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Perfect observer resistant to the changes of dynamics of the standard linear system

M. A. SŁAWIŃSKI

Warsaw University of Technology, 75 Koszykowa St., 00-662 Warsaw, Poland

Abstract. The paper presents a class of perfect observers for standard continuous-time linear systems. The proposed observers are resistant to significant changes in the dynamics of the system. The value of poles of the system can be unlimited as long as their number does not change and the conditions presented in this paper are satisfied. Numerical examples and simulation results are also presented.

Key words: linear systems, perfect observers, robust observers.

1. Introduction

Necessary and sufficient conditions of existence of perfect observers for standard linear continuous-time systems are described in [2, 4, 6–8, 11]. The dynamics of the observer must be greater than the dynamics of the system [5, 9, 10]. The dynamics of the perfect observer are infinite [3, 5], so it can be used as the perfect observer for any observable standard system, including unstable systems. Is the perfect observer resistant to changes in the dynamics of the system?

Most problems associated with robust observers concern their resistance to the external disturbances [1]. The objective of this paper is to present the perfect observer completely robust to any changes of the dynamics of the system. Necessary and sufficient conditions for complete robustness of the perfect observer for continuous-time linear system will be proposed. It will be complemented with a numerical example with simulation results using one perfect observer for two systems with fundamentally different dynamics.

2. Preliminaries

Let us consider continuous-time linear system described by the equation

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}, \tag{1}$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$ are the state, input and output vectors, respectively, and $A \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{p \times n}$, $D \in R^{p \times m}$.

Definition 1. The system described by the equation

$$\hat{E}\dot{\hat{x}} = F\hat{x} + Gy + Hu \tag{2}$$

is called perfect observer of the system (1) if and only if

$$\hat{x}(t) = x(t), \forall t > 0. \tag{3}$$

It is well known that the perfect observer (2) of the system (1) exists if and only if system is observable [7, 8, 11]. The standard system (1) is observable if and only if pair (A, C) is observable or, equivalently, if the observability matrix O has full column rank.

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \dots \\ CA^{n-1} \end{bmatrix} \tag{4}$$

Lemma 1. If and only if the standard system (1) is observable, there exists matrix T such that

$$\begin{aligned} \bar{A} &= TAT^{-1} = \begin{bmatrix} & \bar{A}_1 \\ 0 & I_{n-p} \\ & \bar{a}_3 \end{bmatrix} \\ \hat{B} &= TB = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{b}_3 \end{bmatrix} \\ \bar{C} &= CT^{-1} = \begin{bmatrix} \bar{C}_1 & 0 \end{bmatrix} \end{aligned} \tag{5}$$

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*e-mail: m.slawinski@ee.pw.edu.pl

If matrix A is singular, there may exist a multisystem canonical form

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & & \bar{A}_{1l} \\ 0 \quad I_{n_1} & 0 & \dots & 0 \\ \bar{a}_{11} & \bar{a}_{12} & & \bar{a}_{1l} \\ \bar{A}_{21} & \bar{A}_{22} & & \bar{A}_{2l} \\ 0 & 0 \quad I_{n_2} & \dots & 0 \\ \bar{a}_{21} & \bar{a}_{22} & & \bar{a}_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{A}_{l1} & \bar{A}_{l2} & & \bar{A}_{ll} \\ 0 & 0 & \dots & 0 \quad I_{n_l} \\ \bar{a}_{l1} & \bar{a}_{l2} & & \bar{a}_{ll} \end{bmatrix}, \tag{6}$$

$$\hat{B} = \begin{bmatrix} \bar{B}_{11} \\ \bar{B}_{12} \\ \bar{b}_{13} \\ \vdots \\ \bar{B}_{l1} \\ \bar{B}_{l2} \\ \bar{b}_{l3} \end{bmatrix}$$

$$\bar{C} = diag\left(\begin{bmatrix} \bar{C}_1 & 0 \end{bmatrix}, \begin{bmatrix} \bar{C}_2 & 0 \end{bmatrix}, \dots, \begin{bmatrix} \bar{C}_l & 0 \end{bmatrix}\right)$$

where $1 \le l \le n - rankA + 1$, $\sum_{i=1}^{l} n_i = n - p$, $\sum_{i=1}^{l} rank\bar{C}_i = p$ and every row and column of the matrix C_i have only one element not equal zero. Capital letters mean a matrix and lowercase letters mean a vector or a scalar value.

