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INTL JOURNAL OF ELECTRONICS AND TELECOMMUNICATIONS, 2018, VOL. 64, NO. 2, PP. 217-222
Manuscript received December 17, 2017; revised April, 2018. DOI: 10.24425/119373

Detailed Consideration of Graphical Calculation of Min-Plus Convolution in Deterministic Network Calculus

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***Abstract*—The convolution operation used in deterministic network calculus differs from its counterpart known from the classic systems theory. A reason for this lies in the fact that the former is defined in terms of the so-called min-plus algebra. Therefore, it is oft difficult to realize how it really works. In these cases, its graphical interpretation can be very helpful. This paper is devoted to a topic of construction of the min-plus convolution curve. This is done here in a systematic way to avoid arriving at non-transparent figures that are presented in publications. Contrary to this, our procedure is very transparent and removes shortcomings of constructions known in the literature. Some examples illustrate its usefulness.**

***Keywords*—Convolution, network calculus, min-plus algebra, graphical construction of min-plus convolution.**

I. INTRODUCTION

QUEUEING theory, called also a mass service theory in some publications, is used in analyses of traffic in packet networks. Nowadays, it is not however only tool exploited in this area. The seminal works of R. Cruz [1], [2] initiated emergence of the so-called deterministic network calculus [3], being a new instrument facilitating analysis of packet flows in communication networks, their performance calculations, and also design of components of which they consist.

This paper is devoted to a particular problem of a graphical visualization of the min-plus convolution used in the deterministic network calculus [3]. Admittedly, the main parts of the material presented here were already presented at a telecommunications symposium in Poland, in September 2017. However, the proceedings of this symposium are hardly available to a wider audience because they are only in a form of a CD attachment to [4], and in Polish language. On the other hand, the topic considered therein has a fundamental importance in the deterministic network calculus. Therefore, we want to interest also the readers of this international journal in the subject mentioned above.

The deterministic network calculus used in an analysis of teletraffic systems can be viewed, as shown in [3], as a counterpart of the classic systems theory [5], [6] that deals with processing and transmission of signals. And, as known in this theory (or more precisely, in the theory of linear systems), a key operation is the convolution operation – in form of a convolution integral (in the case of analog systems) or in form of a convolution sum (for digital systems). According to it, a system input signal is „convoluted" – with the use of the so-called system's impulse response – giving a system output signal as a result. A similar operation called also convolution and denoted here by a symbol $\otimes$ occurs in the network calculus. Its defining equation is the following:

$$(A \otimes \beta)(t) \overset{df}{=} \inf_{0 \le \tau \le t} \{A(\tau) + \beta(t-\tau)\} , \qquad (1)$$

where $A(t)$ means the so-called cumulative traffic at the input of a system servicing packets (whereby the word "system" used here may also mean the following: network, network node, communication link, and so on). Furthermore, the symbol $\beta(t)$ in (1) means the so-called service curve of a given system. Note that this curve is a counterpart of the system's impulse response used in the classic systems theory [3]. It is always a non-decreasing function, similarly as the function $A(t)$. Moreover, the symbol inf in (1) denotes the operation of calculation of a lower bound of the sum of functions $A(\tau)$ and $\beta(t-\tau)$ in an indicated range of values of a time variable τ.

In the network calculus, the notion of cumulative traffic is a counterpart of the notion of a signal that is used in the classic systems theory. The former means simply a sum of the packets (bits), which entered or leaved a teletraffic system in a time period from 0 to t, where 0 stands for an assumed initial moment. Note that the value of the sum defined in such a way depends upon the value of t, which changes. Therefore, this sum can be regarded as a function of a time variable t. Further, if this function regards the system input it is then named the input traffic. It has been denoted above as $A(t)$. But when the cumulative traffic regards the system output it is called the output traffic, and denoted here as $D(t)$. This function is always a non-decreasing one, similarly as the functions $A(t)$ and $\beta(t)$.

