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BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 67, No. 5, 2019
DOI: 10.24425/bpasts.2019.130879

# The binary algorithm of cascade connection of nonlinear digital filters described in functional series

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**Abstract.** The article presents an example of the use of functional series for the analysis of nonlinear systems for discrete time signals. The homogeneous operator is defined and it is decomposed into three component operators: the multiplying operator, the convolution operator and the alignment operator. An important case from a practical point of view is considered – a cascade connection of two polynomial systems. A new, binary algorithm for determining the sequence of complex kernels of cascade from two sequences of kernels of component systems is presented. Due to its simplicity, it can be used during iterative processes in the analysis of nonlinear systems (e.g. feedback systems)..

**Key words:** functional series, operators, cascade connection, binary algorithms.

## 1. Introduction

The Volterra and Fredholm functional series are a universal tool for identification and analysis of nonlinear systems [1–17]. In the iterative algorithms, a basic thing is the process of determining the operator of cascade of the systems described by the homogeneous operators. This is a tedious and time consuming procedure, which is why this paper proposes a new, fast algorithm consisting in generating all binary words with a given number of bits. Then these words are sorted by the number of "ones" ("1") contained. On this basis, using the procedure presented in this study, it is possible to determine the sequence of kernels of cascade from two sequences of kernels of component systems.

## 2. Time-discrete functional series and their multidimensional complex representations

A functional series could be defined as a universal description of the nonlinear system transition function and is a generalization of the convolution operator:

$$(Hx)_n = \sum_m h_{n-m} x_m \tag{1}$$

The homogeneous operator, as in the transition function of the digital filter, is called the operator:

$$(H^p x)_n = \sum_{n_1}\sum_{n_2}\cdots\sum_{n_p} h_{n-n_1, n-n_2, \ldots, n-n_p} x_{n_1} x_{n_2} \cdots x_{n_p} \tag{2}$$

where the summation is extended to the full axes of integers. Its further generalization is the polynomial operator $P^m$, defined as:

$$(P^m x)_n = \sum_{p=1}^{m} (H^p x)_n \tag{3}$$

and finally the analytical operator. The basis for the functioning of the analytical operator is the homogeneous operator composed of three operators:

- the multiplying operator $\left(\text{the operation symbol} - ^{\textcircled{p}}\right)$.
  The multiplying operation is the transform of the sequence of one variable $\{x_n\}$ into the sequence of $p$-variables according to the rule:

$$\{x_n\} \to x_{n_1} x_{n_2} \cdots x_{n_{p-1}} x_{n_p} \equiv x^{\textcircled{p}}_{n_1, \ldots, n_p} \tag{4}$$

- the convolution operator $\left(\text{the operation symbol} - *\right)$

$$\left(h * x^{\textcircled{p}}\right)_{n_1, \ldots, n_p} = \sum_{m_1}\cdots\sum_{m_p} h_{n_1-m_1, \ldots, n_p-m_p} x^{\textcircled{p}}_{m_1, \ldots, m_p} \tag{5}$$

- the alignment operator $\left(\text{the operation symbol} - ^{+}\right)$

$$\left(h * x^{\textcircled{p}}\right)^+_n = \left(h * x^{\textcircled{p}}\right)_{n, \ldots, n}. \tag{6}$$

This is illustrated by the block diagram shown in Fig. 1.

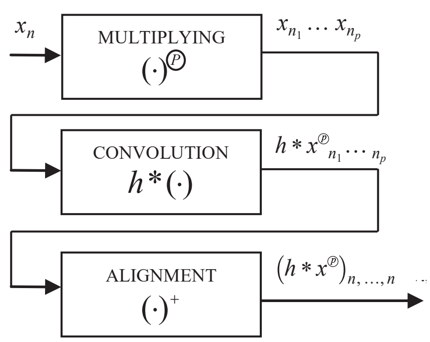


Fig. 1. Block diagram of three component operators of the homogeneous operator

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Manuscript submitted 2018-10-15, revised 2019-01-29, initially accepted for publication 2019-02-07, published in October 2019

Introducing to the considerations operations of the multidimensional $Z$-transform of the signal:

$$\hat{x}\left(z_1, \ldots, z_p\right) = \sum_{n_1} \ldots \sum_{n_p} z_1^{n_1} \ldots z_p^{n_p} x_{n_1, \ldots, n_p} \tag{7}$$

and the inverse transformation:

$$\begin{aligned} x_{n_1, \ldots, n_p} = \frac{1}{(2\pi j)^p} \int\limits_{|z_1|=1} \ldots \int\limits_{|z_p|=1} z_1^{-n_1} \ldots z_p^{-n_p} \cdot \\ \cdot\, \hat{x}\left(z_1, \ldots, z_p\right) d\ln z_1 \ldots d\ln z_p \end{aligned} \tag{8}$$

the individual components of the homogeneous operator can be introduced into the $Z$-domain.