3. Solution

The perfect observer for an observable system (1) can be obtained using canonical form of (5). Without a loss of generality, we can assume that $l = 1$. Using middle $n - p$ rows of the state equation and p rows of the output equation, we obtain

$$\begin{aligned} &\left[\begin{array}{c|c|c} 0^{n-p \times p-1} & I_{n-p} & 0^{n-p \times 1} \\ \hline \multicolumn{3}{c}{0^{p \times n}} \end{array}\right] T\dot{\hat{x}} = \\ &= \begin{bmatrix} 0 & I_{n-p} \\ I_p & 0 \end{bmatrix} T\hat{x} + \begin{bmatrix} \hat{B}_2 \\ \hat{C}_1^{-1} D \end{bmatrix} u + \\ &+ \begin{bmatrix} 0^{n-p \times p} \\ \hat{C}_1^{-1} \end{bmatrix} y \end{aligned} \tag{7}$$

Finally, matrices of the perfect observer (2) are equal

$$\begin{aligned} &\hat{E} = \left[\begin{array}{c|c|c} 0^{n-p \times p-1} & I_{n-p} & 0^{n-p \times 1} \\ \hline \multicolumn{3}{c}{0^{p \times n}} \end{array}\right] T \\ &F = \begin{bmatrix} 0 & I_{n-p} \\ I_p & 0 \end{bmatrix} T, G = \begin{bmatrix} \hat{B}_2 \\ \hat{C}_1^{-1} D \end{bmatrix}, H = \begin{bmatrix} 0 \\ \hat{C}_1^{-1} \end{bmatrix} \end{aligned} \tag{8}$$

Error of the observer (7) can be described by the equation

$$e = x - \hat{x} = T^{-1}\left(\bar{x} - \hat{\bar{x}}\right) = T^{-1}\begin{bmatrix} \bar{x}_1 - \hat{\bar{x}}_1 \\ \bar{x}_2 - \hat{\bar{x}}_2 \end{bmatrix}. \tag{9}$$

From the canonical form of the output equation y, we obtain $\bar{x}_1 = \bar{C}_1^{-1} y - \bar{C}_1^{-1} Du$ and using the last p rows of the perfect observer (7), we obtain

$$\bar{e}_1 = \bar{x}_1 - \hat{\bar{x}}_1 = 0 \tag{10}$$

From first $n - p + 1$ rows of the perfect observer (7) we obtain

$$\begin{aligned} &\begin{bmatrix} I_{n-p} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\hat{\bar{x}}}_{1(p)} \\ \dot{\hat{\bar{x}}}_2 \end{bmatrix} = \\ &= \begin{bmatrix} 0 & I_{n-p} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{\bar{x}}_{1(p)} \\ \hat{\bar{x}}_2 \end{bmatrix} + \quad . \\ &+ \begin{bmatrix} \hat{B}_2 \\ \hat{C}_{1(p)}^{-1} D \end{bmatrix} u + \begin{bmatrix} 0 \\ \hat{C}_{1(p)}^{-1} \end{bmatrix} y \end{aligned} \tag{11}$$

The system described by (11) is completely singular, so its characteristic polynomial is independent of s and not equal zero.

$$det[Es - A] = \begin{vmatrix} s & -1 & \dots & 0 & 0 \\ 0 & s & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & s & -1 \\ -1 & 0 & \dots & 0 & 0 \end{vmatrix} = -1 \tag{12}$$

If the characteristic polynomial is independent of s and not equal zero, the response signal of the system is independent of the initial conditions [3, 5]

$$\hat{x}_2(t) = \sum_{j=1}^{p} \Phi_j B_2 u^{(j-1)}(t), \tag{13}$$

where $u^{(j-1)}$ denotes $(j-1)$ derivative of the input signal u.

The perfect observer reconstructs the state vector $x(t)$ of the system with error $e(t) = x(t) - \hat{x}(t)$ equal zero for time greater than zero. When the dynamics of the perfect observer is infinite and the perfect observer is robust for changes of the system dynamics, even in unstable cases, then we can state that the observer is completely robust.

