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BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES 66(1, 2018
DOI: 10.24425/119054

Suboptimal control of nonlinear continuous-time locally positive systems using input-state linearization and SDRE approach

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Abstract. The infinite time suboptimal control problem for continuous-time nonlinear positive systems is formulated and solved. A solution to the problem using input-state linearization and state-dependent Riccati equation method (SDRE) is established, a procedure for solving the problem is proposed and illustrated with a numerical example.

Key words: positive nonlinear systems, feedback control, Riccati equation.

1. Introduction

Mathematical models for biological, medical systems [1, 2], as well as those for computer science [3] and engineering systems [7, 8], economics and social sciences [4], aircraft and satellite systems [5, 15, 16], are inherently nonlinear and have externally or internally positive properties.

An overview of the state of the art in positive system theory is given by Farina and Rinaldi [21] as well as Kaczorek [11, 20, 23]. The optimal control problems of standard and fractional linear positive continuous-time systems are formulated and solved by Kaczorek and Klamka [11, 22]. In this research area, also the reachability and controllability conditions of standard and singular internally positive linear systems are analysed [11]. The work related to locally positive systems [20] presents necessary and sufficient conditions for the local positiveness of nonlinear time-varying systems and is an inspiration for searching optimal or suboptimal control methods for such systems.

In literature, there are many techniques for nonlinear systems, such as Jacobian linearization [17] and feedback linearization used in conjunction with gain scheduling [12], Hamiltonian system [26], dynamic inversion methods [10], L1-optimal feedback controller synthesis for positive systems with given weighing vectors [27], recursive backstepping [18], sliding mode control and adaptive control [10]. Recently, one of the promising and rapidly emerging methodology for designing nonlinear controllers is the state-dependent Riccati equation (SDRE) approach [6–8, 13]. It is dedicated for designing the suboptimal controller and compensator for a certain class of nonlinear systems. By using Taylor series numerical technique, this controller design method can significantly reduce the online computational burden like the recently popular SDRE methods [6].

This paper presents an SDRE-based suboptimal control strategy for locally positive continuous-time nonlinear systems with separation of linear and nonlinear state functions. The approach to the suboptimal control using nonlinear feedback controller enables determination of suboptimal control allowing nonlinear control theory.

2. Controllability of nonlinear positive systems

Consider the continuous-time nonlinear system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u} \tag{1}$$

where $\mathbf{x} \in \Re^n$, $\mathbf{u} \in \Re^m$ are the state and input vectors, respectively, $\mathbf{F}(\mathbf{x}) \in \Re^n$ and $\mathbf{B}(\mathbf{x}) \in \Re^{n \times m}$ are nonlinear function of $\mathbf{x}$. Let the system (1) be locally positive. The system is called locally positive in the neighborhood of zero $(\mathbf{x} = 0, \mathbf{u} = 0)$ if there exists a neighbourhood of the zero $\mathbf{U}_0$ such that for any $\mathbf{x}_0 \in \mathbf{U}_0 \cap \Re^n_+$ we have $\mathbf{x}(t) \in \mathbf{U}_0 \cap \Re^n_+$ for $t \geq 0$ [20]. This definition performs systems for which

$$\int_0^t \frac{\partial f_i}{\partial x_j} d\tau \geq 0 \tag{2}$$

for $i \neq j$, $i, j = 1, \ldots, n$ and $t \geq 0$. Nonlinear functions f_i are elements of the vector $\mathbf{F}(\mathbf{x})$ and are smooth in their arguments, i.e., they are real-valued functions of state variables x_i.

Let us rewrite the system (1) as a sum of state-independent and nonlinear state-dependent coefficient (SDC) forms [6]:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{\Psi}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u}, \tag{3}$$

where $\mathbf{A}\mathbf{x} + \mathbf{\Psi}(\mathbf{x})\mathbf{x}$ is a sum of parametrized linear and nonlinear state equations (obtained by Taylor series, for instance).

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Manuscript submitted 2017-04-06, revised 2017-08-29, initially accepted for publication 2017-09-08, published in February 2018.

