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Cayley-Hamilton theorem for Drazin inverse matrix and standard inverse matrices

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Abstract. The classical Cayley-Hamilton theorem is extended to Drazin inverse matrices and to standard inverse matrices. It is shown that knowing the characteristic polynomial of the singular matrix or nonsingular matrix, it is possible to write the analog Cayley-Hamilton equations for Drazin inverse matrix and for standard inverse matrices.

Key words: extension, Cayley-Hamilton theorem, characteristic equation, Drazin inverse, inverse matrix.

1. Introduction

The classical Cayley-Hamilton theorem [2, 14, 20] says that every square matrix satisfies its own characteristic equation. The Cayley-Hamilton theorem has been extended to rectangular matrices [3, 11], block matrices [3, 5], pairs of block matrices [5] and standard and singular two-dimensional linear (2-D) systems [4, 9].

In [12] the Cayley-Hamilton theorem has been extended to n-dimensional (n-D) real polynomial matrices. An extension of the Cayley-Hamilton theorem for continuous-time linear systems with delays has been given in [8].

In [7, 10] the Cayley-Hamilton theorem has been extended to the fractional standard and descriptor continuous-time and discrete-time linear systems.

The Cayley-Hamilton theorem and its generalizations have been used in control systems, electrical circuits, systems with delays, singular systems, 2-D linear systems, etc. [1, 6, 13–17, 21–29].

The Drazin inverse matrix method for fractional descriptor continuous-time and discrete-time linear systems has been introduced in [18, 19].

In this paper the Cayley-Hamilton theorem will be extended to the Drazin inverse matrices and standard inverse matrices.

The paper is organized as follows. In Section 2 the basic definitions and theorems concerning Drazin inverse, minimal characteristic polynomials, Lagrange-Sylvester formula and Cayley-Hamilton theorem are recalled. Cayley-Hamilton theorem is extended to the Drazin inverses in Section 3 and to standard inverse matrices in Section 4. Concluding remarks are given in Section 5.

2. Preliminaries

The smallest nonnegative integer q is called the index of the matrix $E \in \Re^{n\times n}$ if

$$\operatorname{rank} E^q = \operatorname{rank} E^{q+1}. \tag{1}$$

Definition 1. A matrix E^D is called the Drazin inverse of the matrix $E \in \Re^{n\times n}$ if it satisfies the conditions

$$EE^D = E^D E, \tag{2a}$$

$$E^D E E^D = E^D, \tag{2b}$$

$$E^D E^{q+1} = E^q, \tag{2c}$$

where q is the index of E.
The Drazin inverse E^D of a square matrix E always exists and is unique [14, 18, 19]. If $\det E \neq 0$ then $E^D = E^{-1}$ (standard inverse matrix).
A procedure for computation of E^D is given in [19].
The characteristic polynomial of the matrix $A \in \Re^{n\times n}$

$$\varphi(\lambda) = \det[I_n\lambda - A] = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 \tag{3}$$

and its minimal polynomial $\psi(\lambda)$ are related by [2, 20]

$$\Psi(\lambda) = \frac{\varphi(\lambda)}{D(\lambda)}, \tag{4}$$

where $D(\lambda)$ is the greatest common divisor of entries of the adjoint matrix $[I_n\lambda - A]_{ad}$. If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the matrix A are distinct, i.e. $\lambda_i \neq \lambda_j$ if $i \neq j$, $i, j = 1, \ldots, n$, then $D(\lambda) = 1$ and $\Psi(\lambda) = \varphi(\lambda)$ [2, 20].

Consider a matrix $A \in \Re^{n\times n}$ with the minimal characteristic polynomial

$$\Psi(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2}\ldots(\lambda - \lambda_r)^{m_r}, \tag{5}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the eigenvalues of the matrix A and $\sum_{i=1}^{r} m_i = m \leq n$. It is assumed that the function $f(\lambda)$ is well-defined on the spectrum $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ of the matrix A, i.e.

