mathcal{M}_{p\times m}(\mathbb{R}_+).$$

A special case of the system (5)–(6) is the following initial value problem without delays

$$^{C}D^{\alpha}x(t) = Ax(t) + Bu(t), \tag{8}$$

$$y(t) = Cx(t) + Du(t), \tag{9}$$

$$x(0) = x_0 \tag{10}$$

for $t \geqslant 0$ and $0 < \alpha < 1$, where B is a $n \times m$ matrix with real elements.

From Theorem 3 it follows that the solution of (8) has the form

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(t)\mathrm{d}\tau. \tag{11}$$

Definition 4 *The fractional system (8)–(10) is called positive if and only if $x(t) \in \mathbb{R}^n_+$ with any initial condition $x(0) \in \mathbb{R}^n_+$ and non-negative control values $u(t) \in \mathbb{R}^m_+$ for $t \geqslant 0$.*

Theorem 5 *[8] The fractional system (8)–(10) is (internally) positive if and only if*

$$A \in M_n, \quad B \in \mathcal{M}_{n\times m}(\mathbb{R}_+), \quad C \in \mathcal{M}_{p\times n}(\mathbb{R}_+), \quad D \in \mathcal{M}_{p\times m}(\mathbb{R}_+).$$

4. Controllability for positive systems without delay

In the case of systems without delays, there are two basic types of positive controllability: approximate and exact [18]. Approximate controllability allows the system to be moved to any small neighborhood of the final state. In turn, exact controllability means that the system can be carried out to any final state. Therefore, it is obvious that the approximate controllability is a weaker concept than the exact one.

We will use the following symbols: let K be any set, then clK is the closure of K, coK is the convex hull of K, while $coconeK$ means the smallest convex cone containing 0 and K. Moreover, let $e(1), \dots, e(m)$ be the basic unit vectors in the space $\mathbb{R}^m$ and $e[1], \dots, e[n]$ in $\mathbb{R}^n$.

Let us define the positive attainable set for the system (8)–(10).

Definition 5 *A set*

$$K^+([0,t], x(0)) = \left\{ x(t) \in \mathbb{R}^n_+ : x(t) = \Phi_0(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)\mathrm{d}\tau \right\} \tag{12}$$

is called the positive attainable set for the system (8)–(10) at $t > 0$.

Lemma 1 *Let $\Phi(t) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ and $B \in \mathcal{M}_{n\times m}(\mathbb{R}_+)$. For the system (8)–(10) and the attainable set (12) the following conditions hold:*

1. $clK^+([0,t], x(0)) = cl\,\{co\{\Phi(s)Bu : 0 \leqslant s \leqslant t, u \in \mathbb{R}^n_+\}\}$
   $cl \bigcup_{t>0} K^+([0,t], x(0)) = cl\{co\{\Phi(s)Bu : 0 \leqslant s, u \in \mathbb{R}^n_+\}\}$

2. $cl K^+([0,t], x(0)) = cl\{cocone\{\Phi(s)Be(k) : 0 \leqslant s \leqslant t, k = 1,\ldots,m\}\}$
$cl \bigcup_{t>0} K^+([0,t], x(0)) = cl\{cocone\{\Phi(s)Be(k) : 0 \leqslant s, k = 1,\ldots,m\}\}.$

Proof. Lemma 1 follows from Theorem 2 and Proposition 4.7 included in [32], for $T(t) = \Phi(t)$. □

Definition 6 *The fractional dynamical system (8)–(10) is called approximately positively controllable on* $[0,t]$ *if* $cl K^+([0,t], x(0)) = \mathbb{R}^n_+$.

Now we may formulate a criterion for approximate controllability of positive fractional systems described by the state equations (8)–(10).

Theorem 6 *Let* A *be the Metzler matrix and* $B \in \mathcal{M}_{n\times m}(\mathbb{R}_+)$ *and assume that the fractional system (8)–(10) is positive. The system is approximately controllable on* $[0,t]$ *if and only if for all* k $(k = 1,\ldots,n)$ *there exist* l $(l = 1,\ldots,m)$ *and* $\mu > 0$*, such that* $e[k] = \mu Be(l)$.