Theorem 1. Perfect observer for standard continuous-time linear system (1) is completely robust if and only if there exists matrix T such that

$$\begin{gathered} QTAT^{-1} = \begin{bmatrix} 0^{n-p\times p} & I_{n-p} \end{bmatrix} \\ QTB = \bar{B}_2 \\ CT^{-1} = \begin{bmatrix} \bar{C}_1 & 0^{p\times n-p} \end{bmatrix} \end{gathered} \tag{14}$$

where

$$Q = \begin{bmatrix} 0^{p\times n-p-1} & I_p & 0^{p\times 1} \end{bmatrix} \tag{15}$$

and the matrices $\bar{B}_2$ and $\bar{C}_1$ are constant.

Proof. From Lemma 1, if and only if system is observable, there exists matrix T and the canonical form of the system. Using canonical form (5) we can create the perfect observer (7). The error of the observer is equal to zero if and only if for the current T such that the matrix $\hat{A}$ consists identity matrix for middle $n-p$ equations and matrices $\hat{C}_1$ and D and $\hat{B}_2$ are constant. If and only if the previous conditions are satisfied, the perfect observer created for current system is the same even if its dynamics is changed. Similar results are obtained for singular matrix A and l greater then 1.

$$\begin{gathered} QTAT^{-1} = diag\begin{pmatrix} \begin{bmatrix} 0 & I_{n_1-p_1} \end{bmatrix} \\ \begin{bmatrix} 0 & I_{n_2-p_2} \end{bmatrix} \\ \dots \\ \begin{bmatrix} 0 & I_{n_l-p_l} \end{bmatrix} \end{pmatrix} \\ QTB = \begin{bmatrix} \bar{B}_{21} \\ \bar{B}_{22} \\ \dots \\ \bar{B}_{2l} \end{bmatrix} \\ diag\begin{pmatrix} \begin{bmatrix} I_{p_1} & 0^{p_1\times p-p_1} \end{bmatrix} \\ \begin{bmatrix} I_{p_2} & 0^{p_2\times p-p_2} \end{bmatrix} \\ \dots \\ \begin{bmatrix} I_{pl} & 0^{p_l\times p-p_l} \end{bmatrix} \end{pmatrix} CT^{-1} = \\ = diag\begin{pmatrix} [\bar{C}_{11}0] \\ \bar{C}_{12} \\ \dots \\ \bar{C}_{1l} \end{pmatrix} \end{gathered} \tag{16}$$

where

$$Q = diag\begin{pmatrix} \begin{bmatrix} 0^{p_1\times n_1-p_1-1} & I_{p_1} & 0^{p_1\times 1} \\ 0^{p_2\times n_2-p_2-1} & I_{p_2} & 0^{p_2\times 1} \end{bmatrix} \\ \dots \\ \begin{bmatrix} 0^{p_l\times n_l-p_l-1} & I_{p_l} & 0^{p_l\times 1} \end{bmatrix} \end{pmatrix} \tag{17}$$

matrices $\hat{B}_{2k}$ and $\hat{C}_{1l}$ for $k = 1, \dots, l$ are constant. □

4. Examples

Example 1. Create the perfect observer for a standard linear system described by equations

$$\begin{gathered} \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} u. \\ y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x \end{gathered} \tag{18}$$

Let us analyse the response of the system (18) for sinusoidal input signal. Fig. 1 shows the input u and the output signal y

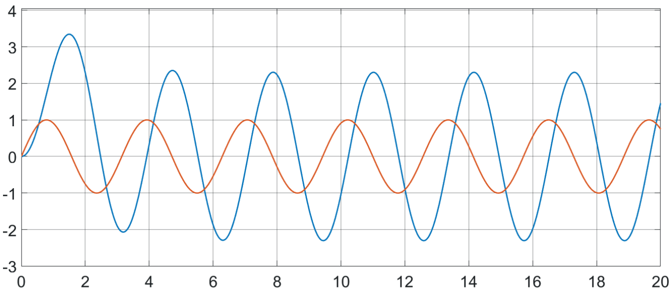

Fig. 1. Input u and output y signal

of the system. Perfect observer for the system (18) is described by the following equation

$$\begin{gathered} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\hat{x}} = \\ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} y \end{gathered} \tag{19}$$

State variables of the system (18) obtained by the use of the perfect observer (19) are shown in Fig. 2. The error of the state variables determined using the perfect observer (19) is shown

