

Stability and stabilization of positive linear dynamical systems: new equivalent conditions and computations

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Abstract. New equivalent conditions of the asymptotical stability and stabilization of positive linear dynamical systems are investigated in this paper. The asymptotical stability of the positive linear systems means that there is a solution for linear inequalities systems. New necessary and sufficient conditions for the existence of solutions of the linear inequalities systems as well as the asymptotical stability of the linear dynamical systems are obtained. New conditions for the stabilization of the resultant closed-loop systems to be asymptotically stable and positive are also presented. Both the stability and the stabilization conditions can be easily checked by the so-called I -rank of a matrix and by solving linear programming (LP). The proposed LP has compact form and is ready to be implemented, which can be considered as an improvement of existing LP methods. Numerical examples are provided in the end to show the effectiveness of the proposed method.

Key words: Positive linear systems, stability, stabilization, linear inequalities systems, consistency, I -rank.

1. Introduction

Positive systems are dynamical systems whose state variables are positive (or nonnegative) in values at all times. Systems with nonnegative states are important because in practice, the non-negativity property can be found frequently in numerous fields such as biology, chemistry, physics, ecology, economy or sociology (specific examples can be found in e.g. [1–5] and references therein). Since positive systems are defined on cones, not on linear spaces, many well-established results of standard linear systems cannot be applied directly to positive systems. Therefore, positive systems have gained increasing interest over the past two decades due to their extensive applications in practice and theoretical complexes in control theory [1–11]. Some developments and applications in positive systems theory are given in [8].

Stability and stabilization are basic issues of standard linear dynamical systems as well as of positive linear systems. Both issues have attracted considerable attention during the last decades. There have been some significant results on the asymptotical stability and stabilization of positive systems. Various approaches have been proposed such as algebra methods [12, 13], methods based on LMI techniques [9], method based on l_1 -induced norm [14] etc. Among the available methods, systems of linear inequalities method is an attractive method [15, 16]. In [15], the stability and stabilization conditions of the positive systems are formulated as a linear inequalities systems and the feedback control law can be obtained by solving linear in-

equalities systems with any standard numerical software. But there are limited discussions available on two problems: (1) On what conditions can one guarantee the existence of the related systems of linear inequalities? (2) The related linear inequalities systems are not standard linear programming, so how to obtain the feedback control law by solving the linear inequalities systems? These two problems are fundamental and there are few papers discussing these two problems. We further investigate these two problems in this paper.

The goal of this paper is to propose new equivalent conditions for the stability and stabilization of discrete-time and continuous-time systems and methods for checking the stability as well as obtaining the feedback control law while maintaining their positivity and stability. To this end, we present new conditions on the consistence and inconsistency of the linear inequalities systems based on Farkas theorem and I -rank of the coefficient matrices. Then we extend the results to the positive linear systems. About the solution of the linear inequalities system, we propose a standard linear programming model. The consistency of the linear inequalities systems and the solvability of the LP are discussed. Numerical checking method is an attractive method which was once used to check the positivity and stability of the dynamical systems because it is easy to implement. We once discussed this method for the positivity and stability checking of singular systems in [17–19]. Numerical examples are given in the end to show the effectiveness of our method. The advantage of the proposed method lies in the fact that they are not only sufficient and necessary, but also can be verified easily with any standard numerical software.

The paper is organized as follows. Section 2 analyzes the stability of the positive systems by means of linear inequalities systems. Equivalent conditions of consistence of the linear inequalities systems as well as the asymptotical stability of the

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linear dynamical systems are presented. We also give a numerical solving method for the the linear inequalities systems in Section 2. Section 3 presents the conditions of the stabilization of the positive linear systems and the numerical methods for finding the feedback control law. Examples are given in Section 4 to verify our conclusions. We end this paper in Section 5 by concluding remarks.

Notations: R_+^n denotes the nonnegative orthant of R^n . A^T denotes the transpose of the real matrix A , $a_i^T, i = 1, 2, \dots, n$ denotes the i th row of matrix A , $J^i, i = 1, 2, 3$ means the i -th matrix. For a real matrix M with elements $m_{ij}, 1 \leq i, j \leq n, M > 0$ means that its elements are positive (i.e., $m_{ij} > 0, 1 \leq i, j \leq n$) and $M \geq 0$ means that the elements are nonnegative (i.e., $m_{ij} \geq 0, 1 \leq i, j \leq n$), M is a Metzler matrix if $m_{ij} \geq 0, i \neq j$, which means all the off-diagonal elements are nonnegative if M is a Metzler matrix, I is the identity matrix with proper dimension.

2. Stability analysis

2.1. Stability analysis. Consider the following autonomous discrete-time and continuous-time linear systems:

$$x(k+1) = A_d x(k), \quad x(0) = x_0 \in R_+^n, \quad (1)$$

$$\dot{x}(t) = A_c x(t), \quad x(0) = x_0 \in R_+^n, \quad (2)$$

where $A_c, A_d \in R^{n \times n}, x \in R^n, k \in Z_+, Z_+ = [0, 1, 2, \dots], t \in [0, +\infty)$. System (1) is a positive system if for any initial conditions $x_0 \in R_+^n, x(k) \geq 0$ for all $k \in Z_+$. System (2) is a positive system if for any initial conditions $x_0 \in R_+^n, x(t) \geq 0$ for any $t \geq 0$. A necessary and sufficient condition for the system (1) and (2) to be positive is given by the following Theorem 1.

Theorem 1. [5, 20]

- (1) System (1) is a positive system for any $x_0 \in R_+^n$ if and only if $A_d \in R_+^{n \times n}$.
- (2) System (2) is a positive system for any $x_0 \in R_+^n$ if and only if A_c is a Metzler matrix.

Theorem 2. [15, 16] Assume that system (1)(or(2)) is positive, or equivalently that the matrix A_d is positive (or matrix A_c is a Metzler matrix), then the following statements are equivalent:

- (i) System (1) is asymptotically stable for every initial condition $x_0 \in R_+^n$ if and only if there exists a $\lambda \in R^n$ such that

$$(A_d - I)\lambda < 0, \quad \lambda > 0. \quad (3)$$

- (ii) System (2) is asymptotically stable for every initial condition $x_0 \in R_+^n$ if and only if there exists a $\lambda \in R^n$ such that

$$A_c \lambda < 0, \quad \lambda > 0. \quad (4)$$

Rewrite (3) into the form of the following system of linear inequalities:

$$\begin{pmatrix} A_d - I \\ -I \end{pmatrix}_{(2n) \times n} \lambda < \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (5)$$

For simplicity, denote $S = \begin{pmatrix} A_d - I \\ -I \end{pmatrix}$ and the set $\Omega_1 = \{\lambda \mid S\lambda < 0, \lambda > 0\}$ for (3) and $\Omega_2 = \{\lambda \mid A_c \lambda < 0, \lambda > 0\}$ for (4), both of them have the same form.