It has been shown [3] that the following inequality:

$$D(t) \ge (A \otimes \beta)(t) \qquad (2)$$

relates the cumulative input traffic of a system with its cumulative output traffic. This inequality determines the amount of traffic, $(A \otimes \beta)(t)$, which will be certainly serviced in the period from 0 to t by a system possessing the service curve $\beta(t)$. Because of this reason, the latter curve is oft called in the literature the minimum service curve [7].

The work presented in this paper was supported partly by the grant AMG DS/450/2018.

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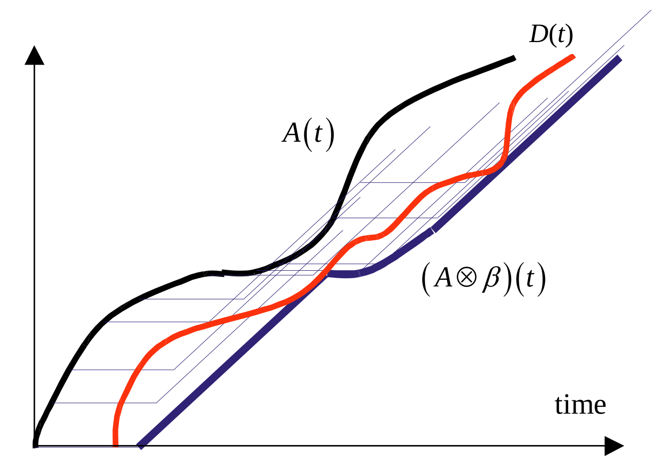
Note that in a particular case, when a traffic system behaves as a linear one, the inequality (2) turns into equality.

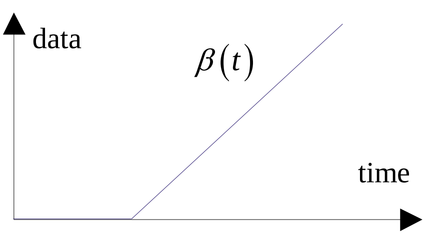
Moreover, observe that the operation of calculation of infimum occurring in the convolution definition (1) is a counterpart of the convolution integral in the convolution operation defined in the classic systems theory. Furthermore, the operation of addition in (1) is a counterpart of the multiplication of the system's impulse response by a time-shifted input signal, which occurs under the convolution integral. So, we see that the algebra used in the network calculus is not an usual algebra. The algebra used therein is called the min-plus algebra [3], while the convolution defined by (1), consistently, the min-plus convolution.

Constructing and operation in the time domain of the convolution defined in the classic systems theory is oft explained graphically [8]. This convention - for illustration purposes - is also used in the monographs [3] (see Fig. 1.8 in [3]) and [10] (see Fig. 2.4 in [10]), and basic articles [7] (Fig. 1 in [7]) and [9] (Fig. 3 in [9]) regarding the network calculus. However, a problem with all these figures mentioned above lies in the fact that they do not explain a basic issue of the min-plus convolution in an enough understandable way. This is mostly because of the lack of visibility on them of the lower limit calculation.

In order not to be groundless, let us take a closer look, successively, at all these figures mentioned above.



(a)



(b)

Fig. 1. This is the Fig. 1.8 in [3] redrawn for discussion in this paper: (a) a graphical illustration to the service curve definition and constructing min-plus convolution – after [3]; (b) a form of the service curve used in calculations for getting figure (a).

In Fig. 1, we see only a bunch of the shifted curves $\beta(t-\tau)$ (blue thin lines), but there is a lack of any visualization of the min-plus convolution calculations, even for a one value of the time variable *t* (denoted as time therein). Moreover, the usage of an auxiliary time variable τ is also not shown in this figure.

Consider now another graphical min-plus convolution presentation after [10] that is visualized in Fig. 2.

