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Analysis of fractional electrical circuit with rectangular input signal using Caputo and conformable derivative definitions

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Abstract: An analysis of a given electrical circuit using a fractional derivative. The state-space equation was developed. The dynamics of tensions described by Kirchhoff's laws equations. The paper used the definition of the integral derivative Caputo and CDF conformable fractional definition. An electrical circuit solution using Caputo and CDF definitions for rectangular with zero initial conditions was developed. The results obtained using the Caputo and CDF definitions were compared. The solutions are shown for capacitor voltages, for fractional derivative orders of 0.6, 0.8, 1. The results were compared using graphs.

Key words: fractional order system, Caputo definition, conformable fractional definition, fractional electrical circuit

1. Introduction

New fabrication technologies and systems have created a need for new mathematical tools to describe the dynamic processes occurring in their components and systems. The solution for that is a fractional order. The fractional calculus was developed mainly in the nineteenth century by Riemann and Liouville, who were the first to present the definition of the fractional derivatives [10]. Currently, many works on the calculus of an incomplete order have been published, such as, for example, the one by Caputo and Grunwald–Letnikov, describing another definition of the fractional order, where the subject has been comprehensively analyzed [9, 17, 18]. Fractional differential equations were analyzed in [17].

The method for determining the stability of nonlinear or non-stationary systems is the stability analysis according to Lyapunov [3–5]. The Lyapunov function allows one to determine the stability without solving state equations. The disadvantage of this method is the problem of determining the Lyapunov function for a given system. There is no general effective approach to determine these functions. To designate the Lyapunov function, a trial and error method was used. Due to this, the solution was achieved by the application of the state equation, in this paper.

The analysis of the transient electrical circuits is depicted using the Caputo definition, described in [6, 10, 13, 15, 18]. The solution of the state equations for the electrical circuits was introduced by means of a linear continuous series of an incomplete order, described by equations contained in [1, 2, 12].

In this paper we will consider the solutions of the fractional circuit equations using the Caputo and CFD definitions. The CDF definition is a new and a simple well-behaved definition of the fractional derivative, called the conformable fractional derivative, described by R. Khalil, M. Horani, A. Yousef and M. Sababheh [2]. For the state-space description of the fractional electrical circuit we will consider solutions for a general case (with non-zero inputs and initial conditions) as well as a non-zero input and zero initial conditions and a zero input and non-zero initial conditions.

The paper is organized as follows: section 2 describes the solution of the electrical circuit equation of the state, obtained by the use of the fractional order derivatives, given by the Caputo and CFD definitions. The result has been used in the calculation part of this paper. The general description of the problem and fractional electrical circuit are considered in section 3. The realization problem for the Caputo and CDF definitions are in section 4. Concluding remarks are given in section 5.

2. Fractional order state-space equations

The equation of state has the following form [16]:

$$D^{\alpha}x(t) = Ax(t) + Bu(t), \qquad 0 < \alpha \leq 1, \tag{1}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$ are the matrices with constant coefficients, $D_t^{\alpha_k} x_k(t)$ is the fractional order derivative of the vector $x(t)$, described by the Caputo or CFD definition.

In the next sections we will use fractional order state-space Equations (1) with fractional order derivatives given by the Caputo and CFD definitions.

The Caputo fractional order derivative is given by [16]:

$$ {}_0^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{a+1-n}}\,\mathrm{d}\tau, \tag{2}$$

where $n-1 < \alpha < n$, $n \in N$, $\Gamma(x)$ is the Euler gamma function and $f^{(n)}(t) = \dfrac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}$.

The solution to state-space Equation (1) with derivative (2) is given by [7, 13, 14, 16].

$$u(t) = \Phi_0(t)u_0 + \int\limits_0^t \Phi(t-\tau)Be(\tau)\,\mathrm{d}\tau, \tag{3a}$$

where

$$\Phi_0(t) = E_\alpha\left(At^\alpha\right) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)},$$
$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma\left[(k+1)\right]\alpha} \tag{3b}$$

and $x(0)$ is the initial condition, $E_\alpha(z)$ is a single parameter of the Mittag–Leffler function.

If $n < \alpha \leq n+1$, $n \in N_0$, then the conformable fractional derivative (CFD) of an n-differentiable function at t function f (where $t > 0$) is defined as [16]:

$${}_{0}^{\mathrm{CFD}}D_t^\alpha(f)(t) = \lim_{\varepsilon\to 0} \frac{f^{([\alpha]-1)}\left(t+\varepsilon t^{[\alpha]-\alpha}\right) - f^{([\alpha]-1)}(t)}{\varepsilon}, \tag{4}$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

The solution to Equation (1) with the CFD definition of fractional order derivative (4) for $0 < \alpha \leq 1$ is given by [1]:

$$u(t) = \Phi_0(t)u_0 + \int_0^t \Phi(t-\tau)\, Be(\tau)\,\mathrm{d}\tau, \tag{5a}$$

where

$$e^{A\frac{t^\alpha}{\alpha}} = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\alpha^k k!}. \tag{5b}$$

3. Fractional electrical circuit and general description of the problem

In this paper we will consider the fractional electrical circuit shown in Fig. 1 with conductance's G_0, G_1, G_2 capacitances C_1, C_2 and source voltage e.