Lemma 1. [21] (Farkas Lemma) Let $M \in R^{m \times n}, b \in R^m$. Then exactly one of the following two statements is true:

- (i) There exists an $x \in R^n$ such that $Mx = b$ and $x \geq 0$.
- (ii) There exists a $y \in R^m$ such that $M^T y \geq 0$ and $b^T y < 0$.

Theorem 3. Let $M \in R^{m \times n}$, then exactly one of the following two statements is true:

$$(i) \quad Mx < 0. \quad (6)$$

$$(ii) \quad M^T y = 0, \quad y \geq 0, \quad \text{and } y \neq 0. \quad (7)$$

Proof. If there exist solutions for (6), i.e., there exists at least a $\bar{x} \in R^n$ such that $M\bar{x} < 0$, then for all $\bar{y} \geq 0, \bar{y} \neq 0$, we have $\bar{y}^T M\bar{x} < 0$, i.e., $\bar{x}^T M^T \bar{y} < 0$, which shows that (7) has no solutions otherwise contradicts to the hypotheses. If there exists no solutions for (6), then there exists no $\alpha < 0, x \in R^n$ such that $Mx \leq (\alpha, \dots, \alpha)$. Denote $\bar{M} = \{M, -e\}, \bar{b} = (0, \dots, 0, -1)$, where $e = (1, \dots, 1)^T \in R^n$. Then there exists no $\alpha < 0, x \in R^n$ such that

$$\bar{M} \begin{pmatrix} x \\ \alpha \end{pmatrix} \leq 0, \quad \bar{b}^T \begin{pmatrix} x \\ \alpha \end{pmatrix} > 0.$$

From Farkas Lemma 1, there exists solution for $\bar{M}^T y = \bar{b}, y \geq 0$, i.e., there exists solution for $M^T y = 0, e^T y = 1, y \geq 0$, which is equivalent to that fact that (7) has solutions. \square

Definition 1. If there exists at least a $x \in R^n$ such that system (6) holds, then system (6) is called consistent.

Theorem 4. System (6) is consistent if and only if system $M^T y = 0, y \geq 0$ in (7) has only one solution $y = 0$.

Proof. From Lemma 1, (6) has solutions if and only if (7) has no solutions, which means if $M^T y = 0, y \geq 0$ in (7) has solution, it holds only when $y = 0$. \square

The above alternative Theorem 3 and Theorem 4 give two sufficient and necessary conditions for the consistency of system (6), it is also a sufficient and necessary conditions for the asymptotical stability of systems (1) and (2). But the algebraic conditions are not easy to verify. Next we present another sufficient and necessary condition for checking the consistency of system (6) by means of the so-called I -rank in [22] of the coefficient matrix M of the linear inequality system. In the sequel, we firstly give a review of the corresponding definitions and results about I -rank of a given matrix.

Definition 2. [22] A real matrix $M \in R^{m \times n}$ is I -positive (or I -negative) with respect to the r th ($1 \leq r \leq n$) column if all elements of the r th column of M are positive (or negative). In both cases (I -positive or I -negative), the matrix M will be said to be I -definite with respect to the r th column. A matrix will be said to be I -positive (or I -negative, or I -definite) if it possesses at least a column whose entries are all positive (or negative).

Definition 3. [22] For the real matrix $M \in R^{n \times n}$, define the I -minors of M according to different cases.

(1) If the matrix M is not I -definite with respect to the given r th column $r = 1, 2, \dots, n$, then the elements of that column are divided into three classes:

- (i) those which are positive: $m_{ir}, i = i_1, i_2, \dots, i_P$;
- (ii) those which are negative: $m_{jr}, j = j_1, j_2, \dots, j_N$;
- (iii) those which are zeroes: $m_{kr}, k = k_1, k_2, \dots, k_Z$;

the number of elements in the respective classes are represented by P, N , and Z .

Define the elements of the matrix $M_1^{(r)}$ from matrix M as the following, for any pair of elements m_{ir}, m_{jr} in the r th column of matrix M (m_{ir} is positive and m_{jr} is negative), we have the second order determinants:

$$\begin{vmatrix} m_{ir} & m_{i1} \\ m_{jr} & m_{j1} \end{vmatrix}, \begin{vmatrix} m_{ir} & m_{i2} \\ m_{jr} & m_{j2} \end{vmatrix}, \dots, \begin{vmatrix} m_{ir} & m_{i(r-1)} \\ m_{jr} & m_{j(r-1)} \end{vmatrix},$$

$$\begin{vmatrix} m_{ir} & m_{i(r+1)} \\ m_{jr} & m_{j(r+1)} \end{vmatrix}, \dots, \begin{vmatrix} m_{ir} & m_{in} \\ m_{jr} & m_{jn} \end{vmatrix}.$$

The above row corresponds to one row in matrix $M_1^{(r)}$, thus we have $PN + Z$ rows in the derived matrix $M_1^{(r)}$, the zero element m_{kr} in matrix M corresponds to one row of the matrix $M_1^{(r)}$, we have Z rows in the matrix. Then the size of the matrix $M_1^{(r)}$ will be $(PN + Z) \times (n - 1)$. The sequential order of the rows will be determined by the convention that: (1) each (ij) row precedes every (k) row; and (2) two (ij) rows that one precedes which has the smaller i or (in case of equal i 's) that one which has the smaller j .

(2) If the matrix M is I -definite with respect to the given r th column, definite the derived matrix as the matrix of one row and $n - 1$ columns, all elements of which are $+1$ or -1 according to M is I -positive or I -negative with respect to the r th column.

In both cases, whether the matrix M is not I -definite or I -definite respect to the given r th ($r = 1, 2, \dots, n$) column, we will have n derived matrices $M_1^{(r)}$ ($r = 1, \dots, n$) with size $(PN + Z) \times (n - 1)$. The n matrices $M_1^{(1)}, M_1^{(2)}, \dots, M_1^{(n)}$ will be called the I -minors of $n - 1$ columns of the matrix M .