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$$f(\lambda_k), f^{(1)}(\lambda_k) = \left.\frac{df(\lambda)}{df}\right|_{\lambda=\lambda_k}, \ldots,$$
$$f^{(m_k-1)}(\lambda_k) = \left.\frac{d^{m_k-1}f(\lambda)}{d\lambda^{m_k-1}}\right|_{\lambda=\lambda_k}, k = 1\ldots, r \quad (6)$$

are finite [2, 17].
In this case the matrix $f(A)$ is well-defined and it is given by the Lagrange-Sylvester formula [2, 17]

$$f(A) = \sum_{i=1}^{r} Z_{i1} f(\lambda_i) + Z_{i2} f^{(1)}(\lambda_i) + \ldots + Z_{im_i} f^{(m_i-1)}(\lambda_i), \quad (7)$$

where

$$Z_{ij} = \sum_{k=j-1}^{m_i-1} \frac{\Psi_i(A)(A-\lambda_i I_n)^k}{(k-j+1)!(j-1)!} \frac{d^{k-j+1}}{d\lambda^{k-j+1}}\left[\frac{1}{\Psi_i(\lambda)}\right]_{\lambda=\lambda_i} \quad (8)$$

and

$$\Psi_i(\lambda) = \frac{\Psi(\lambda)}{(\lambda-\lambda_i)^{m_i}}, \ i = 1,\ldots,r. \quad (9)$$

Theorem 1. Let

$$\Psi(\lambda) = \det[I_n\lambda - f(A)] = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 \quad (10)$$

be the minimal characteristic polynomial of the matrix $f(A)$. Then the matrix $f(A)$ satisfies its characteristic equation, i.e.

$$[f(A)]^n + a_{n-1}[f(A)]^{n-1} + \ldots + a_1[f(A)] + a_0 I_n = 0. \quad (11)$$

Proof. Proof is given in [7].
For $f(A) = A$ we have the classical Cayley-Hamilton theorem [2, 20].
Theorem 2. Let λ_k, $k = 1, \ldots, n$ be the eigenvalues of the matrix $A \in \Re^{n\times n}$ and $f(\lambda)$ be well-defined on the spectrum $\sigma_A = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ of the matrix A, then $f(\lambda_k)$, $k = 1, \ldots, n$ are the eigenvalues of the matrix $f(A)$.
Proof. Proof is given in [2, 20].
In particular case we have the following. If $\lambda_k = \alpha_k + j\beta_k$, $k = 1, \ldots, n$ are the nonzero eigenvalues of $A \in \Re^{n\times n}$, then λ_k^{-1}, $k = 1, \ldots, n$ are the eigenvalues of the inverse matrix A^{-1}.
Theorem 3. If the characteristic equation of the nonsingular matrix $A \in \Re^{n\times n}$ has the form

$$p(\lambda) = \det[I_n\lambda - A] = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0, \quad (12)$$

then the characteristic equation of the inverse matrix $A^{-1} \in \Re^{n\times n}$ is given by

$$a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + 1 = 0. \quad (13)$$

Proof. In [21] it has been shown that if λ_k, $k = 1, \ldots, n$ are the nonzero roots of the equation (12), then λ_k^{-1}, $k = 1, \ldots, n$ are the nonzero roots of the equation (13). Therefore, by Theorem 2 if (12) is the characteristic equation of A, then the characteristic equation of A^{-1} has the form (13). □

3. Cayley-Hamilton theorem for Drazin inverse matrices

In this section the classical Cayley-Hamilton theorem will be extended to Drazin inverse matrices. By assumption the matrix $E \in \Re^{n\times n}$ is singular, i.e. $\det E = a_0 = 0$.
Theorem 4. If

$$\det[I_n s - E] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s, \quad (14)$$

then

$$a_1 E^D + a_2(E^D)^2 + \ldots + a_{n-1}(E^D)^{n-1} + (E^D)^n = 0, \quad (15)$$

where $E^D \in \Re^{n\times n}$ is the Drazin inverse of the matrix E.
Proof. Using (14) and the classical Cayley-Hamilton theorem we obtain