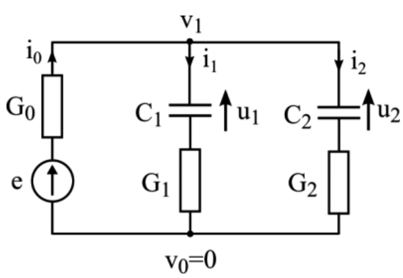

Fig. 1. The fractional electrical circuit (Source: own)

Using Kirchhoff's laws we obtain the equations describing the dynamics of tensions $u_1(t)$, $u_2(t)$ on the corresponding capacitors in response to the control voltage $e(t)$. The corresponding circuit analysis has been concluded in [18].

The electrical circuit is described by state-space Equation (1) with state vector

$$x(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},$$

input vector $u(t) = [e(t)]$ and matrices [18]

$$A = \begin{bmatrix} -\dfrac{G_1(G_0+G_2)}{GC_1} & \dfrac{G_1G_2}{GC_1} \\ \dfrac{G_1G_2}{GC_2} & -\dfrac{G_2(G_0+G_1)}{GC_2} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{6a}$$

$$B = \begin{bmatrix} \dfrac{G_0G_1}{C_1G} \\ \dfrac{G_0G_2}{C_2G} \end{bmatrix}, \tag{6b}$$

where $G = G_0 + G_1 + G_2$.

The initial conditions (the initial voltages across the capacitors) are given in the form:

$$x(0) = \begin{bmatrix} u_{01} \\ u_{02} \end{bmatrix} = \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix}. \tag{7}$$

4. The solutions with fractional definitions

In this section the following lemma will be used.

Lemma 1. For matrix $A = [a_{ij}]_{i,j=1,2}$ with real eigenvalues $\lambda_1 \neq \lambda_2$ there always exists nonsingular similarity matrix $P \in \Re^{2\times 2}$ such that

$$A = P\Lambda P^{-1}, \qquad \Lambda = \mathrm{diag}[\lambda_1, \lambda_2]. \tag{8}$$

The matrix P can be formed using the eigenvectors of the matrix A.

$$P = \begin{bmatrix} \lambda_1 - q_{22} & \lambda_2 - q_{22} \\ a_{21} & a_{21} \end{bmatrix}. \tag{9}$$

Now we will consider the solution to the fractional electrical circuit for two types of derivatives.

4.1. Solution (3a) with zero initial conditions and rectangular function input

Due to the fact that $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the first component of solution (3a) disappears.

The forced response of system (1) for the Caputo definition takes the form:

$$x(t) = \int_0^t \Phi(t-\tau)\, Bu(\tau)\, \mathrm{d}\tau. \tag{10}$$

Consider the source voltage $u(\tau)$, where

$$u(\tau) = \begin{cases} 0 & \text{for} \quad \tau < 0 \\ E & \text{for} \quad nT \le \tau < nT + T_0 \qquad n \in N_0 \\ 0 & \text{for} \quad nT + T_0 \le \tau < (n+1)T \end{cases} \tag{11}$$

and where T is the period of oscillation and occurs as $0 < T_0 < T$.

For times $t \ge 0$, $n \le \frac{t}{T} < n+1$, then $n = \left[\frac{t}{T}\right]$, where $[z]$ is the integer part of the number of z. Then solution (10) takes the form:

$$\begin{aligned} x(t) &= \int_0^{nT} \Phi(t-\tau)\, Bu(\tau)\,\mathrm{d}\tau + \int_{nT}^{t} \Phi(t-\tau)\, Bu(\tau)\,\mathrm{d}\tau = \\ &= \sum_{m=0}^{n-1} \int_{mT}^{(m+1)T} \Phi(t-\tau)\, Bu(\tau)\,\mathrm{d}\tau + \int_{nT}^{t} \Phi(t-\tau)\, Bu(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{12}$$

Each of the integrals under the sign of sum in Formula (12) can be broken down into sums:

$$\int_{mT}^{(m+1)T} \Phi(t-\tau)\, Bu(\tau)\,\mathrm{d}\tau = \int_{mT}^{mT+T_0} \Phi(t-\tau)\, Bu(\tau)\,\mathrm{d}\tau + \int_{mT+T_0}^{(m+1)T} \Phi(t-\tau)\, Bu(\tau)\,\mathrm{d}\tau. \tag{13}$$

After substituting Formula (11) to (13) we obtain:

$$\begin{aligned} \int_{mT}^{(m+1)T} \Phi(t-\tau)\, Bu(\tau)\,\mathrm{d}\tau &= \int_{mT}^{mT+T_0} \Phi(t-\tau) BE\,\mathrm{d}\tau + \int_{mT+T_0}^{(m+1)T} \Phi(t-\tau)\, B0\,\mathrm{d}\tau = \\ &= \left[\int_{mT}^{mT+T_0} \Phi(t-\tau)\,\mathrm{d}\tau\right] BE. \end{aligned} \tag{14}$$

We change variables in order to compute the integral that appeared in Formula (14).

$$\int_{mT}^{mT+T_0} \Phi(t-\tau)\,\mathrm{d}\tau = \int_{t-mT-T_0}^{t-mT} \Phi(\xi)\,\mathrm{d}\xi. \tag{15}$$