In a similar way, for any derived matrix $M_1^{(r)}$, $r = 1, 2, \dots, n$, we can compute all the I -minors, it will denoted by $M_2^{(rs)}$, $s = 1, 2, \dots, n - 1$. Then we have $n(n - 1)$ matrices which are also called I -minors of $n - 2$ columns of matrix M . More generally, define recursively $M_{p+1}^{(r_1 r_2 \dots r_{p+j})}$ as the I -minors of $M_p^{(r_1 r_2 \dots r_p)}$. We thus have relative to the matrix M a well-defined system of I -minors $M_p^{(r_1 r_2 \dots r_p)}$ ($p = 1, 2, \dots, n - 1$). The matrix M itself may be considered as the I -minors of n columns.

Definition 4. [22] The I -rank of a matrix M is said to be k if it possesses at least one I -minor of k columns which is I -definite, but does not possess any I -minors of $k + 1$ columns which is I -definite. If none of its I -minors are I -definite, it will be said to be of I -rank zero.

Proposition 1. The I -rank of a matrix M is not altered if

- (i) Any two rows or any two columns are interchanged.
- (ii) All elements of any row or any column are multiplied by the same positive constant.

Example. Given matrix $M = \begin{pmatrix} 1 & -1 & 2 \\ -5 & 7 & 0 \\ 2 & -6 & -4 \end{pmatrix}$, we compute I -minors of the matrix M as the following.

$$M_1^{(1)} = \begin{pmatrix} 2 & 10 \\ -16 & -20 \end{pmatrix}, \quad M_1^{(2)} = \begin{pmatrix} 2 & 14 \\ -16 & -28 \end{pmatrix}$$

$$M_1^{(3)} = \begin{pmatrix} 8 & -16 \\ 0 & 0 \end{pmatrix}, \quad M_2^{(11)} = \begin{pmatrix} 2 \\ -16 \end{pmatrix},$$

$$M_2^{(12)} = \begin{pmatrix} 10 \\ -20 \end{pmatrix}, \quad M_2^{(21)} = \begin{pmatrix} 2 \\ -16 \end{pmatrix}, \quad M_2^{(22)} = \begin{pmatrix} 14 \\ -28 \end{pmatrix},$$

$$M_2^{(31)} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}, \quad M_2^{(32)} = \begin{pmatrix} -16 \\ 0 \end{pmatrix}.$$

From the above I -minors and the definition of the I -rank, the I -rank of the above matrix M is 0.

Theorem 5. [22] The sufficient and necessary condition of the set Ω_1 (or Ω_2) is nonempty is that the I -rank of the matrix S (or A) is $k > 0$.

Theorem 6. For positive system (1)(or (2)), it is asymptotically stable if and only if the I -rank of the coefficient matrix S (or A) in Ω_1 (or Ω_2) is greater than 0.

Proof. From Theorem 2 (i), (ii) and (5), we know that system (1) (or (2)) is asymptotically stable if and only if the set Ω_1 (or Ω_2) is nonempty. Suppose the I -rank of the matrix S (or A) is k , from Theorem 5, the sufficient and necessary condition of the set Ω_1 (or Ω_2) is empty is that $k > 0$, then we complete the proof. \square

2.2. Numerical methods for checking asymptotical stability.

To transform the linear inequalities systems into linear equalities systems, we need the following Lemma 2.

Lemma 2. Suppose $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)^T \in R^n$, the inequalities $a_i^T \lambda \leq \varepsilon_i, i = 1, 2, \dots, n$ hold if and only if there exists an $\omega \in R^n, \omega \geq 0$, such that $a_i^T \lambda + \omega_i = \varepsilon_i, i = 1, 2, \dots, m$ hold.

Proof. The result is trivial and obvious but important for our conclusion in the following. If there exists an $\omega \in R^n, \omega \geq 0$ such that $a_i^T \lambda + \omega_i = \varepsilon_i, i = 1, 2, \dots, n, a_i^T \lambda \leq \varepsilon_i, i = 1, 2, \dots, m$ hold since $\omega_i \geq 0$. If $a_i^T \lambda \leq \varepsilon_i, i = 1, 2, \dots, n$ hold, there exists $\omega_i, i = 1, 2, \dots, n$ such that $a_i^T \lambda \leq \varepsilon_i, i = 1, 2, \dots, n$, take $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$, thus we get the result. \square

Denote $J^{(1)} = (A, I_n), x = (\lambda^T, \omega^T)^T$, then the asymptotic stability of system (1) (or (2)) is equivalent to the existence of the nonnegative solution for linear equations systems $J^{(1)}x = \omega$, for $\varepsilon < 0$.

Theorem 7. Let $A \in R_+^{m \times n}$. The positive system (1) (or system (2)) is asymptotically stable if and only if for any $\varepsilon \in R^n$, $\varepsilon < 0$, linear equations system

$$(A, I_n) \begin{pmatrix} \lambda \\ \omega \end{pmatrix} = \varepsilon \tag{8}$$

has nonnegative solution $\begin{pmatrix} \lambda \\ \omega \end{pmatrix}$, i.e. the linear equations system is consistent in nonnegative variables $(\lambda, \omega)^T$ (for system (1), A takes $A_d - I$ and for system (2), A takes A_c).

Proof. The results can be obtained by Lemma 2 and Theorem 2 directly if we take (A, I_n) as the coefficient matrix and $\begin{pmatrix} \lambda \\ \omega \end{pmatrix}$ as the unknown, then we have

$$J^{(1)}x = \varepsilon, \tag{9}$$

and then from the matrix equation and linear equalities systems we obtain the result of Theorem 7. \square

Taking into account the problem of nonnegative solution of linear system (9), we can transform (9) into an optimization problem, the nonnegative solution of system (9) can be obtained by solving the following optimization (10).

The nonnegative solution of systems (10) can be obtained by solving the following optimization problem (11). The optimal solution of optimization problem (11) is the nonnegative solution of linear equation systems (10). Rewrite matrix $J^{(1)} = [J_1^{(1)}, J_2^{(1)}, \dots, J_m^{(1)}]$, where $J_i^{(1)}$, $i = 1, 2, \dots, m$ is the column vector of matrix $J^{(1)}$. Define function $f(x) = \sum_{i=1}^m |e_i - (J_i^{(1)})^T x|$, the solution (or approximation solution) of systems (10) can be obtained by finding the solution of the following optimization problem (11).

$$\begin{aligned} \min f(x) \\ \text{s.t. } x \geq 0, \quad \text{and } x \neq 0, \end{aligned} \tag{10}$$

which means finding the nonnegative but not zero solution of (11). We present the specific result in the following Proposition 2.