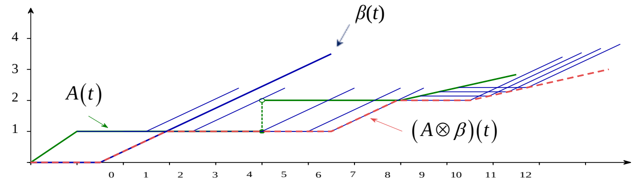


Fig. 2. This is the Fig. 2.4 in [10], which is redrawn here for illustration of the construction of the min-plus convolution that was proposed in [10].

In [10], a caption to Fig. 2 reads as follows: "The min-plus convolution of two functions passing through the origin can be obtained by placing one of the two functions at each point of the other function and taking the minimum of all the resulting functions." Really, it difficult to recognize a construction carried out according to the above description.

A construction of the min-plus convolution, which was proposed in [7], is visualized here in Fig. 3.

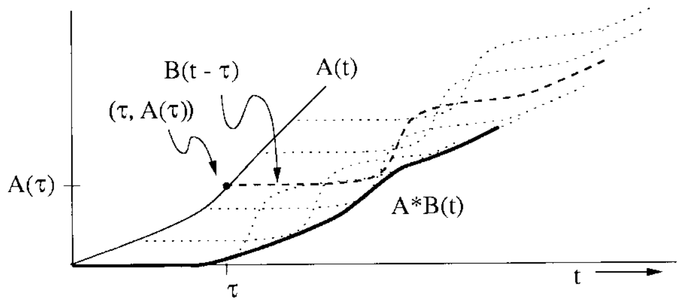


Fig. 3. This is the Fig. 1 in [7], illustrating the construction of the min-plus convolution that was proposed therein.

Some symbols occurring in Fig. 3 need explanation. $\mathrm{A(t)}$ denotes a curve of the cumulative traffic (similarly as $A(t)$ in Figs. 1 and 2) and $\mathrm{B(t)}$ means a service curve (similarly as $\beta(t)$ in Figs. 1 and 2). Moreover, the symbol * denotes the operation of the min-plus convolution (similarly as $\otimes$ in Figs. 1 and 2).

In Fig. 3, one value of an auxiliary time variable τ is indicated explicitly and further ones, for calculations of the time-shifted function $\mathrm{B(t-\tau)}$ and the associated function $\mathrm{A(\tau)}$, are suggested. The main drawback of Fig. 3 is that the operation of calculation of infimum, which is inherently associated with the min-plus convolution (see the definition (1)), is not at all visualized on it. In fact, the same problem is also with Figs. 1 and 2.

Finally, consider also a graphical construction of the min-plus convolution, which was presented in [9]. It shown here in Fig. 4.

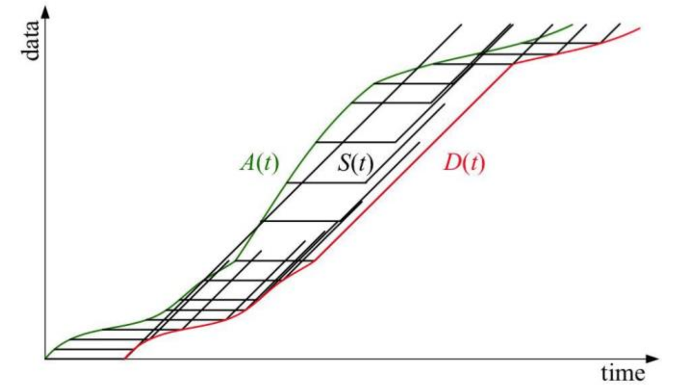


Fig. 4. This is the Fig. 3 in [9], which is redrawn here. It illustrates the graphical construction of the min-plus convolution as proposed in [9].