Proof. Theorem 6 follows from Lemma 1 item 2 and Theorem 4.9 proved in [32], for $T(t) = \Phi(t)$. □

Definition 7 *The fractional dynamical system (8)–(10) is called exactly positively controllable on* $[0,t]$ *if for any vectors* $x_0(t), x_1(t) \in \mathbb{R}^n_+$ *there exists a control* $u(t) \in \mathbb{R}^m_+$*, such that*

$$x_1(t) = \Phi_0(t)x_0 + \int_0^{t_1} \Phi(t-\tau)Bu(\tau)\,\mathrm{d}\tau. \tag{13}$$

Definition 8 *A matrix is called the generalised permutation matrix if its every row and its every column contains only one positive element and the remaining elements are zeros.*

Remark 1 *The generalised permutation matrix is nonsingular. The inverse matrix* A^{-1} *of the generalised permutation matrix* A *is equal to the transpose matrix in which all nonzero elements are replaced by their reciprocal.*

Theorem 7 *Let the fractional system (8)–(10) be positive. The system is exactly controllable on* $[0,t]$*,* $t > 0$ *if the matrix* R *given by the formula*

$$R = \int_0^{t} \Phi(t-\tau)BB^T\Phi^T(t-\tau)\,\mathrm{d}\tau \tag{14}$$

is the generalised permutation matrix.

An input vector that steers the dynamical system (8)–(10) from the state $x_0 = 0$ *at* $t = 0$ *to the state* x_1 *at* $t = t_1$ *has the form*

$$u(t) = B^T \Phi^T (t_1 - t_0) R^{-1} x_1 \,. \tag{15}$$

Proof. If R is the generalised permutation matrix, then it follows from Remark 1 that $R^{-1} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. Let us assume that the system (8)–(10) is positive. Then, based on Theorem 5 A is a Metzler matrix and $B \in \mathcal{M}_{n\times m}(\mathbb{R}_+)$ and, according to Theorem 2, $\Phi(t) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ for $t \geqslant 0$.

For $x_0 = 0$ and $t = t_1$ the equation (11) can be rewritten as

$$x(t_1) = \int\limits_0^{t_1} \Phi(t_1 - \tau) B u(\tau) \mathrm{d}\tau.$$

By inserting (15) into the above equation, we get

$$x(t_1) = \int\limits_0^{t_1} \Phi(t_1 - \tau) B B^T \Phi^T (t_1 - \tau) R^{-1} x_1 \mathrm{d}\tau = R R^{-1} x_1 = x_1 \,,$$

which implies that the control u steers the system (8)–(10) from the initial state x_0 to the final state x_1. □

Remark 2 *Exact controllability implies approximate controllability.*

5. Controllability for positive systems with delay

Many models describing real-life processes require delays in state coordinates or in control. Examples of such a process are, among others, mixing chemicals, business fluctuations or spaceflights. In systems with delays, future states of the system depend not only on the current state but also on the past states.

Two types of controllability of dynamical systems are generally considered for systems with delays in control: relative controllability and absolute controllability [18]. In the case of relative controllability on $[0, t_1]$, the aim is to find a control u such that the state $x(t_1)$ can be reached using the control. In the case of absolute controllability, the aim is to reach a function. This means that the final segment of a trajectory (over the interval $[t_1 - v_M(t_1), t_1]$) should be a given function.

In the paper we present new criteria for relative controllability for fractional positive systems. For this purpose, we transform the solution (7) of the differential equation (5)

$$\begin{aligned}
x(t) &= \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)\left[B_0u(t) + B_1u(t-h)\right]\mathrm{d}\tau = \\
&= \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)\sum_{i=0}^{1} B_iu(\tau-h_i)\mathrm{d}\tau = \\
&= \Phi_0(t)x_0 + \sum_{i=0}^{1}\int_0^t \Phi(t-\tau)B_iu(\tau-h_i)\mathrm{d}\tau = \left|\begin{array}{c} \tau-h_i = z \\ d\tau = dz \\ \tau = 0 \rightarrow z = -h_i \\ \tau = t \rightarrow z = t-h_i \end{array}\right| = \\
&= \Phi_0(t)x_0 + \sum_{i=0}^{1}\int_{-h_i}^{t-h_i} \Phi(t-z-h_i)B_iu(z)dz = \\
&= \Phi_0(t)x_0 + \sum_{i=0}^{1}\int_{-h_i}^{t-h_i} \Phi(t-\tau-h_i)B_iu(\tau)\mathrm{d}\tau.
\end{aligned} \tag{16}$$

Theorem 8 *Let the fractional system with delay in control (5)–(6) be positive. The system is relatively controllable on* $[0, t_1]$ *if and only if the matrix*

$$R(0,t_1) = \sum_{i=0}^{1}\int_{-h_i}^{t_1-h_i} \Phi(t_1-\tau-h_i)B_iB_i^T\Phi^T(t_1-\tau-h_i)\mathrm{d}\tau \tag{17}$$

satisfies the condition $rank\, R(0,t_1) = n$.