Proposition 2. Finding the solution of optimization problem (10) is equivalent to solving the following optimization problem (11).

$$\begin{aligned} \min g(y) = (u^T, \theta^T) \begin{pmatrix} t \\ x \end{pmatrix} = \sum_{i=1}^m t_i \\ \text{s.t. } B \begin{pmatrix} t \\ x \end{pmatrix} \geq \begin{pmatrix} \varepsilon \\ -\varepsilon \end{pmatrix}, \quad \begin{pmatrix} t \\ x \end{pmatrix} \geq 0. \end{aligned} \tag{11}$$

where

$$B = \begin{pmatrix} I & J^{(1)} \\ I & -J^{(1)} \end{pmatrix}, \quad t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

$$y = \begin{pmatrix} t \\ x \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad \text{The optimal solution of (11) is}$$

$$y^{(0)} = \begin{pmatrix} t^{(0)} \\ x^{(0)} \end{pmatrix}, \quad x^{(0)} \text{ is the nonnegative solution of optimization (10), moreover, } f(x^{(0)}) = g(y^{(0)}) = \sum_{i=1}^m t_i^{(0)}.$$

Proof. From (9) and (11), we have

$$t \geq \varepsilon - J^{(1)}x, \quad t \geq -(\varepsilon - J^{(1)}x), \tag{12}$$

thus we have $t_i \geq \varepsilon_i - (J^{(1)})_i^T x$, $t_i \geq -(\varepsilon_i - (J^{(1)})_i^T x)$, such that

$$t_i \geq |\varepsilon_i - (J^{(1)})_i^T x|, \quad i = 1, 2, \dots, m. \tag{13}$$

Suppose $y^{(0)} = \begin{pmatrix} t^{(0)} \\ x^{(0)} \end{pmatrix}$ is the solution of (11), then, we will prove that $x^{(0)}$ is the nonnegative solution of (10) by contradiction. If it is not, there must exist a nonnegative vector x^* such that $f(x^*) < f(x^{(0)})$, let

$$t_i^* = |\varepsilon_i - (J^{(1)})_i^T x^*|, \quad i = 1, 2, \dots, m, \tag{14}$$

then we have

$$y^* = \begin{pmatrix} t^* \\ x^* \end{pmatrix} \geq 0,$$

$$\text{where } t^* = (t_1^*, t_2^*, \dots, t_m^*)^T, \quad x^* = (x_1^*, x_2^*, \dots, x_m^*)^T,$$

$$By^* = \begin{pmatrix} t^* & J_1^{(1)}x^* \\ t^* & -J_1^{(1)}x^* \end{pmatrix} \geq \begin{pmatrix} \varepsilon \\ -\varepsilon \end{pmatrix}, \tag{15}$$

which means y^* satisfies (11) and

$$g(y^*) = \sum_{i=1}^m t_i^* = \sum_{i=1}^m |\varepsilon_i - (J_i^{(1)})^T x^*| = f(x^*).$$

Again because $t^{(0)}$ satisfies (13) and $f(x^*) < f(x^{(0)})$, we have

$$g(y^*) = f(x^*) < f(x^{(0)}) = \sum_{i=1}^n |\varepsilon_i - (J_i^{(1)})^T x^{(0)}| < \sum_{i=1}^n t_i^{(0)} = g(y^{(0)}),$$

which is contradictory to the fact that $y^{(0)}$ is the solution of (11).

In the sequel, we shall prove

$$t_i^{(0)} = |\varepsilon_i - (J_i^{(1)})^T x^{(0)}|, \quad i = 1, 2, \dots, n. \tag{16}$$

If (16) does not hold for all $i = 1, 2, \dots, m$, from (13), we can deduce that

$$g(y^{(0)}) = \sum_{i=1}^m t_i^{(0)} = \sum_{i=1}^m |\varepsilon_i - (J_i^{(1)})^T x^{(0)}| > f(x^{(0)}),$$

then similar to the former proof, we derive a contradiction that $y^{(0)}$ is the solution of (11), then we have (16) holds, i.e.

$$g(y^{(0)}) = \sum_{i=1}^n t_i^{(0)} = \sum_{i=1}^n |\varepsilon_i - (J_i^{(1)})^T x^{(0)}| = f(x^{(0)}),$$

which completes the proof. □

From Theorem 7 and Proposition 2, we have the following Theorem 8 which is a necessary and sufficient condition of the asymptotically stability for linear systems (1) and (2).

Theorem 8. The positive system (1) (or system (2)) is asymptotically stable if and only if for any $\varepsilon \in R^n$, $\varepsilon < 0$, the following optimization has a solution.

$$\begin{aligned} \min \quad & g(y) = (u^T, \theta^T) \begin{pmatrix} t \\ x \end{pmatrix} = \sum_{i=1}^m t_i \\ \text{s.t.} \quad & B \begin{pmatrix} t \\ x \end{pmatrix} \geq \begin{pmatrix} \varepsilon \\ -\varepsilon \end{pmatrix}, \quad \begin{pmatrix} t \\ x \end{pmatrix} \geq 0. \end{aligned} \tag{17}$$

where

$$B = \begin{pmatrix} I & J^{(1)} \\ I & -J^{(1)} \end{pmatrix}, \quad t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

$$y = \begin{pmatrix} t \\ x \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Remark 1. From Theorem 8, the autonomous positive system (1) (or (2)) is asymptotically stable while guaranteeing the positivity if and only if the corresponding optimization problems (17) has solutions. Theorem 8 shows that the asymptotical stability of the system (1) (or (2)) can be checked by the existence of solution for the LP optimization (17). It is an attractive method for checking the asymptotic stability of the positive linear dynamical systems.

3. Stabilization of the positive linear dynamical systems

3.1. Equivalent conditions of stabilization. This section studies the stabilization problem of the positive linear dynamical systems. In the following forced system:

$$x(k+1) = A_d x(k) + B_d u(k), \quad x(0) = x_0 \in R_n^+, \tag{18}$$

$$\dot{x}(t) = A_c x(t) + B_c u(t), \quad x(0) = x_0 \in R_n^+. \tag{19}$$

The control law $u \in R^p$ is assumed to be a constant state-feedback $u(k) = K_d x(k)$ for (18) ($u(t) = K_c x(t)$ for (19)), where

K_d (or K_c) is the gain matrix. The control law must be designed in such way that the resulting governed system is positive and asymptotically stable. The stability synthesis of the systems (18) and (19) can be obtained by the following Theorem 11 and Theorem 12.