The multiplying operation ($^\wedge$ – the Fourier transform symbol)

$$\begin{aligned} &\left(x^{\textcircled{p}}\right)^\wedge \left(z_1, \ldots, z_p\right) = \sum_{n_1} \ldots \sum_{n_p} z_1^{n_1} \ldots z_p^{n_p} x_{n_1, \ldots, n_p} = \\ &= \left[\sum_{n_1} z_1^{n_1} x_{n_1}\right] \ldots \left[\sum_{n_p} z_p^{n_p} x_{n_p}\right] = \hat{x}\left(z_1\right) \ldots \hat{x}\left(z_p\right). \end{aligned} \tag{9}$$

Therefore, the multiplying operator commutates with the Fourier operator $(\cdot)^\wedge$:

$$\left(x^{\textcircled{p}}\right)^\wedge = \left(\hat{x}\right)^{\textcircled{p}} \tag{10}$$

The convolution operation it is performed according to the Borel's theorem:

$$\begin{aligned} &\left(h * x^{\textcircled{p}}\right)^\wedge \left(z_1, \ldots, z_p\right) = \\ &\hat{h}\left(z_1, \ldots, z_p\right)\left(x^{\textcircled{p}}\right)^\wedge \left(z_1, \ldots, z_p\right) = \\ &\hat{h}\left(z_1, \ldots, z_p\right)\hat{x}\left(z_1\right) \ldots \hat{x}\left(z_p\right). \end{aligned} \tag{11}$$

The most difficulties are apparently simple the alignment operation. It should be done first for two variables:

$$y_{n_1, n_2} = \frac{1}{(2\pi j)^2} \int\limits_{|z_1|=1} \int\limits_{|z_2|=1} z_1^{-n_1} z_2^{-n_2} \hat{y}\left(z_1, z_2\right) d\ln z_1 d\ln z_2 \tag{12}$$

whence it follows that

$$y_{n, n} = \frac{1}{(2\pi j)^2} \int\limits_{|z_1|=1} \int\limits_{|z_2|=1} \left(z_1 z_2\right)^{-n} \hat{y}\left(z_1, z_2\right) d\ln z_1 d\ln z_2. \tag{13}$$

Substituting a new variable:

$$z_1 z_2 = z$$

and then after the logarithmic operation

$$\ln z_1 + \ln z_2 = \ln z$$

is obtained

$$y_{n, n} = \frac{1}{(2\pi j)^2} \int\limits_{|z|=1} z^{-n} \left[\int\limits_{|z_1|=1} \hat{y}\left(z_1, z z_1^{-1}\right) d\ln z_1\right] d\ln z. \tag{14}$$

Hence, it follows that:

$$\begin{aligned} \left(y^{+}\right)^\wedge (z) &= \frac{1}{2\pi j} \int\limits_{|z_1|=1} \hat{y}\left(z_1, z z_1^{-1}\right) d\ln z_1 = \\ &= \frac{1}{2\pi j} \int\limits_{|z_2|=1} \hat{y}\left(z z_2^{-1}, z_2\right) d\ln z_2. \end{aligned} \tag{15}$$

Obtained in this way the alignment operator in $Z$-domain $(^\oplus)$ will be denoted with a direct symbol in the following form:

$$\left(\hat{y}\right)^\oplus (z) = \frac{1}{2\pi j} \int\limits_{|z_1|=1} \hat{y}\left(z_1, z z_1^{-1}\right) d\ln z_1. \tag{16}$$

For the signals of several variables, by induction way, the following formula can be obtained:

$$\begin{aligned} \left(\hat{y}\right)^\oplus (z) = \frac{1}{(2\pi j)^{p-1}} \int\limits_{|z_{p-1}|=1} \ldots \int\limits_{|z_2|=1} \int\limits_{|z_1|=1} \hat{y}\Big(z_1, z_2 z_1^{-1}, \\ z_3 z_2^{-1}, \ldots, z z_{p-1}^{-1}\Big) d\ln z_1 \ldots d\ln z_{p-1}. \end{aligned} \tag{17}$$

Thus, the entire homogeneous operator in $Z$-domain is represented by the cascade connection of the three blocks shown in Fig. 2.