Proof. Let $z(0) = (x_0, u_0)$ be any non-negative initial conditions and x_1 be any positive vector. To prove the sufficiency we will show that the control

$$\widetilde{u}(t) = B_i^T\Phi^T(t_1-\tau-h_i)R^{-1}(0,t_1)[x_1 - \Phi_0(t)x_0] \tag{18}$$

steers the fractional system (5)–(6) from the initial complete state $z(0) = \{x(0), u_0\}$ to the state $x(t_1) = x_1$ and the control is non-negative.

Indeed, since we assume that the system is positive, we have $A \in M_n$ and $B_i \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$, $i = 0, 1$. Therefore $\Phi(t) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ for $t \geqslant 0$. Hence, for $x_1(t) \in \mathbb{R}_+^n$ we obtain $\widetilde{u}(t) \in \mathbb{R}_+^m$.

Next, we will verify whether the control (18) steers the positive system (5)–(6) from the initial non-negative complete state $z(0)$ to the final state $x(t_1) = x_1$.

From Theorem 3 and formula (16) it follows that

$$x(t_1) = x(t_1, z(0), \widetilde{u}) = \Phi_0(t_1)x_0 + \sum_{i=0}^{1} \int_{-h_i}^{t_1-h_i} \Phi(t_1 - \tau - h_i)B_i\widetilde{u}(\tau)\mathrm{d}\tau.$$

Substituting $\widetilde{u}$ defined by (18) into the above equality, we have

$$\begin{aligned} x(t_1) = \Phi_0(t_1)x_0 + \sum_{i=0}^{1} \int_{-h_i}^{t_1-h_i} &\Phi(t_1 - \tau - h_i)B_iB_i^T\Phi^T(t_1 - \tau - h_i) \\ &\times R^{-1}(0, t_1)[x_1 - \Phi_0(t)x_0]\mathrm{d}\tau \\ = \Phi_0(t_1)x_0 + R(0, t_1)&R^{-1}(0, t_1)[x_1 - \Phi_0(t)x_0] = x_1\,. \end{aligned}$$

Hence, the positive fractional system (5)–(6) is relatively controllable on $[0, t_1]$.

We will prove the necessary condition by contradiction. We assume, that the positive system (5)–(6) is relatively controllable on $[0, t_1]$ and $rank\ R(0, t_1) < n$. This means that the matrix $R(0, t_1)$ is singular. Therefore, there exists a vector $\widetilde{x} \neq 0$ such that $\widetilde{x}^T R(0, T)\widetilde{x} = 0$. We have

$$\sum_{i=0}^{1} \int_{-h_i}^{t_1-h_i} \widetilde{x}^T\Phi(t_1 - \tau - h_i)B_iB_i^T\Phi^T(t_1 - \tau - h_i)\widetilde{x}\mathrm{d}\tau = 0.$$

It follows that for $t \in [0, t_1]$ we obtain

$$\widetilde{x}^T\Phi(t_1 - t - h_i)B_i = 0. \tag{19}$$

The system is controllable if it can be steered from an initial complete state $z(0)$ to any final state $x(t_1) \in \mathbb{R}^n_+$. Therefore, there exists a control u_0 that steers $z(0)$ to zero, which means that

$$0 = x(t_1, z(0), u_0) = \Phi_0(t_1)x_0 + \sum_{i=0}^{1} \int_{-h_i}^{t_1-h_i} \Phi(t_1 - \tau - h_i)B_iu_0(\tau)\mathrm{d}\tau.$$

Moreover, there exists a positive control $\widetilde{u}$ from the state $z(0)$ to $\widetilde{x}$. Hence

$$\widetilde{x} = x(t_1, z(0), \widetilde{u}) = \Phi_0(t_1)x_0 + \sum_{i=0}^{1} \int_{-h_i}^{t_1-h_i} \Phi(t_1 - \tau - h_i)B_i\widetilde{u}(\tau)\mathrm{d}\tau.$$

From the above dependencies we have

$$\widetilde{x} - \sum_{i=0}^{1} \int_{-h_i}^{t_1-h_i} \Phi(t_1 - \tau - h_i) B_i [\widetilde{u}(\tau) - u_0(\tau)] \mathrm{d}\tau = 0.$$

Multiplying by $\widetilde{x}^T$ and applying (19), we obtain $\widetilde{x}^T \widetilde{x} = 0$. It follows that $\widetilde{x} = 0$, which is a contrary to our assumption. Therefore the matrix $R(0, t_1)$ is nonsingular and hence $rank\ R(0, t_1) = n$. $\square$

6. Examples

Below we present several numerical examples to illustrate the obtained theoretical results.

