ARCHIVES OF ELECTRICAL ENGINEERING VOL. 67(2), pp. 377–389 (2018)

DOI 10.24425/119647

RC-ladder networks with supercapacitors

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(Received: 10.08.2016, revised: 14.03.2018)

Abstract: Nowadays, non-integer systems are a widely researched problem. One of the questions that is of great importance, is the use of mathematical theory of a non-integer order system to the description of supercapacitors (capacitors with very high capacitance). In the description of electronic systems built on a microscale, there are models with distributed parameters of fractional derivatives, which can be successfully approximated by finite-dimensional structures, e.g, in the form of various types of ladder systems (chain). In this paper, we will analyze a ladder system of an *RC* type consisting of supercapacitors.

Key words: supercapacitors, *RC* ladder network, non-integer order system

1. Introduction

The production of supercapacitors (capacitors with very high capacitance) began in 1972. Nowadays, it is possible to develop capacitors with capacitance about 1000 F. It can be observed that for such capacitors the current $i(t)$ through capacitor C is equal to $C\,\mathrm{d}^{\alpha}x(t)/\mathrm{d}t^{\alpha}$, where $x(t)$ denotes the voltage and $\alpha \in (0,\ 2]$ is the non-integer order derivative [1, 2, 9].

In the description of electronic systems built on a microscale, there are models with distributed parameters of fractional derivatives, which can be successfully approximated by finite-dimensional structures, e.g. in the form of various types of ladder systems (chain) [3, 4].

In this paper, we will analyze a ladder system of RC type consisting of supercapacitors.

For better comprehension of further considerations, let us recall the definition of the Caputo derivative of non-integer order [1, 5, 9, 10]. The Caputo definition of non-integer derivative in this case is convenient because has zero initial condition. The Caputo fractional differentia operator of order $\alpha > 1$ is defined by:

$$\mathrm{d}^{\alpha}z(t)/\mathrm{d}t^{\alpha} = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}z^{(n)}(s)\,\mathrm{d}s,$$

where: $n = [\alpha] = \min\{\xi \in N : \ \xi \geq \alpha\}$, Γ is the gamma function

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} \, \mathrm{d}t.$$

The Laplace transform of the Caputo derivative is of the following form:

$$L\{\mathrm{d}^{\alpha} z / \mathrm{d}t^{\alpha}\} = s^{\alpha} Z(S) - \sum_{k=0}^{n-1} z^{(k)}(0^{+}) s^{\alpha-1-k}$$

and may be used for calculating the transfer function of analysed systems.

2. Cyclic and tridiagonal Jacobi matrices

A special case of a cyclic Jacobi matrix is a tridiagonal Jacobi matrix. Tridiagonal matrices are naturally associated with the ladder systems whereas cyclic Jacobi matrices are used in the description of ring ladder network systems. For this reason, we present below the basic properties of the Jacobi matrices [6, pp. 26, 27].

We consider two $n \times n$ real Jacobi matrices denoted by $\boldsymbol{A}_n$ and $\boldsymbol{B}_n$. For example, for $n = 5$ we have:

$$\boldsymbol{A}_5 = \begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 \\ 0 & 0 & a_4 & b_4 & c_4 \\ 0 & 0 & 0 & a_5 & b_5 \end{bmatrix}, \qquad \boldsymbol{B}_5 = \begin{bmatrix} b_1 & c_1 & 0 & 0 & a_1 \\ a_2 & b_2 & c_2 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 \\ 0 & 0 & a_4 & b_4 & c_4 \\ c_5 & 0 & 0 & a_5 & b_5 \end{bmatrix}. \tag{1}$$

If $a_1 c_n > 0$, $a_{i+1} c_i > 0$, $i = 1, 2, 3, \ldots, n-1$ and $a_1 \cdot a_2 \cdots a_n = c_1 \cdot c_2 \cdots c_n$, then $\boldsymbol{B}_n$ is called a cyclic Jacobi matrix. Eigenvalues $\lambda_i(\boldsymbol{B}_n)$ of $\boldsymbol{B}_n$ are real, but not necessarily single [6, p. 139].

If $a_1 = 0$, $c_n = 0$ and $a_{i+1} c_i > 0$ for $i = 1, 2, 3, \ldots, n-1$, then $\boldsymbol{A}_n = \boldsymbol{B}_n$ is following $n \times n$ tridiagonal Jacobi matrix. Tridiagonal real Jacobi matrix $\boldsymbol{A}_n$ has only single (distinct) real eigenvalues $\lambda_1, \ldots, \lambda_n$ [6, pp. 83, 104].