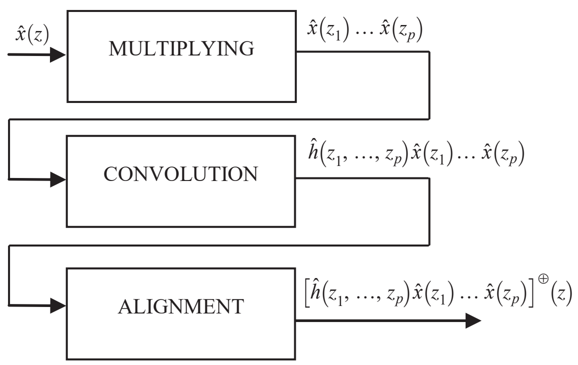


Fig. 2. Block diagram of the homogeneous operator in the Z-domain

## 3. Cascade of polynomial systems

Figure 3 shows an important practical case of the cascade connection of two polynomial systems.

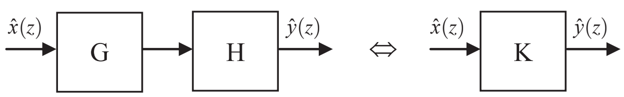


Fig. 3. The cascade connection of polynomial systems $G$, $H$ and the polynomial equivalent system $K$

The following data are given:

- the sequence of kernels of the system $G$ in $Z$-domain

$$\hat{g}_1(z_1),\, \hat{g}_2(z_1, z_2),\, \hat{g}_3(z_1, z_2, z_3),\, \ldots$$

- the sequence of kernels of the system $H$ (also in $Z$-domain)

$$\hat{h}_1(z_1),\, \hat{h}_2(z_1, z_2),\, \hat{h}_3(z_1, z_2, z_3),\, \ldots$$

The sequence of kernels of the equivalent system $K$ is sought:

$$\hat{k}_1(z_1),\, \hat{k}_2(z_1, z_2),\, \hat{k}_3(z_1, z_2, z_3),\, \ldots$$

The output of the cascade is determined by the following formula:

$$\begin{aligned}
\hat{y}(z) = &\sum_{p\geq 1}\Bigg[\hat{h}_p(z_1,\ldots,z_p)\Bigg(\sum_{r\geq 1}\hat{g}_r(u_1,\ldots,u_r)\hat{x}(u_1)\ldots \\
&\ldots\hat{x}(u_r)\Bigg)^{\oplus}(z_1)\ldots\Bigg(\sum_{r\geq 1}\hat{g}_r(u_1,\ldots,u_r)\hat{x}(u_1)\ldots \\
&\ldots\hat{x}(u_r)\Bigg)^{\oplus}(z_p)\Bigg]^{\oplus}(z) = \hat{h}_1(z)\sum_{r\geq 1}\Big[\hat{g}_r(u_1,\ldots,u_r)\cdot \\
&\cdot\hat{x}(u_1)\ldots\hat{x}(u_r)\Big]^{\oplus}(z) + \Bigg[\hat{h}_2(z_1,z_2)\cdot \\
&\cdot\sum_r\Big(\hat{g}_r(u_1,\ldots,u_r)\hat{x}(u_1)\ldots\hat{x}(u_r)\Big)^{\oplus}(z_1)\cdot \\
&\cdot\sum_r\Big(\hat{g}_r(u_1,\ldots,u_r)\hat{x}(u_1)\ldots\hat{x}(u_r)\Big)^{\oplus}(z_2)\Bigg]^{\oplus}(z) + \\
+&\Bigg[\hat{h}_3(z_1,z_2,z_3)\sum_r\Big(\hat{g}_r(u_1,\ldots,u_r)\hat{x}(u_1)\ldots\hat{x}(u_r)\Big)^{\oplus}(z_1)\cdot \\
&\cdot\sum_r\Big(\hat{g}_r(u_1,\ldots,u_r)\hat{x}(u_1)\ldots\hat{x}(u_r)\Big)^{\oplus}(z_2)\cdot \\
&\cdot\sum_r\Big(\hat{g}_r(u_1,\ldots,u_r)\hat{x}(u_1)\ldots\hat{x}(u_r)\Big)^{\oplus}(z_3)\Bigg]^{\oplus}(z) + \ldots
\end{aligned} \tag{18}$$