Example 1 *Let us consider a positive fractional system described by the following differential equation*

$$^{C}D^{\frac{1}{2}} x(t) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u(t), \qquad t > 0. \tag{20}$$

Since A is the 3×3 matrix, we consider following unite base vectors in the space $\mathbb{R}^3$

$$e[1] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad e[2] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad e[3] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Unite base vectors in the space $\mathbb{R}^2$ corresponding to the matrix B have the form

$$e(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad e(2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We are looking for a constant μ, for which the condition $e[k] = \mu B e(l)$ holds, where $k = 1, 2, 3$ and $l = 1, 2$.

$$Be(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e[1],$$

$$Be(2) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e[3].$$

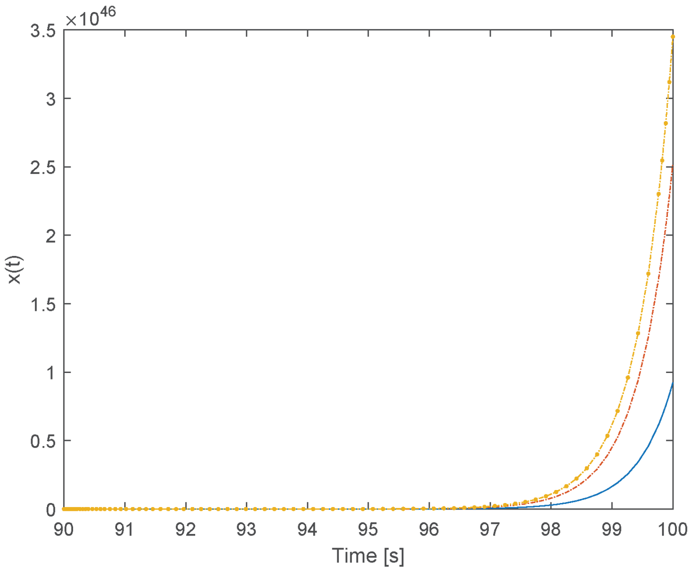

Figure 1: Trajectories of the system (20) for selected controls

By means of vectors $e(1), e(2)$ *and the constant* μ *it is impossible to get the base vector* $e[2]$. *According to Theorem 6, the system (20) is not approximately controllable.*

Example 2 *Let be given a positive fractional system*

$$ {}^{C}D^{\frac{1}{3}}x(t) = Ax(t) + Bu(t) \tag{21}$$

where $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

On the basis of Theorem 7 we will show that the positive system (21) is exactly controlllable on $[0, 1]$ *with zero initial conditions.*

We start with finding the matrix $\Phi(t)$ *for* $\alpha = 1/3$

$$\Phi(t) = \sum_{k=0}^{1} \frac{A^k t^{(k+1)\frac{1}{3}-1}}{\Gamma((k+1)\frac{1}{3})} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{t^{-\frac{2}{3}}}{\Gamma\left(\frac{1}{3}\right)} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \frac{t^{-\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)}$$

$$= \begin{bmatrix} \dfrac{t^{-\frac{2}{3}}}{\Gamma\left(\dfrac{1}{3}\right)} + \dfrac{t^{-\frac{1}{3}}}{\Gamma\left(\dfrac{2}{3}\right)} & 0 \\ 0 & \dfrac{t^{-\frac{2}{3}}}{\Gamma\left(\dfrac{1}{3}\right)} + \dfrac{2t^{-\frac{1}{3}}}{\Gamma\left(\dfrac{2}{3}\right)} \end{bmatrix}.$$