The symbol $S(t)$ in Fig. 4 means the service curve that is denoted by $\beta(t)$ in (1). Furthermore, a caption in [9] to Fig. 4 reads as follows: "System with service curve $S(t)$ being a rate-latency function (as illustrated in Fig. 1b above) and arrivals $A(t)$. A lower bound for the departures $D(t)$ follows by min-plus convolution and is constructed graphically by shifting $S(t)$ along $A(t)$ and taking the infimum." That is a result of the last operation mentioned (i.e. taking the infimum) is not shown at all in Fig. 4. Moreover, the usage of an auxiliary time variable τ in performing the constructions leading to obtaining the min-plus convolution is also not shown in this figure. Also, the meaning of the variable "time" in Fig. 4 is not described. Maybe it is used for both τ and t. But, how? This is not visualized in Fig. 4.

The objective of this paper is to present an alternative approach to the graphical representation of the min-plus convolution. We will show that it is free of shortcomings of any of the methods described above. Our approach allows to achieve a more transparent and understandable picture of the min-plus convolution.

In the next section, principles of constructing graphically the function defined by (1) are discussed in very detail. Then, the successive steps of the procedure developed are presented in section III for different forms of the curves of $A(t)$ and $\beta(t)$. The paper ends with some conclusions.

II. Constructing Curve Representing Min-Plus Convolution

On the right hand side of (1), two time variables, τ and t, occur, however, on its left hand side only one, t. The variable τ is „eliminated" therein through the use of an operation inf while keeping a fixed value of t. So, this fact should be shown in the first instance on the diagram. This is illustrated in Fig. 5.

In Fig. 5, the way of calculation of the expression occurring in the braces in (1) for four values of the varying parameter τ: 0, τ_1, τ_2 and t_0 is shown. The calculated values are marked on a vertical line going through the point $t=t_0$. The point $t=t_0$ is called here a point of „observation" because, loosely saying, the procedure we describe can be viewed as an „observation" of the calculated values (related with the moment indicated above) for the purpose of choosing the lowest element (or calculation of the lower limit of a set consisting of these elements).

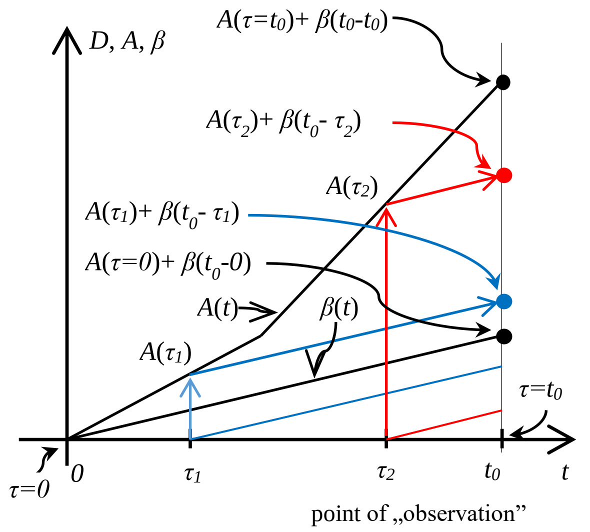


Fig. 5. Illustration of the principle of a calculation of the min-plus convolution for a fixed time.

For an example illustrated in Fig. 5, we see that the set consisting of all these values lies between the black points marked on the „observation" line. The lowest of them equals $A(0)+\beta(t_0)=\beta(t_0)$. That is the convolution value at the time t_0 is equal to $\beta(t_0)$ in the example considered. So, we have $(A\otimes\beta)(t_0)=\beta(t_0)$.

In the example of Fig. 5, the service curve $\beta(t)$ describes a linear system [3]. Hence, in the case of this system, the relation (2) becomes an equality, what means that the system's cumulative output traffic equals $D(t_0)=\beta(t_0)$. And the difference $A(t_0)-D(t_0)=A(t_0)-\beta(t_0)$ is the amount of traffic that the system was not able to service from 0 to the time $t=t_0$.