We use the function deployment $\Phi(\xi)$ in series (3b), and we replace the order of summation with integration

$$\begin{aligned} \int_{mT}^{mT+T_0} \Phi(t-\tau)\,\mathrm{d}\tau &= \int_{t-mT-T_0}^{t-mT} \sum_{k=0}^{\infty} \frac{A^k \xi^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}\,\mathrm{d}\xi = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma[(k+1)\alpha]} \int_{t-mT-T_0}^{t-mT} \xi^{(k+1)\alpha-1}\,\mathrm{d}\xi = \\ &= \sum_{k=0}^{\infty} \frac{A^k \xi^{(k+1)\alpha}}{\Gamma[(k+1)\alpha+1]}\Bigg|_{t-mT-T_0}^{t-mT} = A^{-1} \sum_{k=1}^{\infty} \frac{A^k \xi^{k\alpha}}{\Gamma(k\alpha+1)}\Bigg|_{t-mT-T_0}^{t-mT}, \end{aligned} \tag{16}$$

since $\Gamma[(k+1)\alpha+1] = (k+1)\alpha\Gamma[(k+1)\alpha]$.

In Formula (16), we recognize incomplete functions $\Phi_0(\tau)$.

$$\int\limits_{mT}^{mT+T_0} \Phi(t-\tau)\,\mathrm{d}\tau = A^{-1}\left[\Phi_0(\tau) - \mathrm{I}\right]_{t-mT-T_0}^{t-mT} = \left[\Phi_0(t-mT) - \Phi_0(t-mT-T_0)\right]A^{-1}. \tag{17}$$

Substituting (17) for (14) yields:

$$\int\limits_{mT}^{(m+1)T} \Phi(t-\tau)\,Bu(\tau)\,\mathrm{d}\tau = \left[\Phi_0(t-mT) - \Phi_0(t-mT-T_0)\right]A^{-1}BE. \tag{18}$$

The result of the integration of Formula (18) is inserted into (12).

$$x(t) = \sum_{m=0}^{n-1}\left[\Phi_0(t-mT) - \Phi_0(t-mT-T_0)\right]A^{-1}BE + \left[\int\limits_{nT}^{t}\Phi(t-\tau)u(\tau)\,\mathrm{d}\tau\right]B. \tag{19}$$

Depending on the time t, the integral of Formula (19) is:

$$\int\limits_{nT}^{t}\Phi(t-\tau)u(\tau)\,\mathrm{d}\tau = \begin{cases} E\displaystyle\int\limits_{nT}^{t}\Phi(t-\tau)\,\mathrm{d}\tau & \text{for } 0 \le t-nT < T_0 \\ E\displaystyle\int\limits_{nT}^{nT+T_0}\Phi(t-\tau)\,\mathrm{d}\tau & \text{for } T_0 \le t-nT < T \end{cases}. \tag{20}$$

In order to calculate the integrals we proceed in the same way as in Formula (17).

$$\int\limits_{nT}^{t}\Phi(t-\tau)u(\tau)\,\mathrm{d}\tau = \begin{cases} EA^{-1}\left[\Phi_0(\xi) - \mathrm{I}\right]_{0}^{t-nT} & \text{for } 0 \le t-nT < T_0 \\ EA^{-1}\left[\Phi_0(\xi) - \mathrm{I}\right]_{t-nT-T_0}^{t-nT} & \text{for } T_0 \le t-nT < T \end{cases}. \tag{21}$$

Taking into account the limits and getting additional transformations we obtain:

$$\int\limits_{nT}^{t}\Phi(t-\tau)u(\tau)\,\mathrm{d}\tau = \begin{cases} \left[\Phi_0(t-nT) - \mathrm{I}\right]EA^{-1} & \text{for } 0 \le t-nT < T \\ \left[\Phi_0(t-nT) - \Phi_0(t-nT-T_0)\right]EA^{-1} & \text{for } T_0 \le t-nT < T \end{cases}. \tag{22}$$

The result of (22) is inserted into (19).

$$x(t) = \begin{cases} \left\{\displaystyle\sum_{m=0}^{n-1}\left[\Phi_0(t-mT) - \Phi_0(t-mT-T_0)\right] + \Phi_0(t-nT) - \mathrm{I}\right\}A^{-1}BE \\ \qquad\qquad \text{for } 0 \le t-nT < T_0 \\ \left\{\displaystyle\sum_{m=0}^{n-1}\left[\Phi_0(t-mT) - \Phi_0(t-mT-T_0)\right] + \Phi_0(t-nT) - \Phi_0(t-nT-T_0)\right\}A^{-1}BE \\ \qquad\qquad \text{for } T_0 \le t-nT < T \end{cases}. \tag{23}$$

We simplify Equation (23) .

$$x(t) = \begin{cases} \left[\sum\limits_{m=0}^{n} \Phi_0(t-mT) - \sum\limits_{m=0}^{n-1} \Phi_0(t-mT-T_0) - \mathrm{I}\right] A^{-1}BE & \text{for } 0 \le t-nT < T_0 \\ \sum\limits_{m=0}^{n} \left[\Phi_0(t-mT) - \Phi_0(t-mT-T_0)\right] A^{-1}BE & \text{for } T_0 \le t-nT < T \end{cases}. \tag{24}$$

Using (6a) and (6b) we can show that

$$-A^{-1}BE = \begin{bmatrix} E \\ E \end{bmatrix}. \tag{25}$$

We substitute (25) for (24) and substitute Mittag-Leffler functions (3b).