There are many ways of SDC parameterization. Each parametrization should be true for all $0 \leq \alpha \leq 1$.

$$\begin{aligned}
&\alpha\left[\mathbf{A}_1 + \mathbf{\Psi}_1(\mathbf{x})\right]\mathbf{x} + (1-\alpha)\left[\mathbf{A}_2 + \mathbf{\Psi}_2(\mathbf{x})\right]\mathbf{x} = \\
&\alpha\mathbf{A}_1\mathbf{x} + (1-\alpha)\mathbf{A}_2\mathbf{x} + \alpha\mathbf{\Psi}_1(\mathbf{x})\mathbf{x} + (1-\alpha)\mathbf{\Psi}_2(\mathbf{x})\mathbf{x} = \\
&\alpha\mathbf{F}_1(\mathbf{x}) + (1-\alpha)\mathbf{F}_2(\mathbf{x}) = \mathbf{F}(\mathbf{x}).
\end{aligned} \tag{4}$$

To design a control system and define control law, first we should determine whether or not control of the complete state of the dynamical system is possible. This information can be obtained checking controllability of the dynamical system. Roughly, controllability informs about control possibility of the dynamical system from an arbitrary initial state to an arbitrary final state using set of admissible controls. However, the trajectory of the dynamical system (1) between initial and final state is not specified. Furthermore, there are no constraints posed on the control vector $\mathbf{u}$ and the state vector $\mathbf{x}$. In order to formulate easily computable algebraic controllability criteria, let us introduce the state-dependent controllability matrix $\mathbf{W}(\mathbf{x})$ [6]:

$$\begin{aligned}
\mathbf{W}(\mathbf{x}) = \Big[\mathbf{B}(\mathbf{x}) \quad &\big(\mathbf{A} + \mathbf{\Psi}(\mathbf{x})\big)\mathbf{B}(\mathbf{x}) \quad \ldots \\
&\big(\mathbf{A} + \mathbf{\Psi}(\mathbf{x})\big)^{n-1}\mathbf{B}(\mathbf{x})\Big].
\end{aligned} \tag{5}$$

If $\mathbf{W}(\mathbf{x})$ (a state-dependent in this case) has full rank then the system is controllable for all $\mathbf{x} \in \Re^n$. In practice, we must seek a parameterization that gives $\mathbf{W}(\mathbf{x})$ full rank and monomial for the entire domain for which the system is to be controlled.

3. Control problem solution

We wish to find an admissible control $\mathbf{u}(t) \in \Re^m$ that minimizes performance index [9, 14]

$$J(\mathbf{u}) = \frac{1}{2}\int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right)dt \tag{6}$$

where $\mathbf{Q} \in \Re^{n\times n}$ is a symmetric semi positive-definite matrix, $\mathbf{R} \in \Re^{m\times m}$ is a symmetric positive-definite matrix.

The control problem for the nonlinear continuous-time systems (1) can be stated as follows. Given nonlinear functions $\mathbf{F}(\mathbf{x}) \in \Re^n$, $\mathbf{B}(\mathbf{x}) \in \Re^{n\times m}$ and $\mathbf{Q} \in \Re^{n\times n}$, $\mathbf{R} \in \Re^{m\times m}$ of the performance index (2), find a control $\mathbf{u}(t) \in \Re^m$ for $[t_0, \infty]$ that controls the system state vector from $\mathbf{x}_0$ to $\mathbf{x}_\infty$ while minimizing the performance index (6).

If the integral function $\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}$ and $\mathbf{A}\mathbf{x} + \mathbf{\Psi}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u}$ be continuously differentiable functions of each of their arguments, then we may suppose that $\mathbf{u} \in C[t_0, \infty]$ is an control that minimizes the functional $J(\mathbf{x})$: $C[t_0, \infty] \to \Re_+$. To solve the problem, we define the Hamiltonian

$$H = \frac{1}{2}\left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right) + \mathbf{p}^T\left(\mathbf{A}\mathbf{x} + \mathbf{\Psi}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u}\right). \tag{7}$$

Let $\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}$ and $\mathbf{A}\mathbf{x} + \mathbf{\Psi}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u}$ be continuously differentiable functions of each of their arguments. If $\mathbf{u} \in C[t_0, \infty]$ is a control for the functional (6) subject to the state equation (3) and if $\mathbf{x}$ denotes the corresponding state, then there exists a $\mathbf{p} \in C[t_0, \infty]$ such that

$$\frac{\partial H}{\partial \mathbf{u}}(\mathbf{p}, \mathbf{x}, \mathbf{u}, t) = \mathbf{0} \quad \text{for} \quad t \in [t_0, \infty] \tag{8}$$

and

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{u}}(\mathbf{p}, \mathbf{x}, \mathbf{u}, t) \ \text{ for } \ t \in [t_0, \infty] \ \text{ and } \ \mathbf{p}(\infty) = 0. \tag{9}$$

where $\mathbf{p}$ is the co-state nonlinear function and (9) is the adjoint differential equation.