Definition 5. [20]

(1) The discrete-time (18) is called (internally) positive systems if $x(k) \in R_+^n$, $k \in Z_+$ for any initial conditions $x_0 \in R_+^n$ and all inputs $u(k) \in R_+^m$, $k \in Z_+$.

(2) The continuous-time (19) is called (internally) positive systems if $x(t) \in R_+^n$, $t \geq 0$ for any initial conditions $x_0 \in R_+^n$ and all inputs $u(t) \in R_+^m$, $t \geq 0$.

Theorem 9. [5, 20] For the systems (18) and (19), the positivity conditions of the systems are as the following:

(1) The systems (18) is a (internally) positive systems for any $x_0 \in R_+^n$ if and only if $A_d \in R_+^{n \times n}$, $B_d \in R_+^{n \times m}$.

(2) The systems (19) is a (internally) positive systems for any $x_0 \in R_+^n$ if and only if A_c is a Metzler matrix and $B_c \in R_+^{n \times m}$.

Theorem 10. [15] For the positive discrete-time system (18), the following statements are equivalent:

(1) There exist $n+1$ vectors $d = (d_1, d_2, \dots, d_n)^T \in R^n$ and $z_1, z_2, \dots, z_n \in R^p$ such that

$$\begin{aligned} (A_d - I)d + B_d \sum_{i=1}^n z_i &< 0; \\ d &> 0; \\ a_{ij}d_j + (b_d)_i z_j &\geq 0, \quad 1 \leq i, \quad j \leq n. \end{aligned} \tag{20}$$

with $B_d^T = ((b_d)_1^T, (b_d)_2^T, \dots, (b_d)_n^T)$, where $(b_d)_i$, $i = 1, 2, \dots, n$ denotes the i -th row of matrix B_d in system (18).

(2) There exists a state-feedback law $u(k) = Kx(k)$ such that the closed-loop system is positive and asymptotically stable, where K can be calculated as follows:

$$K = (d_1^{-1}z_1, d_2^{-1}z_2, \dots, d_n^{-1}z_n).$$

For the positive continuous-time systems (19), we have the following analogous Theorem 11 as Theorem 10 for discrete-time systems.

Theorem 11. [16] For the positive continuous-time systems (19), the following statements are equivalent:

(1) There exist $n+1$ vectors $d = [d_1, d_2, \dots, d_n]^T \in R^n$ and $z_1, z_2, \dots, z_n \in R^p$ such that

$$\begin{aligned} A_c d + B_c \sum_{i=1}^n z_i &< 0; \\ d &> 0; \\ a_{ij}d_j + (b_c)_i z_j &\geq 0, \quad 1 \leq i, \quad j \leq n, \quad i \neq j. \end{aligned} \tag{21}$$

with $B_c^T = ((b_c)_1^T, (b_c)_2^T, \dots, (b_c)_n^T)$, where $(b_c)_i$, $i = 1, 2, \dots, n$ denotes the i -th row of matrix B_c in system (19).

(2) There exists a state-feedback law $u(t) = Kx(t)$ such that the closed-loop system is positive and asymptotically stable,

where K can be calculated as follows:

$$K = (d_1^{-1}z_1, d_2^{-1}z_2, \dots, d_n^{-1}z_n).$$

The proof in above Theorem 11 and Theorem 12 can be obtained by replacing A_d in system (1) with $A_d + B_d K d$ and let $B_d K d = B_d \sum_{i=1}^n z_i$ (for continuous-time system by replacing A_c in system (2) with $A_c + B_c K d$ and let $B_c K d = B_c \sum_{i=1}^n z_i$), detailed proofs can be found in reference [15] and [16]. In the sequel, we derive the new conditions according to these two theorems.

We can rewrite the form of (20) in Theorem 11 as the following:

$$\begin{pmatrix} A_d - I & B_d & \cdots & B_d \\ -I & 0 & \cdots & 0 \end{pmatrix} \cdot (d_1, d_2, \dots, d_n, z_1^T, z_2^T, \dots, z_n^T)^T < 0; \quad (22)$$

$$\begin{pmatrix} -(a_d)_{11} & 0 & \cdots & 0 & -(b_d)_1 & 0 & \cdots & 0 \\ 0 & -(a_d)_{12} & \cdots & 0 & 0 & -(b_d)_1 & \cdots & 0 \\ \cdots & \cdots \\ 0 & \cdots & 0 & -(a_d)_{1n} & 0 & \cdots & 0 & -(b_d)_1 \\ -(a_d)_{21} & 0 & \cdots & 0 & -(b_d)_2 & 0 & \cdots & 0 \\ 0 & -(a_d)_{22} & \cdots & 0 & 0 & -(b_d)_2 & \cdots & 0 \\ \cdots & \cdots \\ 0 & \cdots & 0 & -(a_d)_{2n} & 0 & \cdots & 0 & -(b_d)_2 \\ \cdots & \cdots \\ -(a_d)_{n1} & 0 & \cdots & 0 & -(b_d)_n & 0 & \cdots & 0 \\ 0 & -(a_d)_{n2} & \cdots & 0 & 0 & -(b_d)_n & \cdots & 0 \\ \cdots & \cdots \\ 0 & \cdots & 0 & -(a_d)_{nm} & 0 & \cdots & 0 & -(b_d)_n \end{pmatrix} \cdot (d_1, d_2, \dots, d_n, z_1^T, z_2^T, \dots, z_n^T)^T \leq 0. \quad (23)$$

It should be kept in mind that in the above equation (22), the size of the coefficient matrix is $(2n) \times (n + np)$, and the variable $\omega = (d_1, d_2, \dots, d_n, z_1^T, z_2^T, \dots, z_n^T)^T$ is $(n + np)$ dimensional, in brevity, we write the coefficient matrix as M_d and R_d in (22), (23). The size of the coefficient matrix R_d is $n^2 \times (n + np)$ dimensional and the variable is the same as that in (22), we rewrite the coefficient matrix as R_d . Then we can reformulate the equations (22) and (23) as:

$$\Omega_3 = \{\omega, \omega \in R^{(p+1)n} \mid M_d \omega < 0, R_d \omega \leq 0\}. \quad (24)$$