Knowing that $BB^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $\Phi(t) = \Phi^T(t)$, *we determine the matrix* R

$$R = \int_0^1 \Phi(t-\tau)BB^T\Phi^T(t-\tau)\,\mathrm{d}\tau = \begin{bmatrix} \frac{-3}{\left(\Gamma\left(\frac{1}{3}\right)\right)^2} + \frac{3}{\left(\Gamma\left(\frac{2}{3}\right)\right)^2} & 0 \\ 0 & \frac{-3}{\left(\Gamma\left(\frac{1}{3}\right)\right)^2} + \frac{12}{\left(\Gamma\left(\frac{2}{3}\right)\right)^2} \end{bmatrix}.$$

Using approximate values of the gamma function: $\Gamma\left(\frac{1}{3}\right) \approx 2.68$ *and* $\Gamma\left(\frac{2}{3}\right) \approx 1.35$, *we get the following approximate result*

$$\begin{bmatrix} 1.23 & 0 \\ 0 & 6.16 \end{bmatrix}.$$

Each row and each column of the above matrix contains only one positive element and the others are zeros. Therefore, it is the generalized permutation matrix. It follows that the positive system (21) is exactly controllable.

Now we will show that the system is also approximately controllable (see Remark 2).

Base unit vectors in the space $\mathbb{R}^2$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *correspond to vectors* $e[1], e[2]$ *as well as to vectors* $e(1), e(2)$.

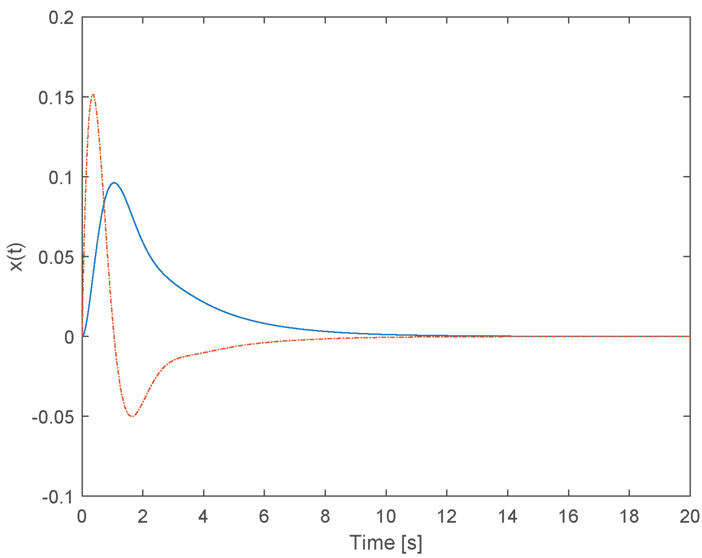

Figure 2: Trajectories of the system (21) for selected controls

For $\mu = 1$ the following inequalities are satisfied:

$$Be(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e[2],$$
$$Be(2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e[1].$$

Therefore, there exist $\mu = 1 > 0$, $k = 1, 2$ and $l = 1, 2$, such that the condition $e[k] = \mu Be(l)$ from Theorem 6 holds. It follows that the positive fractional system (21) is approximately controllable which agrees with Remark 2.

Example 3 *For a positive fractional system with delay in control*

$$^{C}D^{\frac{1}{2}}x(t) = Ax(t) + B_0u(t) + B_1u(t-3), \tag{22}$$

where $t \in [0, 1]$, $z(0) = (0, 0)$, and the system matrices are as follows

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

we will verify whether the condition from Theorem 8 holds, that is if $rank\, R(0, 1) = 2$.

We start with finding the pseudo-transition matrix $\Phi(t)$ and its transpose $\Phi^T(t)$:

$$\Phi(t) = \sum_{k=0}^{1} \frac{A^k t^{(k+1)\frac{1}{2}-1}}{\Gamma\left((k+1)\frac{1}{2}\right)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \frac{t^0}{\Gamma(1)} = \begin{bmatrix} \frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} & 0 \\ 0 & \frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} + 1 \end{bmatrix},$$

$$\Phi^T(t) = \Phi(t) = \begin{bmatrix} \frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} & 0 \\ 0 & \frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} + 1 \end{bmatrix}.$$

For $t_1 = 1$, $h_0 = 0$, and $h_1 = 3$the controllability matrix $R(0, 1)$ has the form

$$R(0, 1) = \int_0^1 \Phi(1-\tau)B_0B_0^T\Phi^T(1-\tau)d\tau + \int_{-3}^{-2} \Phi(-2-\tau)B_1B_1^T\Phi^T(-2-\tau)\mathrm{d}\tau,$$

so the next step is computing products:

$$\Phi(1-\tau)B_0B_0^T\Phi^T(1-\tau),$$
$$\Phi(-2-\tau)B_1B_1^T\Phi^T(-2-\tau),$$

where

$$B_0B_0^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_1B_1^T = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}.$$