The individual components are further developed as follows:

- The first component:

$$\sum_{r\geq 1}\Big[\hat{h}_1(z_1,\ldots,z_r)\,\hat{g}_r(z_1,\ldots,z_r)\,\hat{x}(z_1)\ldots\hat{x}(z_r)\Big]^{\oplus}(z)$$

which gives the sequence of kernels:

$$\begin{aligned}
&\hat{h}_1(z)\,\hat{g}_1(z),\;\hat{h}_1(z_1,z_2)\,\hat{g}_2(z_1,z_2), \\
&\hat{h}_1(z_1,z_2,z_3)\,\hat{g}_3(z_1,z_2,z_3),\;\ldots
\end{aligned}$$

- The second component:

$$\begin{aligned}
&\Big[\hat{h}_2(z_1,z_2)\,\hat{g}_1(z_1)\,\hat{g}_1(z_2)\,\hat{x}(z_1)\,\hat{x}(z_2)\Big]^{\oplus}(z) + \\
&+\Big[\hat{h}_2(z_1,z_2)\Big(\hat{g}_2(u_1,u_2)\,\hat{x}(u_1)\,\hat{x}(u_2)\Big)^{\oplus}(z_1)\,\hat{g}_1(z_2)\,\hat{x}(z_2)\Big]^{\oplus}(z) + \\
&+\Big[\hat{h}_2(z_1,z_2)\,\hat{g}_1(z_1)\,\hat{x}(z_1)\Big(\hat{g}_2(u_1,u_2)\,\hat{x}(u_1)\,\hat{x}(u_2)\Big)^{\oplus}(z_2)\Big]^{\oplus}(z) + \ldots
\end{aligned}$$

$$\begin{aligned}
&=\Big[\hat{h}_2(z_1,z_2)\,\hat{g}_1(z_1)\,\hat{g}_1(z_2)\,\hat{x}(z_1)\,\hat{x}(z_2)\Big]^{\oplus}(z) + \\
&+\Big[\Big(\hat{h}_2(u_1u_2,z_2)\,\hat{g}_2(u_1,u_2)\,\hat{g}_1(z_2)\cdot \\
&\quad\cdot\hat{x}(u_1)\,\hat{x}(u_2)\,\hat{x}(z_2)\Big)^{\oplus u_1=u_2=z_2}(z_1,z_2)\Big]^{\oplus}(z) + \\
&+\Big[\Big(\hat{h}_2(z_1,u_1u_2)\,\hat{g}_1(z_1)\,\hat{g}_2(u_1,u_2)\,\hat{g}_1(z_2)\cdot \\
&\quad\cdot\hat{x}(u_1)\,\hat{x}(u_2)\,\hat{x}(z_1)\Big)^{\oplus u_1=u_2=z_2}(z_1,z_2)\Big]^{\oplus}(z) + \ldots
\end{aligned}$$

$$\begin{aligned}
&=\Big[\hat{h}_2(z_1,z_2)\,\hat{g}_1(z_1)\,\hat{g}_1(z_2)\,\hat{x}(z_1)\,\hat{x}(z_2)\Big]^{\oplus}(z) + \\
&+\Big[\Big(\hat{h}_2(z_1z_2,z_3)\,\hat{g}_2(z_1,z_2)\,\hat{g}_1(z_3)\,\hat{x}(z_1)\,\hat{x}(z_2)\,\hat{x}(z_3)\Big]^{\oplus}(z) + \\
&+\Big[\Big(\hat{h}_2(z_1,z_2z_3)\,\hat{g}_1(z_1)\,\hat{g}_2(z_2,z_3)\,\hat{x}(z_1)\,\hat{x}(z_2)\,\hat{x}(z_3)\Big]^{\oplus}(z) + \ldots
\end{aligned}$$

which gives the sequence of kernels:

$\Theta$, $\hat{h}_2(z_1,z_2)\,\hat{g}_1(z_1)\,\hat{g}_1(z_2)$,
$\hat{h}_2(z_1z_2,z_3)\,\hat{g}_2(z_1,z_2)\,\hat{g}_1(z_3) + \hat{h}_2(z_1,z_2z_3)\,\hat{g}_1(z_1)\,\hat{g}_2(z_2,z_3)$, …
($\Theta$ – the zero function)