It follows that any optimal input $\mathbf{u}(t) \in \Re^m$ and the corresponding state $\mathbf{x} \in \Re^n$ satisfies (8) that is

$$\frac{\partial H}{\partial \mathbf{u}} = \mathbf{R}\mathbf{u} + \mathbf{B}^T(\mathbf{x})\mathbf{p} = \mathbf{0}. \tag{10}$$

Thus optimal control is

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x})\mathbf{p} \tag{11}$$

the adjoint differential equation is following

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}} = -\left[\mathbf{A} + \frac{\partial(\mathbf{\Psi}(\mathbf{x})\mathbf{x})}{\partial \mathbf{x}} + \frac{\partial(\mathbf{B}(\mathbf{x})\mathbf{u})}{\partial \mathbf{x}}\right]^T \mathbf{p} - \mathbf{Q}\mathbf{x}. \tag{12}$$

Consequently, we have nonlinear differential system of equation

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{\Psi}(\mathbf{x})\mathbf{x} - \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x})\mathbf{p} \\
\dot{\mathbf{p}} &= -\frac{\partial H}{\partial \mathbf{x}} = -\left[\mathbf{A} + \frac{\partial(\mathbf{\Psi}(\mathbf{x})\mathbf{x})}{\partial \mathbf{x}} + \frac{\partial(\mathbf{B}(\mathbf{x})\mathbf{u})}{\partial \mathbf{x}}\right]^T \mathbf{p} - \mathbf{Q}\mathbf{x}
\end{aligned} \tag{13}$$

for $t \in [t_0, \infty]$, $\mathbf{x}(t_0) = \mathbf{x}_0$, $\mathbf{\Psi}(\mathbf{x}_0) = \mathbf{\Psi}_0$ and $\mathbf{p}(\infty) = 0$, where:

$$\frac{\partial(\mathbf{\Psi}(\mathbf{x})\mathbf{x})}{\partial \mathbf{x}} = \mathbf{\Psi}(\mathbf{x}) + \frac{\partial \mathbf{\Psi}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{x}, \tag{14}$$

$$\frac{\partial(\mathbf{B}(\mathbf{x})\mathbf{u})}{\partial \mathbf{x}} = \mathbf{\Psi}(\mathbf{x}) + \frac{\partial \mathbf{B}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{u}. \tag{15}$$

Let $\mathbf{p}$ be a combination of linear and nonlinear state of the system (3)

$$\mathbf{p} = \mathbf{K}_1(\mathbf{x})\mathbf{x} + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\mathbf{x}, \tag{16}$$

and let $\mathbf{x}$ be the solution of nonlinear state equation

$$\begin{aligned}
\dot{\mathbf{x}} = &\left[\mathbf{A} - \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x})\mathbf{K}_1(\mathbf{x})\right]\mathbf{x} + \\
&+ \left[\mathbf{I} - \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x})\mathbf{K}_2(\mathbf{x})\right]\mathbf{x} + \mathbf{\Psi}(\mathbf{x})\mathbf{x}
\end{aligned} \tag{17}$$

for $t \in [t_0, \infty]$, $\mathbf{x}(t_0) = \mathbf{x}_0$, $\mathbf{\Psi}(\mathbf{x}_0) = \mathbf{\Psi}_0$.

Employed approach (16) splits the feedback effort into two components. It enables to eliminate $\mathbf{\Psi}(\mathbf{x})\mathbf{x}$ from the system (17) by solving state-dependent gain matrix

$$\mathbf{K}_2(\mathbf{x}) = \left[\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x})\right]^{-1}. \tag{18}$$

Matrix $\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x})$ is singular, thus state-dependent matrix gain $\mathbf{K}_2(\mathbf{x})$ may be computed only by the pseudoinverse operation. This way we have nonlinear state equation (17) reduced to

$$\dot{\mathbf{x}} = \left[\mathbf{A} - \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x})\mathbf{K}_1(\mathbf{x})\right]\mathbf{x}. \tag{19}$$

Equating adjoint differential equation (12) and differential form of (16) we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\right]\mathbf{x} + \mathbf{K}_2\left[\mathbf{K}_1(\mathbf{x}) +\right.\\
&\left. + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\right]\dot{\mathbf{x}} =\\
&-\left[\mathbf{A} + \mathbf{\Psi}(\mathbf{x}) + \frac{\partial \mathbf{\Psi}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{x}\right]^T\left[\mathbf{K}_1(\mathbf{x})\mathbf{x} + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\mathbf{x}\right] -\\
&-\left[\frac{\partial \mathbf{B}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{u}\right]^T\left[\mathbf{K}_1(\mathbf{x})\mathbf{x} + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\mathbf{x}\right] - \mathbf{Q}\mathbf{x}.
\end{aligned} \tag{20}$$