We can also rewrite the form of (21) in Theorem 12 as the following:

$$\begin{pmatrix} A_c & B_c & \cdots & B_c \\ -I & 0 & \cdots & 0 \end{pmatrix} \cdot (d_1, d_2, \dots, d_n, z_1^T, z_2^T, \dots, z_n^T)^T < 0; \quad (25)$$

$$\begin{pmatrix} 0 & -(a_c)_{11} & \cdots & 0 & 0 & -(b_c)_1 & \cdots & 0 \\ \cdots & \cdots \\ 0 & \cdots & 0 & -(a_c)_{1n} & 0 & \cdots & 0 & -(b_c)_1 \\ -(a_c)_{21} & 0 & \cdots & 0 & -(b_c)_2 & 0 & \cdots & 0 \\ \cdots & \cdots \\ 0 & \cdots & 0 & -(a_c)_{2n} & 0 & \cdots & 0 & -(b_c)_2 \\ \cdots & \cdots \\ -(a_c)_{n1} & 0 & \cdots & 0 & -(b_c)_n & 0 & \cdots & 0 \\ 0 & -(a_c)_{n2} & \cdots & 0 & 0 & -(b_c)_n & \cdots & 0 \\ \cdots & \cdots \\ 0 & 0 & \cdots & -(a_c)_{nm} & 0 & -(b_c)_{(n-1)} & \cdots & 0 \end{pmatrix} \cdot (d_1, d_2, \dots, d_n, z_1^T, z_2^T, \dots, z_n^T)^T \leq 0. \quad (26)$$

The size of the coefficient matrix in (25) is $(2n) \times (n + np)$, and the variable $\omega = (d_1, d_2, \dots, d_n, z_1^T, z_2^T, \dots, z_n^T)^T$ is $(n + np)$ dimensional, in brevity, we write the coefficient matrix as M_c and R_c in (25), (26). The size of the coefficient matrix R_c is $(n - 1)^2 \times (n + np)$ dimensional and the variable is the same as that in (25), we rewrite the coefficient matrix as R_c . Then we can reformulate the equations (25) and (26) as:

$$\Omega_4 = \{\omega, \omega \in R^{(p+1)n} \mid M_c \omega < 0, R_c \omega \leq 0\}. \quad (27)$$

Lemma 3. [23] (Motzkin, 1936) Let A, C and D be given matrices, with A being nonvacuous. Then exactly one of the following is true.

- (i) $Ax > 0, Cx \geq 0, Dx = 0$ has solution.
- (ii) $A^T y_1 + C^T y_2 + D^T y_3 = 0, y_1 \geq 0, y_3 \geq 0, y_3 \neq 0$ has a solution y_1, y_2, y_3 .

Lemma 4. [24] Let A, B be given matrices, with A being nonvacuous. Then exactly one of the following is true.

- (i) $Ax > 0, Bx \geq 0$ has a solution x .
- (ii) $A^T y_1 + B^T y_2 = 0, y_1 \geq 0, y_2 \geq 0, y_2 \neq 0$ has a solution y_1, y_2 .

Proof. This lemma can be obtained by taking $D = 0$ and replace A, B in the above Motzkin Lemma 3 as $-A, -B$. \square

Theorem 12. The existence condition of a state-feedback law $u_d(k) = K_d x(k)$ for discrete-time systems (18) ($u_c(t) = K_c x(t)$ for continuous-time systems (19)) while guaranteeing the closed-loop system is positive and asymptotically stable, where K_d (or K_c) can be calculated as in Theorem 11(2) (or in Theorem 12(2)) is if and only if the I -rank of matrix $J^{(2)} = \begin{pmatrix} M_d \\ R_d \end{pmatrix}$ (or $J^{(3)} = \begin{pmatrix} M_c \\ R_c \end{pmatrix}$) is greater than 0.

Proof. From (2) in Theorem 11 (or Theorem 11 for continuous-time systems), we know that the existence condition of a state-feedback law $u_d(k) = K_d x(k)$ for discrete-time systems (18) ($u_c(t) = K_c x(t)$ for continuous-time systems (19)) is equivalent to the consistence of (20) for discrete-time systems (or (21) for continuous-time systems (19)), which is equivalent to the nonempty of the set Ω_3 (or Ω_4) in (24), and in (27) for continuous-time systems (19). From Theorem 5, the nonempty

of the set Ω_3 (or Ω_4) is equivalent to the I -rank of matrix $J^{(2)}$ (or $J^{(3)}$) is greater than 0, which completes the proof. \square

3.2. Numerical solving methods. From (23) and (26), we can see that the coefficient matrices don't have the compact form and are not readily implemented when the state vector is large. Analogous to the above Theorem 8, we have the following theorem to compute the feedback matrix K , the methods here have compact form and are ready to be implemented.

Denote $J^{(2)} = \begin{pmatrix} M \\ R \end{pmatrix}$, $x = \omega$, $\varepsilon' = \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}$. By means of Proposition 3, we have

Theorem 13. For the positive systems (18), the following statements are equivalent:

(1) There exists a nonnegative solution for linear equations systems $J^{(2)}x = \varepsilon'$, the nonnegative solution can be obtained by solving the following optimization problem.

$$\begin{aligned} \min g(y) &= (u^T, \theta^T) \begin{pmatrix} t \\ x \end{pmatrix} = \sum_{i=1}^m t_i \\ \text{s.t. } B \begin{pmatrix} t \\ x \end{pmatrix} &\geq \begin{pmatrix} \varepsilon' \\ -\varepsilon' \end{pmatrix}, \quad \begin{pmatrix} t \\ x \end{pmatrix} \geq 0, \end{aligned} \quad (28)$$

where B , u , t , x , θ are as that in (17). The optimal solution of (28) is $y^{(0)} = \begin{pmatrix} t^{(0)} \\ x^{(0)} \end{pmatrix}$, $x^{(0)}$ is the nonnegative solution of systems $J^{(2)}x = \varepsilon'$, where

$$B = \begin{pmatrix} I & J^{(2)} \\ I & -J^{(2)} \end{pmatrix}, \quad t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix},$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad y = \begin{pmatrix} t \\ x \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(2) There exists a state-feedback law $u(k) = Kx(k)$ such that the closed-loop system is positive and asymptotically stable, where K can be calculated as follows:

$$K_c = (d_1^{-1}z_1, d_2^{-1}z_2, \dots, d_n^{-1}z_n),$$

where $d_i, z_i, i = 1, 2, \dots, n$ can be obtained from the solution x^* of the optimization (28).