$$x(t) = \begin{cases} \left[\sum\limits_{m=0}^{n-1} E_\alpha(A(t-mT-T_0)^\alpha) - \sum\limits_{m=0}^{n} E_\alpha(A(t-mT)^\alpha) + \mathrm{I}\right] \begin{bmatrix} E \\ E \end{bmatrix} & \text{for } 0 \le t-nT < T_0 \\ \sum\limits_{m=0}^{n} \left[E_\alpha(A(t-mT-T_0)^\alpha) - E_\alpha(A(t-mT)^\alpha)\right] \begin{bmatrix} E \\ E \end{bmatrix} & \text{for } T_0 \le t-nT < T \end{cases}. \tag{26}$$

In order to describe the vector$x(t)$by coordinates, we use the dependence derived from (8) and (9).

$$A^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \left(P\Lambda P^{-1}\right)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} = P\Lambda^k P^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \gamma_1\lambda_1^k + \gamma_2\lambda_2^k \\ \gamma_3\lambda_1^k + \gamma_4\lambda_2^k \end{bmatrix}, \tag{27}$$

where coefficients γ_1, γ_2, γ_3, γ_4 of (27) can substituted ($k = 0, 1$), as:

$$\begin{aligned} \gamma_1 &= \frac{a_{22} - a_{12} - \lambda_1}{\lambda_2 - \lambda_1}, & \gamma_2 &= \frac{\lambda_2 + a_{12} - a_{22}}{\lambda_2 - \lambda_1}, \\ \gamma_3 &= \frac{\lambda_2 + a_{21} - a_{22}}{\lambda_2 - \lambda_1}, & \gamma_4 &= \frac{a_{21} + a_{22} - \lambda_1}{\lambda_2 - \lambda_1}. \end{aligned} \tag{28}$$

With

$$\begin{aligned} E_\alpha\left(At^\alpha\right) \begin{bmatrix} E \\ E \end{bmatrix} &= E \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = E \sum_{k=0}^{\infty} \begin{bmatrix} \gamma_1\lambda_1^k + \gamma_2\lambda_2^k \\ \gamma_3\lambda_1^k + \gamma_4\lambda_2^k \end{bmatrix} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} = \\ &= E \begin{bmatrix} \gamma_1 \\ \gamma_3 \end{bmatrix} \sum_{k=0}^{\infty} \frac{\lambda_1^k t^{k\alpha}}{\Gamma(k\alpha+1)} + E \begin{bmatrix} \gamma_2 \\ \gamma_4 \end{bmatrix} \sum_{k=0}^{\infty} \frac{\lambda_2^k t^{k\alpha}}{\Gamma(k\alpha+1)} = E \begin{bmatrix} \gamma_1 \\ \gamma_3 \end{bmatrix} E_\alpha(\lambda_1 t^\alpha) + E \begin{bmatrix} \gamma_2 \\ \gamma_4 \end{bmatrix} E_\alpha(\lambda_2 t^\alpha) = \\ &= E \begin{bmatrix} \gamma_1 E_\alpha\left(\lambda_1 t^\alpha\right) + \gamma_2 E_\alpha\left(\lambda_2 t^\alpha\right) \\ \gamma_3 E_\alpha\left(\lambda_1 t^\alpha\right) + \gamma_4 E_\alpha\left(\lambda_2 t^\alpha\right) \end{bmatrix} \end{aligned} \tag{29}$$

and using (29) to solve (26) we obtain:

$$u_1(t)=\begin{cases} E\left\{\begin{array}{l} 1+\gamma_1\left[\sum\limits_{m=0}^{n-1}E_\alpha\left(\lambda_1(t-mT-T_0)^\alpha\right)-\sum\limits_{m=0}^{n}E_\alpha\left(\lambda_1(t-mT)^\alpha\right)\right]+ \\ +\gamma_2\left[\sum\limits_{m=0}^{n-1}E_\alpha\left(\lambda_2(t-mT-T_0)^\alpha\right)-\sum\limits_{m=0}^{n}E_\alpha\left(\lambda_2(t-mT)^\alpha\right)\right]\end{array}\right\} \\ \qquad\qquad\qquad\qquad \text{for } 0\le t-nT<T_0 \\ E\sum\limits_{m=0}^{n}\left\{\begin{array}{l}\gamma_1\left[E_\alpha\left(\lambda_1(t-mT-T_0)^\alpha\right)-E_\alpha\left(\lambda_1(t-mT)^\alpha\right)\right]+ \\ +\gamma_2\left[E_\alpha\left(\lambda_2(t-mT-T_0)^\alpha\right)-E_\alpha\left(\lambda_2(t-mT)^\alpha\right)\right]\end{array}\right\} \\ \qquad\qquad\qquad\qquad \text{for } T_0\le t-nT<T\end{cases}, \tag{30a}$$