Changing the location of the point $t=t_0$ on the time axis and carrying out the construction shown on Fig. 5 for each of them, we obtain a set of points that connected with each another build up the curve $(A\otimes\beta)(t)$. This curve for the example of Fig. 5, for the reasons given above, coincides with the curves $D(t)$ and $\beta(t)$. Obviously, this is a specific case. In the next section, we consider other examples of the arrangements of the curves mentioned above.

III. Examples of Min-Plus Convolution Curves

A. First Example

In this example, illustrated in Fig. 6, we have such an arrangement, in which the cumulative input traffic curve lies above the service curve in the period from 0 to $t=\tau_1$. While for the next times, it does not change, remaining constant equal to $A(\tau_1)$, what means the lack of further packet supply to the system considered. Packets staying in a buffer are all the time serviced by the service curve until the moment, when all of them will be served. The latter occurs at the time $t = t_0$. From this moment on, the service curve passes above the cumulative input traffic and the difference $\beta(t) - A(t)$ increases over time. This difference determines the amount of traffic, which could be serviced if it appeared at the system's input.

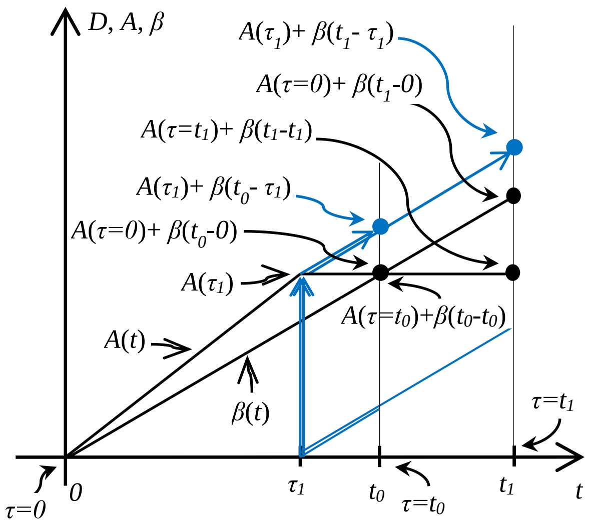


Fig. 6. An example of a calculation of the min-plus convolution, when the curves of $A(t)$ and $\beta(t)$ intersect.

Moreover, with regard to Fig. 6, the following remarks may be made:

1. For the time $t = t_0$, the values of the expression $A(\tau) + \beta(t - \tau)$ for the extreme values of time variable $\tau = 0$ and $\tau = t_0$ are the same. Therefore, we have only one black point lying on the „observation" line for $t = t_0$. However, we have two black points on the „observation" line for $t = t_1$ because the values of $A(\tau) + \beta(t - \tau)$ differ then from each other.
2. The value of the expression $A(\tau) + \beta(t - \tau)$ is shown only for one intermediate value of the variable $\tau = \tau_1$ on the "observation" lines $t = t_0$ and $t = t_1$ (blue point). Just this is a value for which the expression mentioned above has the largest value.
3. The operation inf in either of these cases searches for the lowest value in a set of points lying between the lower black point and the blue one. The lowest value is of course indicated by the lower black point.
4. In this example, similarly as in the previous one shown in Fig. 5, the convolution curve $(A \otimes \beta)(t)$ - passing through the two black points mentioned above – is at the same time the curve of the cumulative output traffic $D(t)$. It overlaps with the curve of $\beta(t)$ only partly. This is shown in Fig. 7.

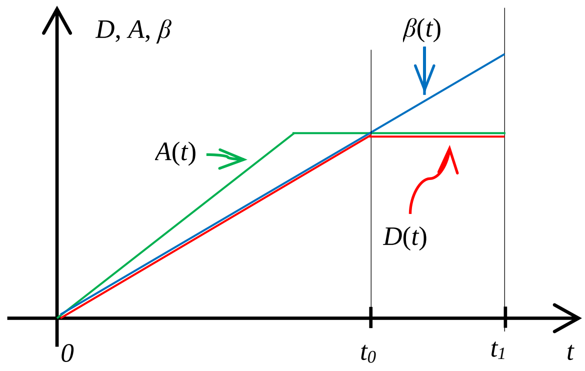


Fig. 7. The curve of $D(t)$ against the curves of $A(t)$ and $\beta(t)$ for the example illustrated in Fig. 6.