Remark 1. The matrix $\boldsymbol{A}_n$ is similar to the diagonal canonical Jordan form:

$$\boldsymbol{J} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

That is to say that there exists $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1} \boldsymbol{A}_n \boldsymbol{P} = \boldsymbol{J} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. In other words, the matrix $\boldsymbol{A}_n$ is diagonalizable. Additionally, [7] if

$$a_i > 0, \quad c_i > 0 \qquad \text{and} \qquad b_i = -(a_i + c_i),$$

then

$$\lambda_k \in [-m, 0), \ \ k = 1, \ldots, n, \quad m = 2 \max_k (a_k + c_k).$$

Thus the matrix $\boldsymbol{A}_n$ is asymptotically stable.

Now we consider cyclic Jacobi matrix $\boldsymbol{B}_n$ with $b_i = b$, $a_i = c_i = 1$ and Jacobi matrix $\boldsymbol{A}_n$ with $b_i = b$, $a_i = c_i = 1$ given in following equalities and denoted by:

$$\boldsymbol{J}(n;b) = \begin{bmatrix} b & 1 & 0 & \dots & 0 \\ 1 & b & 1 & \dots & 0 \\ 0 & 1 & b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b \end{bmatrix}, \qquad \boldsymbol{J}_c(n;b) = \begin{bmatrix} b & 1 & 0 & \dots & 1 \\ 1 & b & 1 & \dots & 0 \\ 0 & 1 & b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & b \end{bmatrix}, \tag{2}$$

where b is real. The eigenvalues λ_k of cyclic matrix $\boldsymbol{J}_c(n;b)$ [6, p. 159] are given by:

$$\lambda_k(\boldsymbol{J}_c(n;b)) = b + 2\cos\phi_k, \qquad k = 1,2,3,\dots n, \qquad \phi_k = k2\pi/n \tag{3}$$

and the eigenvalues of Jacobi matrix $\boldsymbol{J}(n;b)$ are given in the formula:

$$\lambda_k(\boldsymbol{J}(n;b)) = b + 2\cos\phi_k\,, \qquad \phi_k = k\pi/(n{+}1), \qquad k = 1,2,3,\dots n. \tag{4}$$

Example 1. Consider the matrix $\boldsymbol{J}(n;b)$ given in (2). Let [6, p. 159]

$$\boldsymbol{P} = \sqrt{\frac{2}{n+1}} \begin{bmatrix} \sin\phi_1 & \sin\phi_1 & \dots & \sin n\phi_1 \\ \sin\phi_2 & \sin\phi_2 & \dots & \sin n\phi_2 \\ \vdots & \vdots & \ddots & \vdots \\ \sin\phi_n & \sin\phi_n & \dots & \sin n\phi_n \end{bmatrix}, \tag{5}$$

where $\phi_k = k\pi/(n+1)$, $k = 1,2,3,\dots,n$.

In this case $\boldsymbol{P} = \boldsymbol{P}^{-1}$ and $\boldsymbol{P}^{-1}\boldsymbol{J}(n;b)\boldsymbol{P} = \boldsymbol{J} = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, $\lambda_k(\boldsymbol{J}(n;b)) = b + 2\cos\phi_k$.

Remark 2. Consider the matrix $\boldsymbol{J}(n;b)$ given in (2). Let e and g be real numbers. Note that

$$\boldsymbol{J}(n;e+g) = \boldsymbol{J}(n;e) + g\boldsymbol{I},$$

where $\boldsymbol{I}$ is the identity matrix $n \times n$.

Consequently, $\lambda_k(\boldsymbol{J}(n;e+g)) = e + g + 2\cos\phi_k$. Matrix $\boldsymbol{J}(n;e+g) = \boldsymbol{J}(n;e) + g\boldsymbol{I}$ is diagonalizable by $\boldsymbol{P}$ given in (5).

3. RC-ladder network with supercapacitors

Let us now consider a ladder *RC* system depicted in Fig. 1. For simplicity we chose $n = 3$.

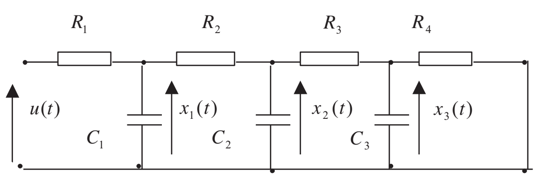

Fig. 1. *RC*-ladder network for $n = 3$

The capacitance of supercapacitors is denoted with C_k, respectively. The current $i(t)$ through capacitor C_k is equal to $C_k\,\mathrm{d}^{\alpha}x_k(t)/\mathrm{d}t^{\alpha}$, where $x_k(t)$ denotes the voltage for C_k, $\alpha \in (0,\ 2]$ is the non-integer order derivative [1, 2] and $u(t)$ is the voltage source.