$$u_2(t)=\begin{cases} E\left\{\begin{array}{l} 1+\gamma_3\left[\sum\limits_{m=0}^{n-1}E_\alpha\left(\lambda_1(t-mT-T_0)^\alpha\right)-\sum\limits_{m=0}^{n}E_\alpha\left(\lambda_1(t-mT)^\alpha\right)\right]+ \\ +\gamma_4\left[\sum\limits_{m=0}^{n-1}E_\alpha\left(\lambda_2(t-mT-T_0)^\alpha\right)-\sum\limits_{m=0}^{n}E_\alpha\left(\lambda_2(t-mT)^\alpha\right)\right]\end{array}\right\} \\ \qquad\qquad\qquad\qquad \text{for } 0\le t-nT<T_0 \\ E\sum\limits_{m=0}^{n}\left\{\begin{array}{l}\gamma_3\left[E_\alpha\left(\lambda_1(t-mT-T_0)^\alpha\right)-E_\alpha\left(\lambda_1(t-mT)^\alpha\right)\right]+ \\ +\gamma_4\left[E_\alpha\left(\lambda_2(t-mT-T_0)^\alpha\right)-E_\alpha\left(\lambda_2(t-mT)^\alpha\right)\right]\end{array}\right\} \\ \qquad\qquad\qquad\qquad \text{for } T_0\le t-nT<T\end{cases}. \tag{30b}$$

4.2. CFD definition for zero initial conditions $u_{01} = u_{02} = 0$

Due to the fact that $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the first component of solution (5a) disappears.

The forced response of system (1) for the CFD definition takes the form:

$$x(t) = e^{A\frac{t^\alpha}{\alpha}} \int\limits_0^t e^{-A\frac{\tau^\alpha}{\alpha}} Bu(\tau)\tau^{\alpha-1}\,\mathrm{d}\tau. \tag{31}$$

Then solution (31) takes the form:

$$\begin{aligned} x(t) &= e^{A\frac{t^\alpha}{\alpha}}\left[\int\limits_0^{nT} e^{-A\frac{\tau^\alpha}{\alpha}} Bu(\tau)\tau^{\alpha-1}\,\mathrm{d}\tau + \int\limits_{nT}^{t} e^{-A\frac{\tau^\alpha}{\alpha}} Bu(\tau)\tau^{\alpha-1}\,\mathrm{d}\tau\right] = \\ &= e^{A\frac{t^\alpha}{\alpha}}\left[\sum_{m=0}^{n-1}\int\limits_{mT}^{(m+1)T} e^{-A\frac{\tau^\alpha}{\alpha}} Bu(\tau)\tau^{\alpha-1}\,\mathrm{d}\tau + \int\limits_{nT}^{t} e^{-A\frac{\tau^\alpha}{\alpha}} Bu(\tau)\tau^{\alpha-1}\,\mathrm{d}\tau\right]. \end{aligned} \tag{32}$$

Each of the integrals under the sign of sum can be written as:

$$\int_{mT}^{(m+1)T} e^{-A\frac{\tau^\alpha}{\alpha}} Bu(\tau)\tau^{\alpha-1}\,\mathrm{d}\tau = \int_{mT}^{mT+T_0} e^{-A\frac{\tau^\alpha}{\alpha}} Bu(\tau)^{\alpha-1}\,\mathrm{d}\tau + \int_{mT+T_0}^{(m+1)T} e^{-A\frac{\tau^\alpha}{\alpha}} Bu(\tau)\tau^{\alpha-1}\,\mathrm{d}\tau. \tag{33}$$

Then, Formula (11) is inserted into (33).

$$\begin{aligned}\int_{mT}^{(m+1)T} e^{-A\frac{\tau^\alpha}{\alpha}} Bu(\tau)\tau^{\alpha-1}\,\mathrm{d}\tau &= \int_{mT}^{mT+T_0} e^{-A\frac{\tau^\alpha}{\alpha}} BE\tau^{\alpha-1}\,\mathrm{d}\tau + \int_{mT+T_0}^{(m+1)T} e^{-A\frac{\tau^\alpha}{\alpha}} B0\tau^{\alpha-1}\,\mathrm{d}\tau = \\ &= \left[\int_{mT}^{mT+T_0} e^{-A\frac{\tau^\alpha}{\alpha}} \tau^{\alpha-1}\,\mathrm{d}\tau\right] BE.\end{aligned} \tag{34}$$

We change variables in order to compute the integral that appeared in Formula (34).

$$\begin{aligned}\int_{mT}^{mT+T_0} e^{-A\frac{\tau^\alpha}{\alpha}} \tau^{\alpha-1}\,\mathrm{d}\tau &= \int_{\frac{(mT)^\alpha}{\alpha}}^{\frac{(mT+T_0)^\alpha}{\alpha}} e^{-A\zeta}\,\mathrm{d}\zeta = -A^{-1}e^{-A\zeta}\Big|_{\frac{(mT)^\alpha}{\alpha}}^{\frac{(mT+T_0)^\alpha}{\alpha}} = \\ &= \left[e^{-A\frac{(mT)^\alpha}{\alpha}} - e^{-A\frac{(mT+T_0)^\alpha}{\alpha}}\right] A^{-1}.\end{aligned} \tag{35}$$

The result of injection from Formula (35) is replaced by Formula (34).

$$\int_{mT}^{(m+1)T} e^{-A\frac{\tau^\alpha}{\alpha}} Bu(\tau)\tau^{\alpha-1}\,\mathrm{d}\tau = \left[e^{-A\frac{(mT)^\alpha}{\alpha}} - e^{-A\frac{(mT+T_0)^\alpha}{\alpha}}\right] A^{-1}BE \tag{36}$$

The result of the integration of Formula (36) is inserted into (32).