$$E^n + a_{n-1}E^{n-1} + \ldots + a_2E^2 + a_1E = 0. \quad (16)$$

Premultiplying and postmultiplying (16) by the Drazin inverse matrix E^D we obtain

$$E^D E^n E^D + a_{n-1}E^D E^{n-1} E^D + \ldots + a_2 E^D E^2 E^D + a_1 E^D E E^D = 0 \quad (17)$$

and using (2a) and (2b)

$$E^D E^{n-1} + a_{n-1}E^D E^{n-2} + \ldots + a_2 E^D E + a_1 E^D = 0 \quad (18)$$

since

$$E^D E^k E^D = E^D E E^D E^{k-1} = E^D E^{k-1} \quad \text{for } k = 1,2,\ldots,n. \quad (19)$$

Postmultiplying (18) by E^D and using (19) we obtain

$$E^D E^{n-2} + a_{n-1}E^D E^{n-3} + \ldots + a_2 E^D + a_1(E^D)^2 = 0. \quad (20)$$

Repeating $n-2$ times this procedure we obtain (15). □
Example 1. The Drazin inverse of the singular matrix

$$E = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad (21)$$

has the form [14]

$$E^D = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}. \tag{22}$$

The characteristic polynomial of (21) is

$$\det[I_3 s - E] = \begin{vmatrix} s-1 & 0 & 1 \\ 0 & s-1 & 0 \\ 0 & 1 & s \end{vmatrix} = s^3 - 2s^2 + s. \tag{23}$$

From the classical Cayley-Hamilton theorem we have

$$\begin{aligned} E^3 - 2E^2 + E &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}^3 - 2\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}^2 \\ &+ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{24}$$

Applying Theorem 4 to (22) we obtain

$$\begin{aligned} E^D - 2(E^D)^2 + (E^D)^3 &= \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \\ -2\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}^2 &+ \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{25}$$

Postmultiplying (15) by $(E^D)^k$, $k = 1, 2, \ldots$ we obtain the following corollary.

Corollary 1. If (14) is the characteristic polynomial of E, then

$$\begin{aligned} a_1(E^D)^{k+1} + a_2(E^D)^{k+2} + \ldots + a_{n-1}(E^D)^{n+k-1} \\ + (E^D)^{n+k} = 0 \text{ for } k = 1,2,\ldots. \end{aligned} \tag{26}$$

4. Cayley-Hamilton theorem for inverse matrices

Theorem 5. If the characteristic equation of the matrix $A \in \Re^{n\times n}$ has the form

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0, \tag{27}$$

then the inverse matrix A^{-1} satisfies the equation

$$\begin{aligned} a_0(A^{-1})^n + a_1(A^{-1})^{n-1} + \ldots + a_{n-1}A^{-1} + I_n \\ = a_0 A^{-n} + a_1 A^{1-n} + \ldots + a_{n-1}A^{-1} + I_n = 0. \end{aligned} \tag{28}$$

Proof. From classical Cayley-Hamilton theorem and (27) we have

$$A^n + a_{n-1}A^{n-1} + \ldots + a_1 A + a_0 I_n = 0. \tag{29}$$

Postmultiplication of (29) by $(A^{-1})^n$ yields

$$I_n + a_{n-1}(A^{-1}) + \ldots + a_1(A^{-1})^{n-1} + a_0(A^{-1})^n = 0 \tag{30}$$

since $A^k(A^{-1})^k = I_n$ and $A^{-k} = (A^{-1})^k = (A^k)^{-1}$ for $k = 0, 1, \ldots, n$. □

Remark 1. Proof of Theorem 5 follows also from Theorem 3 and Cayley-Hamilton theorem applied to the matrix A^{-1} and to the characteristic equation (13).