- The third component gives the sequence of kernels:

$$\Theta,\;\Theta,\;\hat{h}_3(z_1,z_2,z_3)\,\hat{g}_1(z_1)\,\hat{g}_1(z_2)\hat{g}_1(z_3),\;\ldots$$

The whole sum gives individual kernels of the equivalent system:

$$\begin{aligned}
&\hat{k}_1(z) = \hat{h}_1(z)\,\hat{g}_1(z) \\
&\hat{k}_2(z_1,z_2) = \hat{h}_1(z_1z_2)\,\hat{g}_2(z_1,z_2) + \hat{h}_2(z_1,z_2)\,\hat{g}_1(z_1)\,\hat{g}_1(z_2) \\
&\hat{k}_3(z_1,z_2,z_3) = \hat{h}_1(z_1z_2z_3)\,\hat{g}_3(z_1,z_2,z_3) + \\
&\qquad + \hat{h}_2(z_1z_2,z_3)\,\hat{g}_2(z_1,z_2)\,\hat{g}_1(z_3) + \\
&\qquad + \hat{h}_2(z_1,z_2z_3)\,\hat{g}_1(z_1)\,\hat{g}_2(z_2,z_3) + \\
&\qquad + \hat{h}_3(z_1,z_2,z_3)\,\hat{g}_1(z_1)\,\hat{g}_1(z_2)\,\hat{g}_1(z_3) \\
&\ldots
\end{aligned} \tag{19}$$

This is a tedious and time consuming process, so before the next expressions are written $\big(\hat{k}_4(z_1,z_2,z_3,z_4), \hat{k}_5(z_1,z_2,z_3,z_4,z_5), \hat{k}_6(z_1,z_2,z_3,z_4,z_5,z_6), \ldots\big)$, a new way of determining expressions for individual kernels is proposed.

## 4. The binary algorithm of cascade

The following binary algorithm for creating the combination of $z$ variables for the $h$ function (the second one in the cascade) applies:

Table 1
The order of the $h$ function – 2

| | The number of commas | |
|---|---|---|
| **NBC** | **0** | **1** |
| 0 | 0 | |
| 1 | | 1 |
| | $h_1$ | $h_2$ |

Table 2
The order of the $h$ function – 3

| | The number of commas | | |
|---|---|---|---|
| **NBC** | **0** | **1** | **2** |
| 00 | 00 | | |
| 01 | | 01 | |
| 10 | | 10 | |
| 11 | | | 11 |
| | $h_1$ | $h_2$ | $h_3$ |

Table 3
The order of the $h$ function – 4

| | The number of commas | | | |
|---|---|---|---|---|
| **NBC** | **0** | **1** | **2** | **3** |
| 000 | 000 | | | |
| 001 | | 001 | | |
| 010 | | 010 | | |
| 011 | | | 011 | |
| 100 | | 100 | | |
| 101 | | | 101 | |
| 110 | | | 110 | |
| 111 | | | | 111 |
| | $h_1$ | $h_2$ | $h_2$ | $h_4$ |

In the NBC column, we write in natural binary code all combinations of $m$-bits, where $m$ is 1 smaller than the order of the $h$ function. Such a set of zero-one words is divided into subsets containing 0, 1, 2, .., "ones" ("1") (to individual columns representing the number of commas, we enter the combinations of zero-one words whose number of ones is equal to the number of the column). In the subsets of the above-mentioned words, "1" means a comma between the $z$ variables, and "0" – no comma (the action of multiplication the $z$ variables). Each subset (column) defines the appropriate combination of the variable of $h$ function, e.g.:

- *the order* – 3, the bit word 00 is in column 0 (does not contain any 1), hence the entry for the $h_1$ component:

$$00 \;\rightarrow\; \hat{h}_1\left(z_1^0 z_2^0 z_3\right) \;\rightarrow\; \hat{h}_1\left(z_1 z_2 z_3\right)$$

- *the order* – 3, the bit word 11 is in column 2 (contains two 1), hence the entry for the $h_3$ component:

$$11 \;\rightarrow\; \hat{h}_3\left(z_1^1 z_2^1 z_3\right) \;\rightarrow\; \hat{h}_3\left(z_1, z_2, z_3\right)$$