B. Second Example

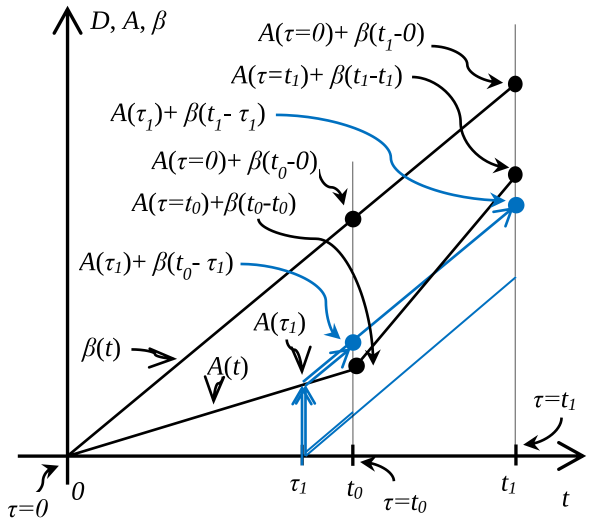


Fig. 8. An example of the convolution calculation in a case, when the curve of the cumulative input traffic lies beneath the service curve.

The service curve of a system illustrated in Fig. 8 lies above the curve of the cumulative input traffic applied to it, in the time interval from 0 to t_1. Moreover, it is a linear function. On the other hand, the curve of the cumulative input traffic is a polyline consisting of two segments of straight lines. The first of them (in the interval from 0 to t_0) possesses a slope, which is smaller than the one of the service curve. However, the slope of the second segment (in the interval from t_0 to t_1) is larger. In Fig. 8, two „observation" lines are shown that belong to the points t_0 and t_1 on the time axis. Also, it is shown one intermediate point τ_1 for which the value of the expression in the braces in (1) is calculated. The calculated values are marked by a black or blue color – according to the convention assumed in the previous example.

Moreover, with regard to Fig. 8, the following remarks may be made:

1. A blue point can also lie beneath a black one. Such a case occurs on the „observation" line passing through the point t_1.
2. The black points do not overlap in this case on any of the „observation" lines shown.

Note that in the configuration visualized in Fig. 8, similarly as in the previous ones shown in Figs. 5 and 6, the convolution curve $(A \otimes \beta)(t)$ represents at the same time also the curve of the cumulative output traffic $D(t)$. However, now, it coincides on none of its segments with the curve of $\beta(t)$. But, it overlaps with the curve of the cumulative input traffic $A(t)$ on the time interval from 0 to t_0. While on the interval from t_0 to t_1, it lies beneath the latter curve and possesses a slope of the service curve $\beta(t)$. This is visualized in Fig. 9.

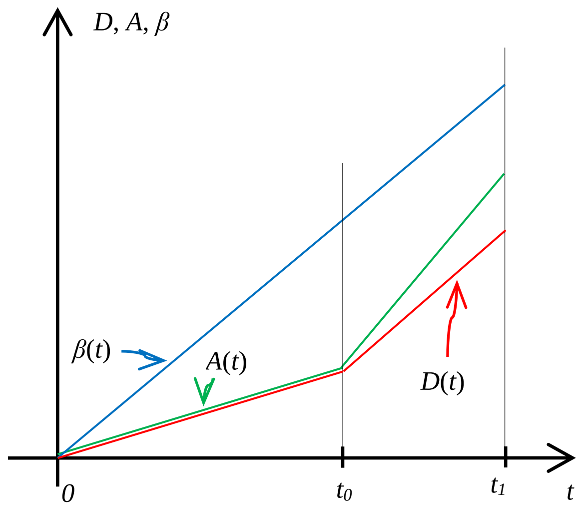


Fig. 9. The curve of $D(t)$ against the curves of $A(t)$ and $\beta(t)$ for the example illustrated in Fig. 8.