$$x(t) = e^{A\frac{t^\alpha}{\alpha}} \left\{\sum_{m=0}^{n-1}\left[e^{-A\frac{(mT)^\alpha}{\alpha}} - e^{-A\frac{(mT+T_0)^\alpha}{\alpha}}\right] A^{-1}BE + \left[\int_{nT}^{t} e^{-A\frac{\tau^\alpha}{\alpha}} u(\tau)\tau^{\alpha-1}\,\mathrm{d}\tau\right] B\right\}. \tag{37}$$

Depending on time t the integral of Formula (37) is:

$$\int_{nT}^{t} e^{-A\frac{\tau^\alpha}{\alpha}} u(\tau)\tau^{\alpha-1}\,\mathrm{d}\tau = \begin{cases} E\displaystyle\int_{nT}^{t} e^{-A\frac{\tau^\alpha}{\alpha}} \tau^{\alpha-1}\,\mathrm{d}\tau & \text{for } 0 \le t - nT < T_0 \\ E\displaystyle\int_{nT}^{nT+T_0} e^{-A\frac{\tau^\alpha}{\alpha}} \tau^{\alpha-1}\,\mathrm{d}\tau & \text{for } T_0 \le t - nT < T \end{cases}. \tag{38}$$

In order to calculate the integrals we proceed in the same way as in Formula (35).

$$\int_{nT}^{t} e^{-A\frac{\tau^\alpha}{\alpha}} u(\tau)\tau^{\alpha-1}\,\mathrm{d}\tau = \begin{cases} -EA^{-1}e^{-A\zeta}\Big|_{\frac{(nT)^\alpha}{\alpha}}^{\frac{t^\alpha}{\alpha}} & \text{for } 0 \le t - nT < T_0 \\ -EA^{-1}e^{-A\zeta}\Big|_{\frac{(nT)^\alpha}{\alpha}}^{\frac{(nT+T_0)^\alpha}{\alpha}} & \text{for } T_0 \le t - nT < T \end{cases}. \tag{39}$$

After replacing borders and additional transformations we get:

$$\int_{nT}^{t} e^{-A\frac{\tau^\alpha}{\alpha}} u(\tau)\tau^{\alpha-1}\,\mathrm{d}\,\tau = \begin{cases} EA^{-1}\left[e^{-A\frac{(nT)^\alpha}{\alpha}} - e^{-A\frac{t^\alpha}{\alpha}}\right] & \text{for } 0 \le t - nT < T_0 \\ EA^{-1}\left[e^{-A\frac{(nT)^\alpha}{\alpha}} - e^{-A\frac{(nT+T_0)^\alpha}{\alpha}}\right] & \text{for } T_0 \le t - nT < T \end{cases}. \tag{40}$$

The result of (40) is inserted into (37).

$$x(t) = \begin{cases} e^{A\frac{t^\alpha}{\alpha}} \left\{\sum_{m=0}^{n-1}\left[e^{-A\frac{(mT)^\alpha}{\alpha}} - e^{-A\frac{(mT+T_0)^\alpha}{\alpha}}\right] + e^{-A\frac{(nT)^\alpha}{\alpha}} - e^{-A\frac{t^\alpha}{\alpha}}\right\} A^{-1}BE \\ \qquad \text{for } 0 \le t - nT < T_0 \\ e^{A\frac{t^\alpha}{\alpha}} \left\{\sum_{m=0}^{n-1}\left[e^{-A\frac{(mT)^\alpha}{\alpha}} - e^{-A\frac{(mT+T_0)^\alpha}{\alpha}}\right] + e^{-A\frac{(nT)^\alpha}{\alpha}} - e^{-A\frac{(nT+T_0)^\alpha}{\alpha}}\right\} A^{-1}BE \\ \qquad \text{for } T_0 \le t - nT < T \end{cases}. \tag{41}$$

We simplify Equation (41).

$$x(t) = \begin{cases} e^{A\frac{t^\alpha}{\alpha}} \left[\sum_{m=0}^{n} e^{-A\frac{(mT)^\alpha}{\alpha}} - \sum_{m=0}^{n-1} e^{-A\frac{(mT+T_0)^\alpha}{\alpha}} - e^{-A\frac{t^\alpha}{\alpha}}\right] A^{-1}BE & \text{for } 0 \le t - nT < T_0 \\ e^{A\frac{t^\alpha}{\alpha}} \sum_{m=0}^{n}\left[e^{-A\frac{(mT)^\alpha}{\alpha}} - e^{-A\frac{(mT+T_0)^\alpha}{\alpha}}\right] A^{-1}BE & \text{for } T_0 \le t - nT < T \end{cases}. \tag{42}$$

We use Formula (25) in Formula (42).

$$x(t) = \begin{cases} e^{A\frac{t^\alpha}{\alpha}} \left[\sum_{m=0}^{n-1} e^{-A\frac{(mT+T_0)^\alpha}{\alpha}} - \sum_{m=0}^{n} e^{-A\frac{(mT)^\alpha}{\alpha}} + e^{-A\frac{t^\alpha}{\alpha}}\right]\begin{bmatrix} E \\ E \end{bmatrix} & \text{for } 0 \le t - nT < T_0 \\ e^{A\frac{t^\alpha}{\alpha}} \sum_{m=0}^{n}\left[e^{-A\frac{(mT+T_0)^\alpha}{\alpha}} - e^{-A\frac{(mT)^\alpha}{\alpha}}\right]\begin{bmatrix} E \\ E \end{bmatrix} & \text{for } T_0 \le t - nT < T \end{cases}. \tag{43}$$