- *the order* – 4, the bit word 010 is in column 1 (contains one 1), hence the entry for the $h_2$ component:

$$010 \;\rightarrow\; \hat{h}_2\left(z_1^0 z_2^1 z_3^0 z_4\right) \;\rightarrow\; \hat{h}_2\left(z_1 z_2, z_3 z_4\right)$$

- *the order* – 4, the bit word 101 is in column 2 (contains two 1), hence the entry for the $h_3$ component:

$$101 \;\rightarrow\; \hat{h}_3\left(z_1^1 z_2^0 z_3^1 z_4\right) \;\rightarrow\; \hat{h}_3\left(z_1, z_2 z_3, z_4\right)$$

Then, sets of the function $g$ are appended according to the following examples (the subscript of the function $g$ is the number of $z$ variables – $z$ variables in the $g$ function record are always separated by a comma – of course, except for one variable):

- *the order* – 3

$$\hat{h}_1\left(\underbrace{z_1 z_2 z_3}\right) \;\rightarrow\; \hat{h}_1\left(z_1 z_2 z_3\right)\hat{g}_3\left(z_1, z_2, z_3\right)$$

the multiplication of 3 $z$ variables, will be $g_3$

$$\hat{h}_3\left(z_1, z_2, z_3\right) \;\rightarrow\; \hat{h}_3\left(z_1, z_2, z_3\right)\hat{g}_1\left(z_1\right)\hat{g}_1\left(z_2\right)\hat{g}_1\left(z_3\right)$$

1 $z$ variable, will be $g_1$ | 1 $z$ variable, will be $g_1$ | 1 $z$ variable, will be $g_1$

- *the order* – 4

$$\hat{h}_2\left(\underbrace{z_1 z_2}, \underbrace{z_3 z_4}\right) \;\rightarrow\; \hat{h}_2\left(z_1 z_2, z_3 z_4\right)\hat{g}_2\left(z_1, z_2\right)\hat{g}_2\left(z_3, z_4\right)$$

the multiplication of 2 $z$ variables, will be $g_2$ | the multiplication of 2 $z$ variables, will be $g_2$

$$\hat{h}_3\left(z_1, \underbrace{z_2 z_3}, z_4\right) \;\rightarrow\; \hat{h}_3\left(z_1, z_2 z_3, z_4\right)\hat{g}_1\left(z_1\right)\hat{g}_2\left(z_2, z_3\right)\hat{g}_1\left(z_4\right)$$

1 $z$ variable, will be $g_1$ | the multiplication of 2 $z$ variables, will be $g_2$ | 1 $z$ variable, will be $g_1$

Hence:

$$
\begin{aligned}
&\hat{k}_4(z_1, z_2, z_3, z_4) = \\
&\quad = \hat{h}_1(z_1 z_2 z_3 z_4)\,\hat{g}_4(z_1, z_2, z_3, z_4) + \\
&\quad + \hat{h}_2(z_1 z_2 z_3, z_4)\,\hat{g}_3(z_1, z_2, z_3)\,\hat{g}_1(z_4) + \\
&\quad + \hat{h}_2(z_1 z_2, z_3 z_4)\,\hat{g}_2(z_1, z_2)\,\hat{g}_2(z_3, z_4) + \\
&\quad + \hat{h}_2(z_1, z_2 z_3 z_4)\,\hat{g}_1(z_1)\,\hat{g}_3(z_2, z_3, z_4) + \\
&\quad + \hat{h}_3(z_1 z_2, z_3, z_4)\,\hat{g}_2(z_1, z_2)\,\hat{g}_1(z_3)\,\hat{g}_1(z_4) + \qquad (20) \\
&\quad + \hat{h}_3(z_1, z_2 z_3, z_4)\,\hat{g}_1(z_1)\,\hat{g}_2(z_2, z_3)\,\hat{g}_1(z_4) + \\
&\quad + \hat{h}_3(z_1, z_2, z_3 z_4)\,\hat{g}_1(z_1)\,\hat{g}_1(z_2)\,\hat{g}_2(z_3, z_4) + \\
&\quad + \hat{h}_4(z_1, z_2, z_3, z_4)\,\hat{g}_1(z_1)\,\hat{g}_1(z_2)\,\hat{g}_1(z_3)\,\hat{g}_1(z_4) \\
&\hat{k}_5(z_1, z_2, z_3, z_4, z_5) = \ldots \\
&\ldots
\end{aligned}
$$