The system shown in Fig. 1 is described (for any n) by equations:

$$\begin{gathered}\mathrm{d}^{\alpha}\boldsymbol{x}(t)/\mathrm{d}t^{\alpha} = \boldsymbol{A}_n\boldsymbol{x}(t) + \boldsymbol{B}u(t),\\ \boldsymbol{B}^T = [1, 0, 0, \ldots, 0, 0]/(R_1C_1),\\ \boldsymbol{x}(t) = [x_1(t), \ldots, x_n(t)]^T,\end{gathered} \tag{6}$$

where $\boldsymbol{A}_n$ is the tridiagonal Jacobi matrix given (example for $n = 5$) by (1), with

$$a_i = 1/(R_iC_i), \qquad c_i = 1/(R_{i+1}C_i), \qquad b_i = -(a_i + c_i). \tag{7}$$

Remark 3. The matrix $\boldsymbol{A}_n$ is diagonalizable (see Remark 1). In this case, system (6) is diagonalizable and can be decomposed to the n independent differential equation:

$$\mathrm{d}^{\alpha}z_k(t)/\mathrm{d}t^{\alpha} = \lambda_k z_k(t) + w_k u(t), \tag{8}$$

where λ_k is the eigenvalue of matrix $\boldsymbol{A}_n$, $\boldsymbol{P}^{-1}\boldsymbol{A}_n\boldsymbol{P} = \boldsymbol{J} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, w_k is the value of the vector in the k-th place ($\boldsymbol{w} = \boldsymbol{P}^{-1}\boldsymbol{B}$, see (6)), $z(t) = \boldsymbol{P}^{-1}\boldsymbol{x}(t)$.

Example 2. In further analysis, we will limit the consideration to a uniform ladder system $R_i = R$ and $C_i = C$ (see Fig. 1). Let $x_0(t) = u(t)$, $x_{n+1}(t) = 0$. The system (6), (7) is called ***RC*-uniform ladder network**. The RC-uniform ladder system (6), (7) can be described by the matrix equation:

$$\begin{gathered}RC\frac{\mathrm{d}^{\alpha}\boldsymbol{x}(t)}{\mathrm{d}t^{\alpha}} = \boldsymbol{J}(n;-2)x(t) + \boldsymbol{B}u(t), \qquad \boldsymbol{B}^T = [1, 0, 0, \ldots, 0, 0],\\ \boldsymbol{x}(t) = [x_1(t), \ldots, x_n(t)]^T.\end{gathered} \tag{9}$$

The eigenvalues λ_k of the $n \times n$ matrix $\boldsymbol{J}(n;b)$ are given by (4) with $b = -2$. Let $\boldsymbol{x}(t) = \boldsymbol{P}\boldsymbol{z}(t)$, $\boldsymbol{P}$ is given by (5). Thus $z(t) = \boldsymbol{P}^{-1}\boldsymbol{x}(t)$ and from (9) we have Equation (8) with

$$\begin{gathered}\lambda_k = -\frac{2}{RC}(1 - \cos\phi_k) = -\frac{4}{RC}\sin^2\frac{\phi_k}{2},\\ w_k = \frac{1}{RC}\sqrt{\frac{2}{n+1}}\sin\phi_k, \qquad \phi_k = \frac{k\pi}{n+1},\end{gathered} \tag{10}$$

where $k = 1, 2, 3, \ldots, n$.

The solution of (8) for $\alpha \in (0,\ 1]$ has the form [1]:

$$\begin{gathered}z_k(t) = E_{\alpha}(\lambda_k t^{\alpha})z_k(0) + \int_0^t \Phi(t-\tau)w_k u(\tau)\,\mathrm{d}\tau,\\ \Phi(t) = \sum_{k=0}^{\infty}\lambda_i^k t^{(k+1)\alpha-1}/\Gamma[(k+1)\alpha],\end{gathered} \tag{11}$$

where:

$$E_\alpha(p) = \sum_{k=0}^{\infty} p^k / \Gamma(k\alpha + 1) \tag{12}$$

and $E\alpha$ is the Mittage-Leffler function, Γ is the gamma function. In [1] we can find Equation (11) for $n-1 < \alpha < n$, $n = 1, 2, 3, \ldots$.