Using Formulas (27) and (28) we obtain:

$$\begin{aligned} e^{A\frac{t^\alpha}{\alpha}} e^{-A\frac{\tau^\alpha}{\alpha}} \begin{bmatrix} E \\ E \end{bmatrix} &= e^{A\frac{t^\alpha-\tau^\alpha}{\alpha}} \begin{bmatrix} E \\ E \end{bmatrix} = E\sum_{k=0}^{\infty} \frac{A^k\,(t^\alpha-\tau^\alpha)^k}{\alpha^k k!} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \\ &= E\sum_{k=0}^{\infty} \begin{bmatrix} \gamma_1\lambda_1^k + \gamma_2\lambda_2^k \\ \gamma_3\lambda_1^k + \gamma_4\lambda_2^k \end{bmatrix} \frac{(t^\alpha-\tau^\alpha)^k}{\alpha^k k!} = \\ &= E\begin{bmatrix} \gamma_1 \\ \gamma_3 \end{bmatrix} \sum_{k=0}^{\infty} \frac{\lambda_1^k\,(t^\alpha-\tau^\alpha)^k}{\alpha^k k!} + E\begin{bmatrix} \gamma_2 \\ \gamma_4 \end{bmatrix} \sum_{k=0}^{\infty} \frac{\lambda_2^k\,(t^\alpha-\tau^\alpha)^k}{\alpha^k k!} = \\ &= E\begin{bmatrix} \gamma_1 \\ \gamma_3 \end{bmatrix} e^{\frac{\lambda_1(t^\alpha-\tau^\alpha)}{\alpha}} + E\begin{bmatrix} \gamma_2 \\ \gamma_4 \end{bmatrix} e^{\frac{\lambda_2(t^\alpha-\tau^\alpha)}{\alpha}} = E\begin{bmatrix} \gamma_1 e^{\frac{\lambda_1(t^\alpha-\tau^\alpha)}{\alpha}} + \gamma_2 e^{\frac{\lambda_2(t^\alpha-\tau^\alpha)}{\alpha}} \\ \gamma_3 e^{\frac{\lambda_1(t^\alpha-\tau^\alpha)}{\alpha}} + \gamma_4 e^{\frac{\lambda_2(t^\alpha-\tau^\alpha)}{\alpha}} \end{bmatrix}. \end{aligned} \tag{44}$$

Equation (44) allows us to hold Formula (43) on the coordinates.

$$u_1(t) = \begin{cases} \begin{array}{l} \gamma_1 e^{\lambda_1 \frac{t^\alpha}{\alpha}} \left[\sum\limits_{m=0}^{n-1} e^{-\lambda_1 \frac{(mT+T_0)^\alpha}{\alpha}} - \sum\limits_{m=0}^{n} e^{-\lambda_1 \frac{(mT)^\alpha}{\alpha}} + e^{-\lambda_1 \frac{t^\alpha}{\alpha}} \right] E + \\ + \gamma_2 e^{\lambda_1 \frac{t^\alpha}{\alpha}} \left[\sum\limits_{m=0}^{n-1} e^{-\lambda_1 \frac{(mT+T_0)^\alpha}{\alpha}} - \sum\limits_{m=0}^{n} e^{-\lambda_1 \frac{(mT)^\alpha}{\alpha}} + e^{-\lambda_1 \frac{t^\alpha}{\alpha}} \right] E \end{array} & \text{for } 0 \le t - nT < T_0 \\ \left[\begin{array}{l} \gamma_3 e^{\lambda_1 \frac{t^\alpha}{\alpha}} \sum\limits_{m=0}^{n} \left(e^{-\lambda_1 \frac{(mT+T_0)^\alpha}{\alpha}} - e^{-\lambda_1 \frac{(mT)^\alpha}{\alpha}} \right) + \\ + \gamma_4 e^{\lambda_2 \frac{t^\alpha}{\alpha}} \sum\limits_{m=0}^{n} \left(e^{-\lambda_2 \frac{(mT+T_0)^\alpha}{\alpha}} - e^{-\lambda_2 \frac{(mT)^\alpha}{\alpha}} \right) \end{array} \right] E & \text{for } T_0 \le t - nT < T \end{cases}, \tag{45a}$$

$$u_1(t) = \begin{cases} \begin{array}{l} \gamma_1 e^{\lambda_1 \frac{t^\alpha}{\alpha}} \left[\sum\limits_{m=0}^{n-1} e^{-\lambda_1 \frac{(mT+T_0)^\alpha}{\alpha}} - \sum\limits_{m=0}^{n} e^{-\lambda_1 \frac{(mT)^\alpha}{\alpha}} + e^{-\lambda_1 \frac{t^\alpha}{\alpha}} \right] E + \\ + \gamma_2 e^{\lambda_1 \frac{t^\alpha}{\alpha}} \left[\sum\limits_{m=0}^{n-1} e^{-\lambda_1 \frac{(mT+T_0)^\alpha}{\alpha}} - \sum\limits_{m=0}^{n} e^{-\lambda_1 \frac{(mT)^\alpha}{\alpha}} + e^{-\lambda_1 \frac{t^\alpha}{\alpha}} \right] E \end{array} & \text{for } 0 \le t - nT < T_0 \\ \left[\begin{array}{l} \gamma_3 e^{\lambda_1 \frac{t^\alpha}{\alpha}} \sum\limits_{m=0}^{n} \left(e^{-\lambda_1 \frac{(mT+T_0)^\alpha}{\alpha}} - e^{-\lambda_1 \frac{(mT)^\alpha}{\alpha}} \right) + \\ + \gamma_4 e^{\lambda_2 \frac{t^\alpha}{\alpha}} \sum\limits_{m=0}^{n} \left(e^{-\lambda_2 \frac{(mT+T_0)^\alpha}{\alpha}} - e^{-\lambda_2 \frac{(mT)^\alpha}{\alpha}} \right) \end{array} \right] E & \text{for } T_0 \le t - nT < T \end{cases}, \tag{45b}$$