C. Third Example

In Fig. 10, an example of the min-plus convolution calculation is presented, when the service curve exhibits a delay in the service fulfilment. (Note that the result of this operation is denoted here by *C* because in general (see (2)) it does not have to be equal to the system's cumulative output traffic *D*.) It is seen in the figure that this delay equals t_0. The "observation" lines for the convolution calculation pass through the points t_0 and t_1 in Fig. 10. In each case, the results of calculations are shown for the extreme values of the variable τ (black points) and for the intermediate value τ_1 (blue points).

Note also that the lowest blue point on the "observation" line $t = t_1$ in Fig. 10 occurs for $\tau = t_1 - t_0$. Then, we have

$$A(t_1 - t_0) + \beta(t_1 - (t_1 - t_0)) = A(t_1 - t_0) + \beta(t_0) = \\ = A(t_1 - t_0) \quad . \tag{3}$$

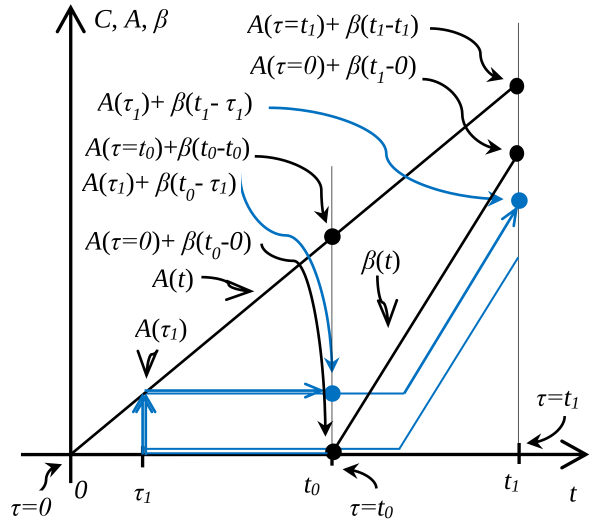


Fig. 10. An example of the convolution calculation in a case, when the service curve exhibits a delay in the service fulfilment.

So, it follows from the results of calculations performed for the "observation" lines of Fig. 10 that the convolution curve for the example presented in this figure has the form shown in Fig. 11.

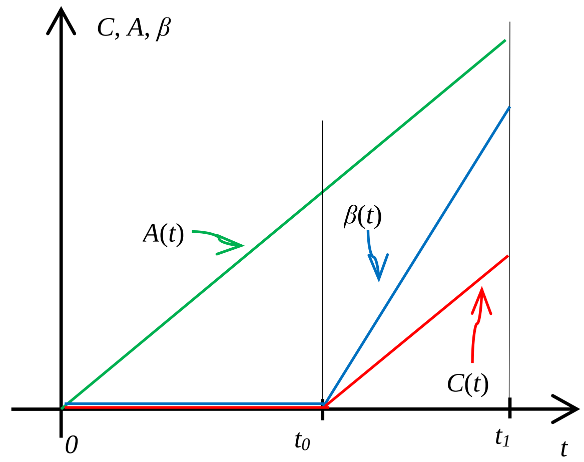


Fig. 11. The convolution curve $C(t)$ against the curves of $A(t)$ and $\beta(t)$ for the example illustrated in Fig. 10.

IV. Conclusions

In this paper, an alternative approach to the graphical presentation of the min-plus convolution operation used in the network calculus has been presented. It is more transparent and removes shortcomings of the existing methods. Its usefulness has been proven on examples of presentations of the min-plus convolutions for some typical, but, on the other hand, simple shapes of the curves of the cumulative input traffic and service curves.

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