Example 3. For $\alpha = 1$ we have $E1(at) = \exp(at)$. We have then the "classic" transfer function of (8) (for system (9), where λ_k is described by (10)) defined above:

$$G(s) = \frac{w_k}{s - \lambda_k} = \frac{Z_k(s)}{U_k(s)}. \tag{13}$$

The unit-step response of the system (13) can be expressed as:

$$y_k(t) = L^{-1}\{G(s)\} = w_k \exp(\lambda_k t). \tag{14}$$

For $\alpha = 2$ we have $E_2(at^2) = \cos(a^{1/2}t)$. The solution of (8) for $\alpha = 2$ is of the form:

$$\begin{aligned} z_k(t) &= \cos\left(\sqrt{\lambda_k t}\right) z_k(0) + \left(\sqrt{\lambda_k}\right)^{-1} \sin\left(\sqrt{\lambda_k t}\right) \dot{z}_k(0) + \\ &+ \left(\sqrt{\lambda_k}\right)^{-1} \int_0^t \sin\left(\sqrt{\lambda_k}(t-\tau)\right) w_k u(\tau)\, \mathrm{d}\tau. \end{aligned} \tag{15}$$

For $\alpha = 2$ the *RC* system becomes a ladder *LC* system.

We present certain results obtained with the MATLAB SIMULINK environment (Figs. 2 and 3). Due to the symmetry of a sine function we depicted responses only for $k = 1, 2, 3$ $(n = 5)$.

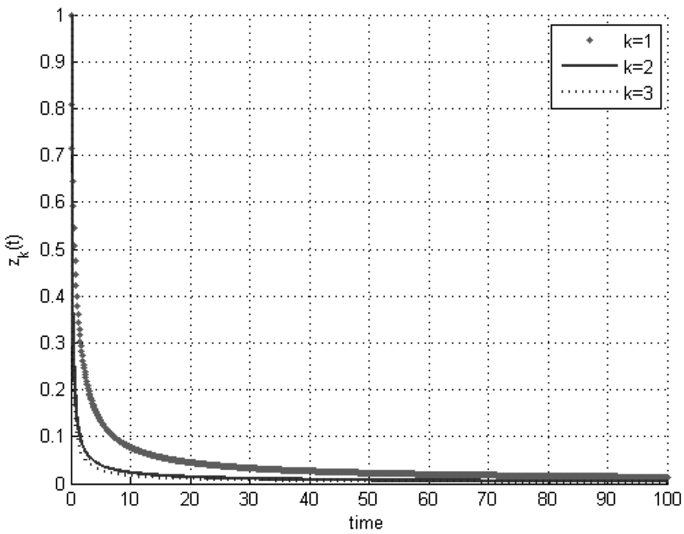

Fig. 2. Responses of subsystems (8) for $\alpha = 0.7$

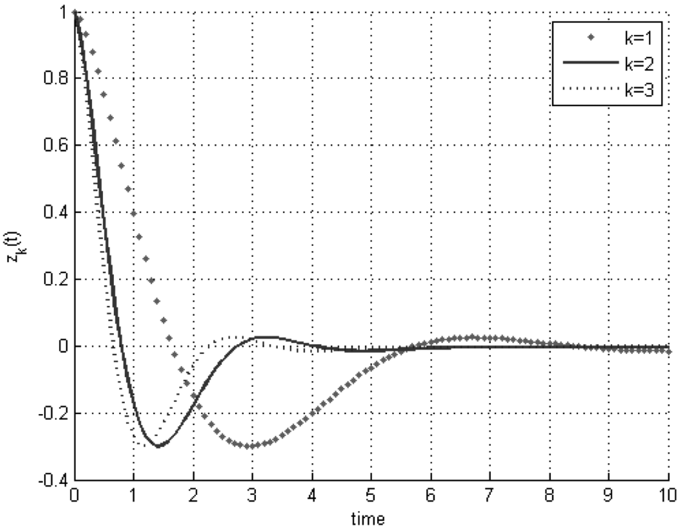

Fig. 3. Responses of subsystems (8) for $\alpha = 1.5$

4. Other RC-ladder network with supercapacitors

In next sections we will described other *RC*-ladder network with supercapacitors. For simplification (but without losing the general solution, see Remark 1) we will limit the consideration to a uniform ladder system. Interested is that the other type of *RC*-ladder networks in many cases can be reduced to n systems of type (8).

***RCR*-uniform ladder network.** Consider an *RCR*-uniform ladder network [3] with supercapacitors. For $n = 3$ the *RCR*-ladder network is shown in Fig. 4.