Theorem 14. For the positive systems (19), the following statements are equivalent:

(1) There exists a nonnegative solution for linear equations systems $J^{(3)}x = \varepsilon'$, the nonnegative solution can be obtained by

solving the following optimization problem.

$$\begin{aligned} \min g(y) &= (u^T, \theta^T) \begin{pmatrix} t \\ x \end{pmatrix} = \sum_{i=1}^m t_i \\ \text{s.t. } B \begin{pmatrix} t \\ x \end{pmatrix} &\geq \begin{pmatrix} \varepsilon' \\ -\varepsilon' \end{pmatrix}, \quad \begin{pmatrix} t \\ x \end{pmatrix} \geq 0, \end{aligned} \quad (29)$$

where B , u , t , x , θ are as that in (17). The optimal solution of (29) is $y^{(0)} = \begin{pmatrix} t^{(0)} \\ x^{(0)} \end{pmatrix}$, $x^{(0)}$ is the nonnegative solution of systems $J^{(3)}x = \varepsilon'$, where

$$B = \begin{pmatrix} I & J^{(3)} \\ I & -J^{(3)} \end{pmatrix}, \quad t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

$$y = \begin{pmatrix} t \\ x \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(2) There exists a state-feedback law $u(k) = Kx(k)$ such that the closed-loop system is positive and asymptotically stable, where K can be calculated as follows:

$$K_d = (d_1^{-1}z_1, d_2^{-1}z_2, \dots, d_n^{-1}z_n),$$

where $d_i, z_i, i = 1, 2, \dots, n$ can be obtained from the solution x^* of the optimization (11).

Remark 2.

- (1) The results of Theorem 13 and Theorem 14 are from Theorem 8 and Theorem 12 directly, we omit the proof here for brevity.
- (2) From Theorem 8 and Theorem 12 (or Theorem 13), the closed-loop system (18) (or (19)) is asymptotically stable while guaranteeing the positivity if and only if the corresponding LP optimization problems (28) (or (29)) has solutions.

4. Examples

In this section, two simple examples are given to illustrate the effectiveness of methods proposed in this paper. We present three methods to check the asymptotic stability of the given positive linear dynamical systems.

Example 1. Consider the discrete-time linear system

$$x(k+1) = A_d x(k) + B_d u(k),$$

where

$$A_d = \begin{pmatrix} 0.5 & 0 & 0.6 \\ 0.6 & 0.8 & 1.2 \\ 0.8 & 1 & 0.8 \end{pmatrix}, \quad B_d = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

From Theorem 10, the linear system is positive. The I -minors of matrix $\begin{pmatrix} A_d - I \\ -I \end{pmatrix}$ are:

$$M_1^{(1)} = \begin{pmatrix} -0.1 & -0.2 & 0.5 & 1 & 0 & 0 \\ 0.96 & -1.2 & 0.38 & -0.2 & 0 & 0 \end{pmatrix}^T,$$

$$M_1^{(2)} = \begin{pmatrix} 0.76 & 0.8 & 0 & 0 & 0 \\ 1.16 & -0.2 & 0 & 0 & 0 \end{pmatrix}^T,$$

$$M_1^{(3)} = \begin{pmatrix} 0.38 & -0.5 & 1.08 & 0.6 & 0 & 0 \\ 0.6 & -0.5 & 1.16 & -0.2 & 0 & 0 \end{pmatrix}^T,$$

it is obvious that the I -rank of matrix $\begin{pmatrix} I - A_d \\ -I \end{pmatrix}$ is 0. From Theorem 2, the autonomous discrete-time system is not asymptotically stable. The stability result can also be obtained from Theorem 6, the autonomous linear system is not asymptotically stable since the spectrum of $\sigma(A_d) = (2.1458, 0.3542, -0.4)$, A_d has the unstable eigenvalue $\lambda_1 = 2.1458$ we know that the linear system is not asymptotically stable. We obtained the gain matrix of the closed loop system (17) by Theorem 13 in this paper with $\varepsilon' = -(0.1, 0.1, 0.1)^T$, the solution $(d_1, d_2, d_3, z_1, z_2, z_3)^T = (580.5420, 394.1752, 392.3399, -300.3974, -266.4273, -266.7070)^T$, then $K_d = (-0.5174, -0.6759, -0.6798)$. The closed-loop system is positive and asymptotically stable since

$$A_d + B_d K_d = \begin{pmatrix} 0.5000 & 0.0000 & 0.6000 \\ 0.0826 & 0.1241 & 0.5202 \\ 0.2826 & 0.3241 & 0.1202 \end{pmatrix},$$

and the eigenvalues are $\lambda_1 = 0.8674$, $\lambda_2 = 0.2517$, $\lambda_3 = -0.3748$.

Remark 3. Throughout the paper, output stabilization refers to output positively asymptotically stabilization. Note, the definition of stabilization ensures that stabilization is regardless of the control input (positive or negative), the plant will maintain nonnegativity of states for all initial conditions $x_0 \in R_+^n$ of the plant (due to the Metzler property (Theorem 1 and Theorem 10)). It is worth pointing out that in “real life systems”, nonnegativity of states occurs quite often; however, the need for the input u to be also nonnegative, as in the original definition (Definition 5) may not always be a necessity, as was also pointed out in [16, 25]. Thus, throughout this paper, we do not restrict ourselves to nonnegative inputs, as it was shown in [26] that such a restriction breaks down the possibility of stabilization.

Example 2. Consider the continuous-time linear system

$$\dot{x}(t) = A_c x(t) + B_c u(t),$$

where

$$A_c = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & -2 & 2 & 0 \\ 2 & 1 & 3 & 1 \\ 0 & 2 & 0 & -1 \end{pmatrix}, \quad B_c = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

From Theorem 9, the linear system is positive and the spectrum of A_c are $\sigma(A_c) = (4.2974, -0.7148, -1.7913 + 0.8353i, -1.7913 - 0.8353i)$, the autonomous system is unstable since A_c has unstable eigenvalue $\lambda = 4.2974$. As that in Example 1, also we can obtain the result from the I -rank of matrix $\begin{pmatrix} A \\ -I \end{pmatrix}$ according to Theorem 2(ii) and Theorem 6 in this paper, we have the same conclusion about the asymptotic stability since the I -rank is 0. By Theorem 14, take $\varepsilon' = -1.0e - 01 * (1, 1, 1, 1)^T$, we obtained the solution

$$(d_1, d_2, d_3, d_4, z_1, z_2, z_3, z_4) = (502.5335, 120.3769, 45.1729, 119.9020, -223.9135, 31.5827, -692.3870, -38.0045, -46.8862, 33.2171, -14.7687, 36.5199, -478.8018, 47.8386, -237.6350, 129.2268),$$

compute the feedback control law

$$K_c = \begin{pmatrix} -0.4456 & -0.3157 & -0.3269 & 0.3990 \\ 0.0628 & -0.3895 & 0.8084 & -1.9819 \\ -1.3778 & 0.2759 & -105993 & 1.0778 \end{pmatrix}$$

and

$$A + B K_c = \begin{pmatrix} -0.3827 & 0.2948 & 1.4815 & 0.4171 \\ 0.1089 & -2.6314 & 1.3461 & 0.7980 \\ 0.2395 & 0.5707 & -7.1178 & 0.4948 \\ 0.0628 & 1.6105 & 0.8084 & -2.9819 \end{pmatrix}$$

with the spectrum

$$\sigma(A_c + B K_c) = (-7.3582, -0.2334, -1.5541, -3.9681)$$

which guarantee the positivity and asymptotic stability of the closed-loop system according to Theorem 14 in this paper.