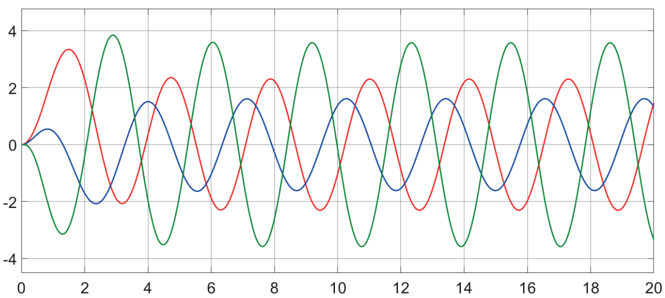

Fig. 2. The system (18) state signal x

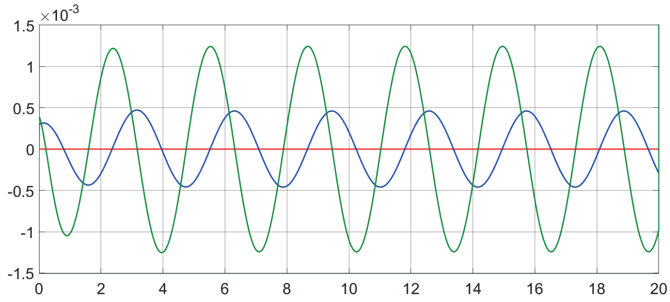

Fig. 3. The error of the observer output signal $\hat{x}$

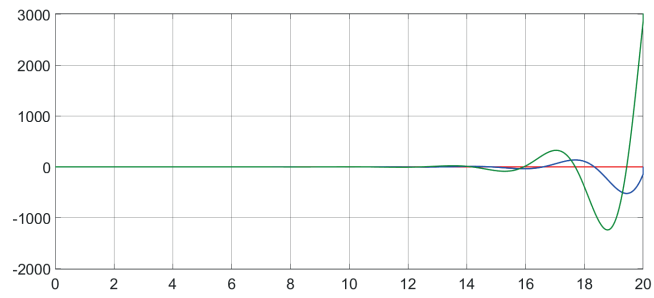

Fig. 5. The error of the observer output signal $\hat{x}$

in Fig. 3. The value of this error is proportional to the input and output signals and their changes [11]. As can be seen, error is less than 0.1%. Designed perfect observer (19) is completely robust for any change in the equations describing the system (18) as long as the conditions of Theorem 1 are satisfied. Thus, the perfect observer is completely robust to any changes in the dynamics of the system (18) if and only if the first and second row of the matrix A and the first and second row of the matrix B and the matrix C remain unchanged.

Example 2. Create the perfect observer for the unstable continuous-time linear system described by the following equations:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & 1 \end{bmatrix} x + \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} u \qquad (20)$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

System (20) is unstable and its response to a sinusoidal input signal is shown in Fig. 4. As can be seen, it is unstable. Error of the perfect observer (19) for the system (20) is shown in Fig. 5.

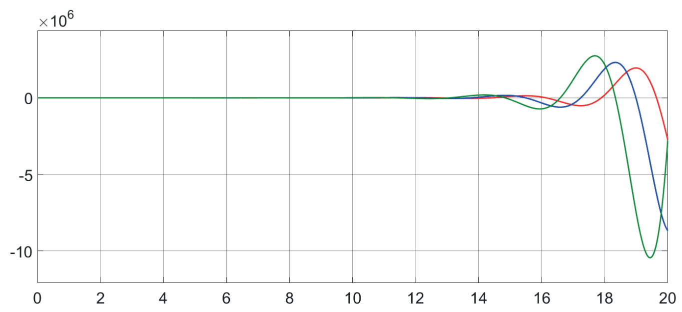

Fig. 4. State of the system (20)

Similarly, as shown in Fig. 3, error of the perfect observer is proportional to the input and output signals and their changes. It is about 0.1%, exactly as in the previous case.

The answer of the perfect observer depends on the extortion derivative. Because of this, there are dependencies between calculation of the input signal derivative and error of the output signal. Derivatives calculated in the computer simulations depend on the changes of the signal, the derivative calculation methods and simulation step time value. The error value is great if the changes of the signal are essential, as in square wave signal.

5. Conclusions

This paper presents the perfect observer completely robust to any changes in the dynamics of the system. If and only if the system is observable, the state vector of the system can be accurately calculated using the input vector and the output vector and derivatives thereof. The dynamics of the system is free to change as long as the conditions of Theorem 1 are satisfied. The examples presented in this paper include two different systems, but for both of them the designed observer is the same.

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