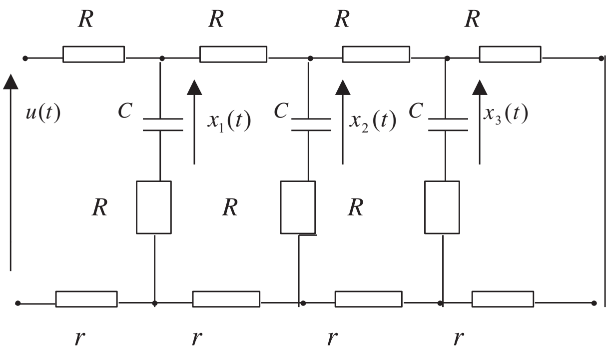

Fig. 4. *RCR*-uniform ladder network for $n = 3$

The *RCR*-uniform ladder network can be described by the following equation:

$$2RC\frac{\mathrm{d}^{\alpha}\boldsymbol{x}(t)}{\mathrm{d}t^{\alpha}} = \boldsymbol{J}(n;-2)\boldsymbol{x}(t) + \boldsymbol{B}u(t), \qquad \boldsymbol{B}^T = [1\,0\,0\ldots0]. \tag{16}$$

Let $\boldsymbol{x}(t)$=$\boldsymbol{P}\boldsymbol{z}(t)$, where $\boldsymbol{P}$ is given in (5), det($\boldsymbol{P}$)$\neq$0. Thus from (16) we have Equation (8) with

$$\lambda_k = -\frac{1}{RC}(1-\cos\phi_k) = -\frac{2}{RC}\sin^2\frac{\phi_k}{2}, \qquad w_k = \frac{1}{2RC}\sqrt{\frac{2}{n+1}}\sin\phi_k, \qquad \phi_k = \frac{k\pi}{n+1}, \tag{17}$$

where $k = 1, 2, 3, \ldots, n$. The system (16) is diagonalizable, i.e. system (16) can be broken down into n scalar systems given by (8) with parameters (17). Simulations results for this type of system have been presented in Figs. 5 and 6.

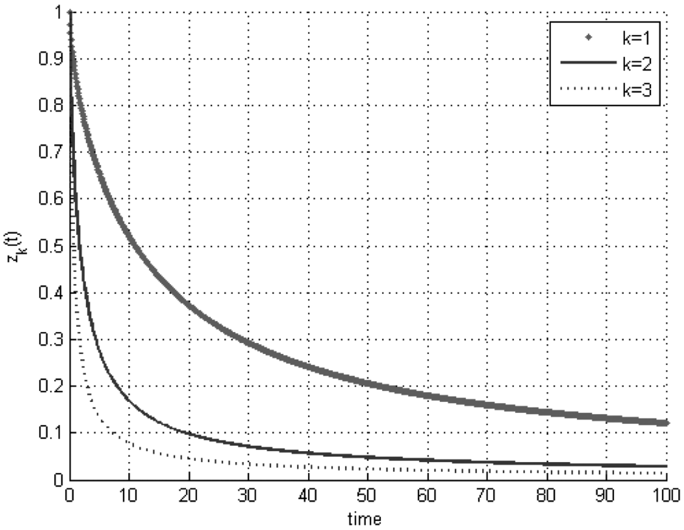

Fig. 5. Responses of subsystems (8) for *RCR* an $\alpha = 0.7$

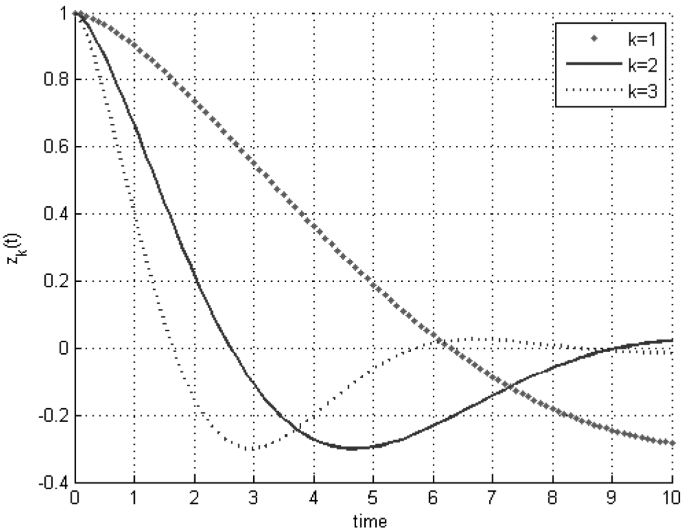

Fig. 6. Responses of subsystems (8) for *RCR* and $\alpha = 1.5$

***RC*-ring network**. Now consider fundamental RC-ladder system (6) and let

$$\boldsymbol{x}_0(t) = \boldsymbol{x}_n(t), \qquad \boldsymbol{x}_{n+1}(t) = \boldsymbol{x}_1(t) \quad \text{and} \quad R_{n+1} = R_1\,. \tag{18}$$

In this case the system (6), (18) is called an electric RC-ring network.