Integrating elements of the matrix and then adding them, we obtain

$$R(0,1) = \begin{bmatrix} \int\limits_0^1 \frac{(1-\tau)^{-1}}{\left(\Gamma(\frac{1}{2})\right)^2}\mathrm{d}\tau & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \int\limits_{-3}^{-2} 4\left(\frac{(-2-\tau)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}+1\right)^2 \mathrm{d}\tau \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & \frac{16}{\Gamma(\frac{1}{2})}+4 \end{bmatrix}.$$

We see that the determinant of the above matrix is equal to zero which means that $rank\, R(0,1) \neq 2$*. Therefore, based on Theorem 8 we conclude that the positive system (22) is not relatively controllable on* $[0,1]$.

Example 4 *Let us consider the positive fractional system with delay in control*

$$^C D^{\frac{1}{2}}x(t) = Ax(t) + B_0u(t) + B_1u(t-1) \tag{23}$$

for $t \in [0,2]$*, with zero initial conditions and the following matrices*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Applying Theorem 8 we will check whether the system is relatively controllable on the time interval $[0,2]$*, that is whether*

$$rank\left(\sum_{i=0}^{1}\int\limits_{-h_i}^{t_1-h_i}\Phi(1-\tau-h_i)B_iB_i^T\Phi^T(1-\tau-h_i)\mathrm{d}\tau\right) = 2.$$

We determine matrices $\Phi(t)$ *and* $\Phi^T(t)$ *first*

$$\Phi(t) = \sum_{k=0}^{1}\frac{A^k t^{(k+1)\frac{1}{2}-1}}{\Gamma\left((k+1)\frac{1}{2}\right)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{t^0}{\Gamma(1)} = \begin{bmatrix} \frac{1}{\sqrt{\pi t}} & 1 \\ 1 & \frac{1}{\sqrt{\pi t}} \end{bmatrix} = \Phi^T(t).$$

Next, we calculate products $B_0B_0^T$ *and* $B_1B_1^T$

$$B_0B_0^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B_1B_1^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Obtained results we substitute into the formula for $R(0,2)$

$$R(0,2) = \int_0^2 \Phi(2-\tau)B_0B_0^T\Phi^T(2-\tau)\,\mathrm{d}\tau + \int_{-1}^1 \Phi(1-\tau)B_1B_1^T\Phi^T(1-\tau)\,\mathrm{d}\tau.$$

Multiplying matrices and integrating elements, we have

$$R(0,2) = \begin{bmatrix} 4 + \dfrac{\ln 2}{\pi} + \dfrac{4\sqrt{2\pi}}{\pi} & 2 + \dfrac{\ln 2}{\pi} + \dfrac{6\sqrt{2\pi}}{\pi} \\ 2 + \dfrac{\ln 2}{\pi} + \dfrac{6\sqrt{2\pi}}{\pi} & 2 + \dfrac{2\ln 2}{\pi} + \dfrac{4\sqrt{2\pi}}{\pi} \end{bmatrix}.$$

Obviously $rank\, R(0,2) = 2$. *Therefore, according to Theorem 8, positive system (22) is relatively controllable on* $[0,2]$.

7. Practical example

Let us consider an electrical circuit of fractional order consisting of resistors R_i, inductances L_i and voltage sources e_i, as shown in Fig. 3 [16].

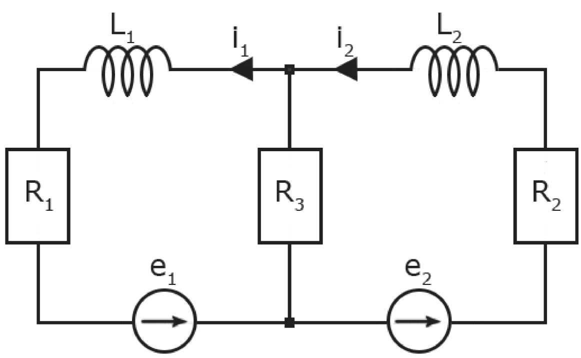

Figure 3: Electrical circuit

There is a voltage drop during current flow through the voltage source. From Ohm's law, the drop in voltage U corresponds to the product of the resistance R

and the current intensity i. For each mesh we write an appropriate equation using Kirchhoff's second law, which says that in a closed electrical circuit the sum of voltages at sources e_i is equal to the sum of voltage drops at all current receivers U_i. There is an inductor in each mesh, so the respective current voltages increase by the electromotive force ε which, according to Faraday's law, is expressed by the formula $\varepsilon = -L\frac{d^\alpha i}{dt^\alpha}$ in the opposite direction to i, as Lenz's law states.