## Example

A cascade connection of analytical systems will be used, among others in a feedback system (Fig. 4), in whose circuit there is a system described by an analytical operator with given kernels:

$$\hat{h}_1(z_1),\ \hat{h}_2(z_1, z_2),\quad \hat{h}_3(z_1, z_2, z_3),\ \ldots$$

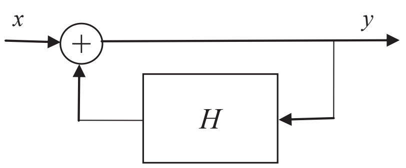


Fig. 4. A feedback system

The feedback system is described by the operator equation:

$$y = x + Hy$$

This is essentially a non-linear integral type equation. Its solution will be sought in the form of:

$$y = Gx$$

where $G$ is a searched analytical operator with unknown kernels:

$$\hat{g}_1(z),\ \hat{g}_2(z_1, z_2),\ \hat{g}_3(z_1, z_2, z_3),\ \ldots$$

Inserting the searched signal into the equation is obtained:

$$(G - I)x = HGx$$

($I$ – the identity operator).

Hence, using the cascade formulas, by equating the kernels, a system of equations is obtained:

$$
\begin{aligned}
&\hat{g}_1(z) - 1 = \hat{h}_1(z)\,\hat{g}_1(z) \\
&\hat{g}_2(z_1, z_2) = \hat{h}_1(z_1 z_2)\hat{g}_2(z_1, z_2) + \hat{h}_2(z_1, z_2)\hat{g}_2(z_1, z_2)\hat{g}_1(z_1)\hat{g}_1(z_2) \\
&\hat{g}_3(z_1, z_2, z_3) = \hat{h}_1(z_1 z_2 z_3)\,\hat{g}_3(z_1, z_2, z_3) + \\
&\qquad + \hat{h}_2(z_1 z_2, z_3)\,\hat{g}_2(z_1, z_2)\,\hat{g}_1(z_3) + \\
&\qquad + \hat{h}_2(z_1, z_2 z_3)\,\hat{g}_1(z_1)\,\hat{g}_2(z_2, z_3) + \\
&\qquad + \hat{h}_3(z_1, z_2, z_3)\,\hat{g}_1(z_1)\,\hat{g}_1(z_2)\,\hat{g}_1(z_3) \\
&\ldots
\end{aligned}
$$

from which successively the searched kernels are obtained:

$$
\begin{aligned}
&\hat{g}_1(z) = \frac{1}{1 - \hat{h}_1(z)} \\
&\hat{g}_2(z_1, z_2) = \frac{\hat{h}_2(z_1, z_2)\,\hat{g}_1(z_1)\,\hat{g}_1(z_2)}{1 - \hat{h}_1(z_1 z_2)} \\
&\hat{g}_3(z_1, z_2, z_3) = \frac{\hat{h}_2(z_1 z_2, z_3)\,\hat{g}_2(z_1, z_2)\,\hat{g}_1(z_3)}{1 - \hat{h}_1(z_1 z_2 z_3)} + \\
&\qquad + \frac{\hat{h}_2(z_1, z_2 z_3)\,\hat{g}_1(z_1)\,\hat{g}_2(z_2, z_3)}{1 - \hat{h}_1(z_1 z_2 z_3)} + \\
&\qquad + \frac{\hat{h}_3(z_1, z_2, z_3)\,\hat{g}_1(z_1)\,\hat{g}_1(z_2)\,\hat{g}_1(z_3)}{1 - \hat{h}_1(z_1 z_2 z_3)} \\
&\ldots
\end{aligned}
$$

## 5. Conclusions

The article concerns the description of nonlinear systems using the Volterra and Fredholm functional series for discrete time signals. After making multidimensional Fourier transforms, the functional series is unambiguously described in the sequence of functions of many complex variables. An important issue is the problem of determining the sequence of complex kernels of the system, which is a cascade connection of two nonlinear systems, each of which is identified by its own sequence of complex kernels. It is called the cascade problem. A new, unknown way to determine the sequence of complex kernels of a cascade from two sequences of kernels of component systems has been developed. It is a simple and clear algorithm consisting in generating multi-bit sets, and then their appropriate sorting. Due to its simplicity, it can be used in the analysis of nonlinear systems during iterative processes, that appear when solving feedback systems. This issue will be presented in a separate paper [18].

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