If $R_i = R$, $C_i = C$, then the RCR-ring system (see Fig. 7 for $n = 6$) can be described by the equation:

$$2RC\frac{\mathrm{d}^{\alpha}\boldsymbol{x}(t)}{\mathrm{d}t^{\alpha}} = \boldsymbol{J}_c(n;-2)\boldsymbol{x}(t), \tag{19}$$

where $\boldsymbol{x}(t) = [\boldsymbol{x}_1(t), \boldsymbol{x}_2(t) = \boldsymbol{x}_n(t)]^T$ and matrix $\boldsymbol{J}_c\ (n;b)$ is given in (2). The system given in (19) is diagonalizable (see eigenvalues of matrix $\boldsymbol{J}_c\ (n;b)$ given by (3)).

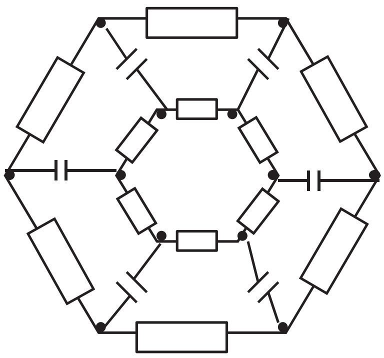

Fig. 7. RCR-ring uniform ladder network for $n = 6$

***RRCr*-ladder network**. Let us consider an $RRCr$-ladder network (see Fig. 4 for $n = 3$). If $r = 0$, then the ladder network is called an electric RRC-uniform ladder network [3, 4].

The RRC-ladder network (see Fig. 8 with $r = 0$) can be described in general by the equation:

$$-RC\boldsymbol{J}(n;-3)\frac{\mathrm{d}^{\alpha}x(t)}{\mathrm{d}t^{\alpha}} = \boldsymbol{J}(n;-2)\boldsymbol{x}(t) + \boldsymbol{B}u(t), \tag{20}$$

where $\boldsymbol{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$, $\boldsymbol{B}^T = [1\,0\,0\ldots0]$.

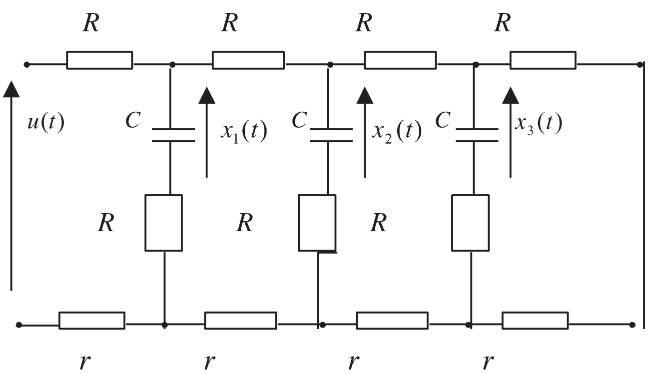

Fig. 8. $RRCr$-uniform ladder network for $n = 3$

System (20) is diagonalizable [4], i.e. system (20) can be broken down into n scalar systems given by the following equation:

$$RCs_k(-\boldsymbol{J}(n;-3))\,\mathrm{d}^{\alpha}z_k(t)/\,\mathrm{d}t^{\alpha} = s_k(\boldsymbol{J}(n;-2))z_k(t) + \sqrt{\frac{2}{n+1}}\sin\phi_k u(t), \tag{21}$$

where $k = 1, 2, 3, \ldots, n$ and $\boldsymbol{x}(t) = \boldsymbol{P}\boldsymbol{z}(t)$, $\boldsymbol{z}(t) = \boldsymbol{P}^{-1}\boldsymbol{x}(t)$, $\boldsymbol{P}$ is given in (5). The eigenvalues s_k of the $n\times n$ matrix $\boldsymbol{J}(n;b)$ are given by (4). Thus we have (see Remark 2)

$$S_k(\boldsymbol{J}(n;-3)) = 1+4\sin^2\frac{\phi_k}{2}, \qquad S_k(-\boldsymbol{J}(n;-2)) = -4\sin^2\frac{\phi_k}{2} < 0. \tag{22}$$

From (21) and (20) we have system (8), where

$$\lambda_k = \{S_k(\boldsymbol{J}(n;-2))/S_k(\boldsymbol{J}(n;-3))\}/RC,$$

$$\lambda_k = \frac{4\sin^2\dfrac{\phi_k}{2}}{RC\left(4\sin^2\dfrac{\phi_k}{2}\right)}, \qquad w_k = \frac{\sqrt{\dfrac{2}{n+1}}\sin\phi_k}{RC\left(1+4\sin^2\dfrac{\phi_k}{2}\right)}. \tag{23}$$

Simulations results for this type of system have been presented in Figs. 9 and 10.