For $\alpha \in (0, 1]$ and the Caputo fractional derivative $\frac{\mathrm{d}^\alpha i}{\mathrm{d}t^\alpha} = {}^C D^\alpha i$, we obtain the following equalities

$$
\begin{aligned}
e_1 &= R_1 i_1 + L_1 {}^C D^\alpha i_1 + R_3(i_1 - i_2),\\
e_2 &= R_2 i_1 + L_2 {}^C D^\alpha i_2 + R_3(i_2 - i_1).
\end{aligned}
$$

The above equations can be expressed in the matrix form as

$$
{}^C D^\alpha \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{24}
$$

where

$$
A = \begin{bmatrix} \frac{-(R_1 + R_3)}{L_1} & \frac{R_3}{L_1} \\ \frac{R_3}{L_2} & \frac{-(R_2 + R_3)}{L_2} \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}.
$$

The system defined in this way is positive because A is a Metzler matrix and B is a matrix with non-negative elements.

We take the following values: $R_1 = 1$, $R_2 = 2$, $R_3 = 0$, $L_1 = 1$, $L_2 = 1$. Then the matrices A and B have the forms

$$
A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
$$

We will show that for the above matrices A and B the system (24) of order $\alpha = \frac{1}{3}$ is exactly controllable on the time interval [0, 1].

$$\Phi(t) = \sum_{k=0}^{1} \frac{A^k t^{(k+1)\frac{1}{3}-1}}{\Gamma\left((k+1)\frac{1}{3}\right)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{t^{-\frac{2}{3}}}{\Gamma\left(\frac{1}{3}\right)} + \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \frac{t^{-\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)} =$$

$$= \begin{bmatrix} \frac{t^{-\frac{2}{3}}}{\Gamma\left(\frac{1}{3}\right)} - \frac{t^{-\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)} & 0 \\ 0 & \frac{t^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} - 2\frac{t^{-\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)} \end{bmatrix} = \Phi^T(t), \tag{25}$$

$$BB^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For this purpose we verify that

$$R = \int_0^1 \begin{bmatrix} \frac{(t-\tau)^{-\frac{2}{3}}}{\Gamma\left(\frac{1}{3}\right)} - \frac{(t-\tau)^{-\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)} & 0 \\ 0 & \frac{(t-\tau)^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} - 2\frac{(t-\tau)^{-\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} \frac{(t-\tau)^{-\frac{2}{3}}}{\Gamma\left(\frac{1}{3}\right)} - \frac{(t-\tau)^{-\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)} & 0 \\ 0 & \frac{(t-\tau)^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} - 2\frac{(t-\tau)^{-\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)} \end{bmatrix} \mathrm{d}\tau$$

is the generalized permutation matrix.

$$R = \begin{bmatrix} -\frac{3}{\left(\Gamma\left(\frac{1}{3}\right)\right)^2} + \frac{3}{\left(\Gamma\left(\frac{2}{3}\right)\right)^2} & 0 \\ 0 & -\frac{3}{\left(\Gamma\left(\frac{1}{3}\right)\right)^2} + \frac{12}{\left(\Gamma\left(\frac{2}{3}\right)\right)^2} \end{bmatrix} \approx \begin{bmatrix} 1.23 & 0 \\ 0 & 6.17 \end{bmatrix}.$$

Hence, on the basis of Theorem 7, the system (24) is exactly controllable on [0, 1].

8. Concluding remarks

Positive fractional systems without delays as well as positive fractional systems with single delay in control have been studied in the paper. The approximate (Theorem 6) and exact (Theorem 7) controllability criteria for positive systems without delays has been established. The new necessary and sufficient conditions for the relative controllability (Theorem 8) of positive fractional systems with single delay have been formulated and proved. The numerical examples have been presented to illustrate the theoretical results. The given example of electrical circuit is one of the possible practical applications of the discusses theoretical issues. The presented results can be extended to positive semilinear fractional systems.

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