5. Concluding remarks

New equivalent conditions of the asymptotic stability and stabilization of positive linear dynamical systems were studied in this paper. The problems of stability and the stabilization for the positive continuous-time and discrete-time linear systems are transformed into consistency conditions of the algebraic linear inequalities systems. We presented some conditions of

consistency and inconsistency for the linear inequalities systems. We also proposed a numerical solving method of the linear inequalities system, then the relation between the consistency for the linear inequalities systems and the solvability of the LP optimization problem were discussed. The advantage of our method is that we transform the inequalities constraints conditions into the equalities constraints condition which can be formulated into a standard LP optimization problem, which avoided solving the strict algebraic linear inequalities systems. The method proposed in this paper can be considered as an improvement of LP method in [15, 16]. Compared with existing LP approaches in the literature, our design approach constructs a controller with compact form, which is easy to implement, especially when the dimension of the state vector is large. Numerical examples are given in the end to show the effectiveness of our method.

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REFERENCES

- [1] B. Abouzaid, J.J. Winkin, and V. Wertz, "Positive stabilization of infinite-dimensional linear systems", *Proceedings of the 49th IEEE Conference on Decision and Control* 845–850 (2010).
- [2] V. Chellaboina, S.P. Bhat, W.M. Haddad, and D.S. Bernstein, "Modeling and analysis of massaction kinetics: nonnegativity, realizability, reducibility, and semistability", *IEEE Contr. Syst. Mag.* 29 (4), 60–78 (2009).
- [3] W.M. Haddad, V. Chellaboina, and Q. Hui, *Nonnegative and Compartmental Dynamical Systems*, Princeton University Press (2010).
- [4] J.J. Winkin, D. Dochain, and P. Ligarius, "Dynamical analysis of distributed parameter tubular reactors" *Automatica* 36, 349–361 (2000).
- [5] T. Kaczorek, *Positive 1D and 2D Systems*, Springer-Verlag, London, 2002.
- [6] R. Bru, S. Romero, and E. Sanchez, "Canonical forms for positive discrete-time linear control systems", *Linear Algebra Appl.* 310 (1), 49–71 (2000).
- [7] T. Kaczorek and M. Buslowicz, "Minimal realization for positive multivariable linear systems with delay", *Int. J. Appl. Math. Comput. Sci.* 14 (2), 181–188 (2004).
- [8] F. Cacace, L. Farina, R. Setola, and A. Germani, *Positive Linear Systems: Theory and Applications*, Springer, New York, 2017.
- [9] H. Gao, J. Lam, Ch. Wang, and S. Xu, "Control for stability and positivity: equivalent conditions and computation", *IEEE T. Circuits-II* 52 (9), 540–544 (2005).
- [10] J. Zhang, X. Zhao, and R. Zhang, "An improved approach to controller design of positive systems using controller gain decomposition", *J. Franklin I.* 354, 1356–1373 (2017).
- [11] S. Du, J. Qiao, X. Zhao, and R. Wang, "Stability and L_1 -gain analysis for switched positive T–S fuzzy systems under asynchronous switching", *J. Franklin I.* 355 (13), 5912–5927 (2018).
- [12] G. James and V. Rumchev, "Stability of positive linear discrete-time systems", *Bull. Pol. Ac.: Tech.* 53 (1), 1–8 (2005).
- [13] T. Kaczorek, "Stabilization of positive linear systems", *Proc. 37th Conference of Decision Control*, Tampa, FL., 620–621 (1998).
- [14] X. Chen, J. Lam, P. Li, and Z. Shu, " l_1 -induced norm and controller synthesis of positive systems", *Automatica* 49 (5), 1377–1385 (2013).
- [15] M.A. Rami, F. Tadeo, and A. Benzaouia, "Control of constrained positive discrete systems", *Proceedings of 2007 American Control Conference* 5851–5856 (2007).
- [16] M. Ait Rami and F. Tadeo, "Controller Synthesis for Positive Linear Systems With Bounded Controls", *IEEE T. Circuits-II.* 54 (2), 641–642 (2007).
- [17] H. Yang and Y. Jia, "New Conditions and Numerical Checking Method for the Practical Stability of Fractional Order Positive Discrete-Time Linear Systems", *Int. J. Nonlin. Sci. Num.* 20 (3), 315–323 (2019).
- [18] H. Yang, M. Zhou, M. Zhao, and P. Pan, "Nonnegativity, stability analysis of linear discrete-time positive descriptor systems: an optimization approach", *Bull. Pol. Ac.: Tech.* 66 (1), 23–29 (2018).
- [19] M. Zhou, H. Yang, and P. Pan, "A novel approach to positivity analysis of continuous-time positive singular linear systems", *Journal of Shandong University of Science and Technology (Natural Science)* 37 (4), 83–91 (2018).
- [20] L. Farina and S. Rinaldi, *Positive Linear Systems; Theory and Applications*, J. Wiley, New York, 2000.
- [21] N. Dinh and V. Jeyakumar, "Farkas' lemma: three decades of generalizations for mathematical optimization", *TOP* 22, 1–22 (2014).
- [22] L.L. Dines, "Systems of Linear Inequalities", *Annals of Mathematics, Second Series* 20(3), 191–199 (1919).
- [23] T.S. Motzkin, "Two consequences of the transposition theorem on linear inequalities", *Econometrica* 19, 184–185 (1951).
- [24] H.A. Antosiewicz, "A Theorem on Alternatives for Pairs of Matrices", *Pac. J. Math.* 5 (5), 641–642 (1955).
- [25] P. de Leenheer and D. Aeyels, "Stabilization of positive linear systems", *Syst. Control Lett.* 44, 259–271 (2001).
- [26] A.M. Rami, "Solvability of static output-feedback stabilization for lti positive systems", *Syst. Control Lett.* 60 (9), 704–708 (2011).