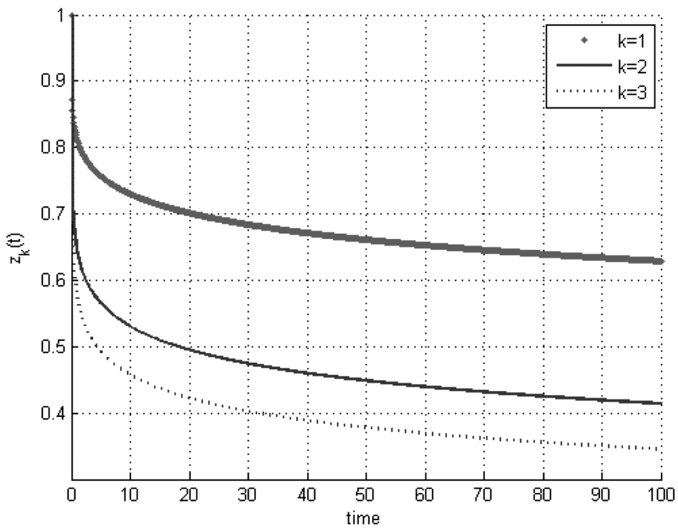

Fig. 9. Responses of subsystems (8) for *RCR*r and $\alpha = 0.7$

***RRCR*- uniform ladder network**. If $r = R$ (see Fig. 8), then the ladder network is called *RRCR*-uniform ladder network. The *RRCR*-ladder network can be described by the equation:

$$-RC\boldsymbol{J}(n;-4)\frac{\mathrm{d}^{\alpha}x(t)}{\mathrm{d}t^{\alpha}} = \boldsymbol{J}(n;-2)x(t) + \boldsymbol{B}u(t). \tag{24}$$

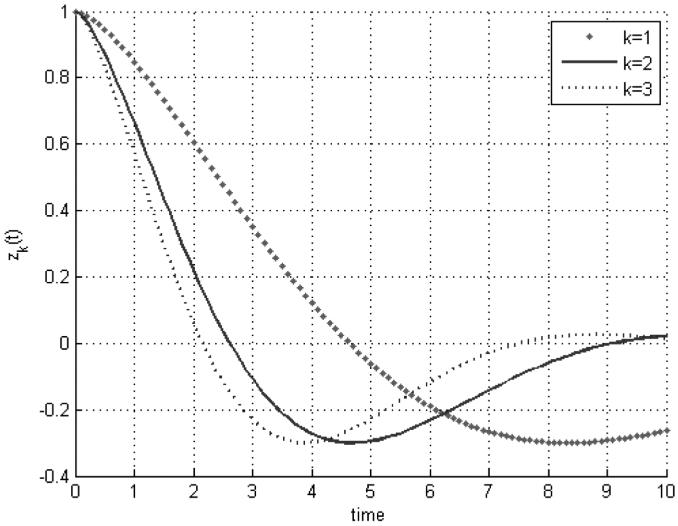

Fig. 10. Responses of subsystems (8) for $RCRr$ and $\alpha = 1.5$

System (24) is diagonalizable, i.e. system (24) can be broken down into n scalar systems given by following equation:

$$-RCs_k(-\boldsymbol{J}(n;-4))\frac{\mathrm{d}^{\alpha}z_k(t)}{\mathrm{d}t^{\alpha}} = s_k(\boldsymbol{J}(n;-2))z_k(t) + \sqrt{\frac{2}{n+1}}\sin\phi_k u(t), \tag{25}$$

where $k = 1, 2, 3, \ldots, n$ and the eigenvalues of $\boldsymbol{J}(n;b)$ are given by (4):

$$\begin{aligned} s_k(-\boldsymbol{J}(n;-4)) &= 2+4\sin^2\frac{\phi_k}{2} > 0,\\ s_k(-\boldsymbol{J}(n;-2)) &= -4\sin^2\frac{\phi_k}{2} > 0. \end{aligned} \tag{26}$$

From (25) and (26) we have (see (8)):

$$\frac{\mathrm{d}^{\alpha}z_k(t)}{\mathrm{d}t^{\alpha}} = \lambda_k z_k(t) + w_k u(t), \qquad k = 1,\ 2,\ 3,\ \ldots,\ n, \tag{27}$$

where

$$\begin{aligned} \lambda_k &= \frac{s_k(\boldsymbol{J}(n;-2)}{RCs_k(\boldsymbol{J}(n;-3)}\,,\\ w_k &= \frac{\sqrt{\frac{2}{n+1}}\sin\phi_k}{RCs_k(\boldsymbol{J}(n;-3)}\,. \end{aligned} \tag{28}$$