where γ_1, γ_2, γ_3, γ_4 are given by Formula (28).

5. Numerical analysis

In this section we will present numerical solutions of the fractional electrical circuit shown in Fig. 2 and Fig. 3 described by state Equation (1) with the matrices (13). The parameters of simulations are conductances – $G_0 = 1.1\ \Omega^{-1}$, $G_1 = 2.1\ \Omega^{-1}$, $G_2 = 1.5\ \Omega^{-1}$, capacitances – $C_1 = 1.0$ F, $C_2 = 2.0$ F, initial voltages – $u_{01} = 0.0$ V, $u_{02} = 0.0$ V, times – $T = 4$ s, $T_0 = 2$ s.

The graphs show voltage variations on the first supercapacitor during alternating charging and discharging.

Maximum and minimum values of the voltage on the capacitors, suitable for long times t, have been growing steadily in subsequent periods at times t, ascon templated by the Caputo definition.

When using the CDF definition for the sustained times t, the maxima of the large voltage across the capacitor decrease and the minima grow.

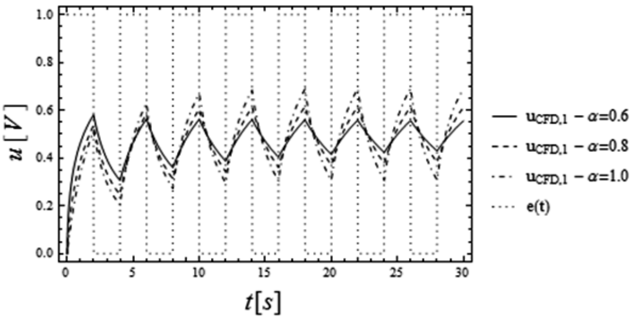

Fig. 2. Solution using the Caputo definition for the first capacitor for $\alpha = 0.6, 0.8, 1$

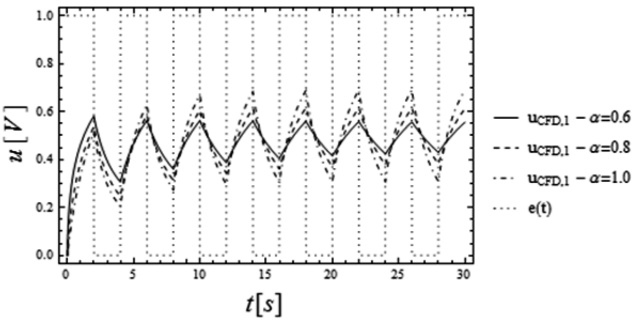

Fig. 3. Solution using the CFD definition for the first capacitor for $\alpha = 0.6, 0.8, 1$

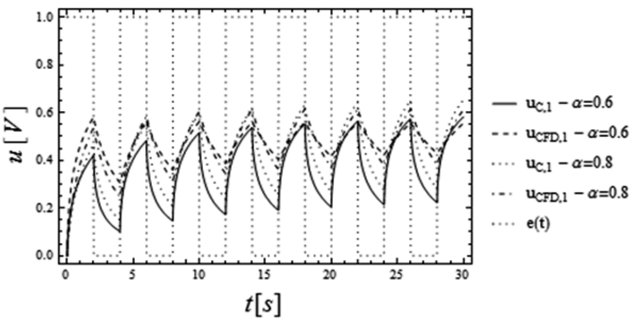

Fig. 4. Comparison of solutions using Caputo and CFD definitions for the first capacitor for $\alpha = 0.6, 0.8$

6. Conclusions

The study investigated the electrical circuit using fractional derivatives in the case of a control signal in the form of a non-symmetric rectangular. The solutions have been achieved by the use of fractional derivatives, according to the Caputo and new CDF definitions. The obtained results were compared with each other and the following conclusions were drawn.

The maximum local voltages on the superconductor (the end of charging, and the beginning of discharge) increased their values in subsequent periods with time. The same applies to the minimum voltages on the supercondenser (the end of discharge and beginning of charging). In subsequent periods as the derivative order increases, the values of the respective maxima and minima increase as well.

Comparing the results of the maxima of the voltages obtained with the Caputo definition it was concluded that they had higher values than the maxima of the voltages obtained by the CDF definitions for the same order of the derivative. The smallest values of the solutions for the same order of the derivative have lower values of the achieved minimum voltages obtained with the Caputo definition from the CFD definition (fractional derivative of alpha 0.6). In summary, the maximum and minimum voltage values are the higher the greater is the derivative order.

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