Example 2. The characteristic equation of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \tag{31}$$

has the form

$$\det[I_2 s - A] = \begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} = s^2 + 3s + 2 = 0. \tag{32}$$

The inverse matrix of (31) is

$$A^{-1} = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} \tag{33}$$

and by Theorem 5 it satisfies the equation

$$\begin{aligned} 2A^{-2} + 3A^{-1} + I_2 &= 2(A^{-1})^2 + 3A^{-1} + I_2 \\ &= 2\begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix}^2 + 3\begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{34}$$

Premultiplying (28) by A^{-1}, $k = 1, 2, \ldots$ we obtain the following corollary.

Corollary 2. If (27) is the characteristic equation of the matrix A, then

$$\begin{aligned} a_0 A^{-(n+k)} + a_1 A^{1-n-k} + \ldots + a_{n-1}A^{-(k+1)} + A^{-k} = 0 \\ \text{for } k = 1,2,\ldots. \end{aligned} \tag{35}$$

Example 3. (Continuation of Example 2)
The characteristic equation of the matrix (31) is given by (32). Using (35) for $k = 1$ and (32) we obtain

$$\begin{aligned} &2A^{-3} + 3A^{-2} + A^{-1} \\ &= 2\begin{bmatrix} -\frac{15}{8} & -\frac{7}{8} \\ \frac{7}{4} & \frac{3}{4} \end{bmatrix} + 3\begin{bmatrix} \frac{7}{4} & \frac{3}{4} \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{36}$$

The considerations presented in this section for A can be easily extended to A^k for $k = 2, 3, \ldots$. For example Theorem 3 can be extended to A^k for $k = 2, 3, \ldots$ as follows.

Theorem 6. If the characteristic equation of the matrix A^k, $k = 2, 3, \ldots$ has the form

$$\begin{aligned}\bar{p}(\lambda) &= \det[I_n\lambda - A^k] \\ &= \lambda^n + \bar{a}_{n-1}\lambda^{n-1} + \ldots + \bar{a}_1\lambda + \bar{a}_0 = 0,\end{aligned} \tag{37}$$

then the characteristic equation of the inverse matrix $A^{-k} \in \Re^{n\times n}$ is given by

$$\bar{a}_0\lambda^n + \bar{a}_1\lambda^{n-1} + \ldots + \bar{a}_{n-1}\lambda + 1 = 0. \tag{38}$$

Proof. Proof is similar to the proof of Theorem 3.

Example 4. For the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \tag{39}$$

we have

$$A^2 = \begin{bmatrix} -3 & -4 \\ 12 & 13 \end{bmatrix} \tag{40}$$

and

$$\det[I_2\lambda - A^2] = \begin{vmatrix} \lambda+3 & 4 \\ -12 & \lambda-13 \end{vmatrix} = \lambda^2 - 10\lambda + 9 = 0. \tag{41}$$

The inverse matrix of (40) has the form

$$(A^2)^{-1} = A^{-2} = \begin{bmatrix} \frac{13}{9} & \frac{4}{9} \\ -\frac{4}{3} & -\frac{1}{3} \end{bmatrix} \tag{42}$$

and

$$\begin{aligned}\det[I_2\lambda - A^{-2}] &= \begin{vmatrix} \lambda - \frac{13}{9} & -\frac{4}{9} \\ \frac{4}{3} & \lambda + \frac{1}{3} \end{vmatrix} \\ &= 9\lambda^2 - 10\lambda + 1 = 0.\end{aligned} \tag{43}$$

5. Concluding remarks

The classical Cayley-Hamilton theorem has been extended to the Drazin inverse matrices and standard inverse matrices.

It has been shown that if the characteristic polynomial of the singular matrix E has the form (14), then the Drazin inverse matrix E^D satisfies the equation (15) (Theorem 4). If the characteristic equation of the nonsingular matrix A has the form (27), then the inverse matrix A^{-1} satisfies the equation (28) (Theorem 5). The theorems can be extended to any integer powers $k = 2, 3, \ldots$ of the matrices (Theorem 6). The theorems have been illustrated by numerical examples.

The considerations can be extended to fractional linear systems.

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