Exponential *RC*-ladder network. Consider the long line [8, pp. 22, 46], [10] of heterogeneous parameters R and C. Let the length of the line be equal to 1, $z \in (0,1)$. Let $h = 1/(n+1)$ be a step discretization variable $z \in (0,1)$. Heterogeneous, exponentially convergent transmission line has the following parameters given by the formulas: $T(z) = R\exp(az)$ and $C(z) = C\exp(-az)$. In this case the suitable *RC*-ladder system similar to that is shown in Fig. 1 with parameters:

$$R_i = k^i R, \qquad C_i = k^{-i} C, \qquad k > 0. \tag{29}$$

Systems (6), (29) are called exponential *RC*-ladder networks. The matrix of systems (6), (29) is given by:

$$\boldsymbol{A}_n = \frac{1}{RC}\begin{bmatrix} -(1+1/k) & 1/k & 0 & \dots & 0 \\ 1 & -(1+1/k) & 1/k & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & -(1+1/k) & 1/k \\ 0 & \dots & 0 & 1 & -(1+1/k) \end{bmatrix}. \tag{30}$$

The eigenvalues of matrix (30) are given by [3]:

$$\lambda_k = -\frac{1}{RC}\left(1 + \frac{1}{k} - 2\sqrt{\frac{1}{k}}\cos\phi_k\right), \qquad \phi_k = \frac{k\pi}{n+1}, \tag{31}$$

where $k = 1, 2, 3, \dots, n$. It is evident that the exponential *RC*-ladder network may be represented in the form of (8). The simulation results for this type of system have been presented in Figs. 11 and 12.

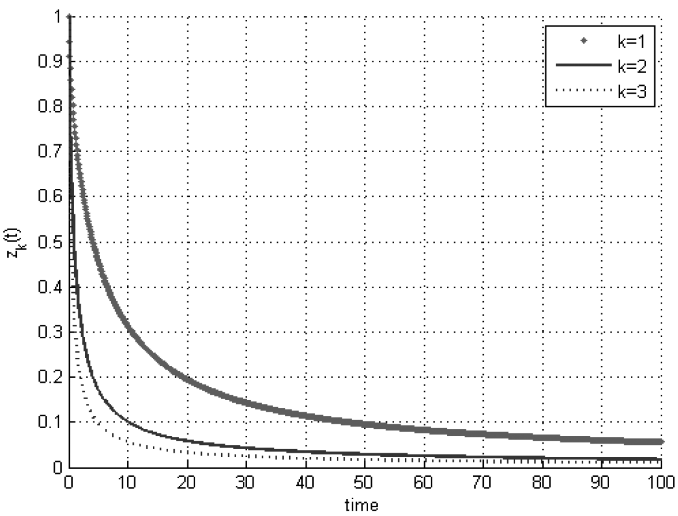

Fig. 11. Responses of subsystems (8) for *RCRR* and $\alpha = 0.7$

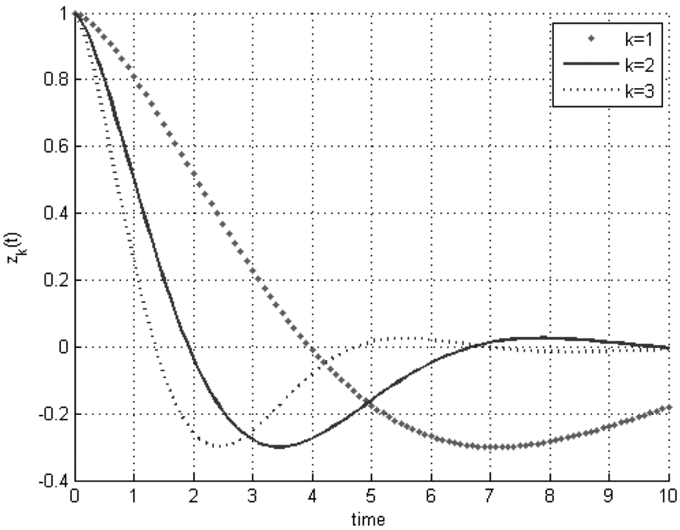

Fig. 12. Responses of subsystems (8) for *RCRR* and $\alpha = 1.5$

5. Conclusions

In this paper the dynamic properties characterized by the eigenvalues of the following structures are considered: ladder systems with supercapacitors of *RC*, *RCR*, *RRC*, *RRCRR* types. The study considered also exponentially convergent ladder networks and RCR ring systems. Similarly to integer order systems, it may be decomposed into n scalar subsystems which simplifies the analysis. However, contrarily to "classic" systems, it displays changing behaviour with varying α. Depending on the order of the equation, it becomes either ab *RC* or *LC* ladder system.

We proved that the analytical approach to complex *RC*-ladder systems with supercapacitors is possible. The *RC* structures can be transformed to (8).

Acknowledgment

This work was supported by the AGH (Poland) – the project No 11.11.120.817.

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