Consequently, employing (19) to (20) we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\right]\mathbf{x} + \left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\right]\mathbf{A}\mathbf{x} -\\
&-\left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\right]\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x})\mathbf{K}_1(\mathbf{x})\mathbf{x} =\\
&-\left[\mathbf{A} + \mathbf{\Psi}(\mathbf{x}) + \frac{\partial \mathbf{\Psi}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{x}\right]^T\left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\right]\mathbf{x} -\\
&-\left[\frac{\partial \mathbf{B}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{u}\right]^T\left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\right]\mathbf{x} - \mathbf{Q}\mathbf{x}
\end{aligned} \tag{21}$$

and rearranging terms in (21) we find

$$\begin{aligned}
&\left(\begin{aligned}
&\frac{\partial}{\partial t}\left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\right] +\\
&\left[\frac{\partial \mathbf{\Psi}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{x}\right]^T\left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\right] -\\
&\left[\frac{\partial \mathbf{B}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{u}\right]^T\left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\right]
\end{aligned}\right)\mathbf{x} +\\
&\left(\begin{aligned}
&\left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\right]\mathbf{A} -\\
&\left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\right]\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x})\mathbf{K}_1(\mathbf{x}) +\\
&\left[\mathbf{A} + \mathbf{\Psi}(\mathbf{x})\right]^T\left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\right] + \mathbf{Q}
\end{aligned}\right)\mathbf{x} = 0.
\end{aligned} \tag{22}$$

If we assume that $\mathbf{K}_1(\mathbf{x})$ solves the state-dependent Riccati equation (SDRE), which using (18) is given by

$$\begin{aligned}
&\mathbf{K}_1(\mathbf{x})\mathbf{A} + \mathbf{A}^T\mathbf{K}_1(\mathbf{x}) -\\
&\mathbf{K}_1(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x})\mathbf{K}_1(\mathbf{x}) +\\
&\left[\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x})\right]^{-1}\mathbf{\Psi}(\mathbf{x})\mathbf{A} + \mathbf{A}^T\mathbf{\Psi}^T(\mathbf{x})\left[\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x})\right]^{-1} +\\
&\mathbf{K}_1(\mathbf{x})\mathbf{\Psi}(\mathbf{x}) + \mathbf{\Psi}^T(\mathbf{x})\mathbf{K}_1(\mathbf{x}) +\\
&\mathbf{\Psi}^T(\mathbf{x})\left[\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x})\right]^{-1}\mathbf{\Psi}(\mathbf{x}) + \mathbf{Q} = 0,
\end{aligned} \tag{23}$$

then the following condition must be satisfied for optimality [6, 13]

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\right] +\\
&\left[\frac{\partial \mathbf{\Psi}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{x}\right]^T\left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\right] -\\
&\left[\frac{\partial \mathbf{B}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{u}\right]^T\left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\right] = 0.
\end{aligned} \tag{24}$$

So, the suboptimal control

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x})\left[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\right]\mathbf{x} \tag{25}$$

for index (6) subject to (1) can be found solving state-dependent Riccati equation (23). In general, the solution of (23) cannot be found analytically. One approach for solving the SDRE is via symbolic software packages such as Matlab [24]. However, for complex systems, the solution may become complicated and then it is necessary to approximate the solution. To approximate we may use interpolation method or Taylor series method, for instance [6].

4. Numerical example

Consider the nonlinear locally positive system (1) with nonlinear state function and input matrix

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} x_2 \\ x_1^2 \end{bmatrix}, \quad \mathbf{B}(\mathbf{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{26}$$

with state vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and performance index (6) with matrices

$$\mathbf{Q} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \mathbf{R} = \left[0.5\right]. \tag{27}$$

Nonlinear system (26) satisfies the condition (2) since

$$\int_0^t \frac{\partial f_1}{\partial x_2} d\tau = \int_0^t 1 d\tau \geq 0, \quad \int_0^t \frac{\partial f_2}{\partial x_1} d\tau = 0. \tag{28}$$

An SDC parameterization of (26) is given by

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ x_1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ x_1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{29}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{\Psi}(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ x_1 & 0 \end{bmatrix}. \tag{30}$$

This parameterization has state-dependent controllability matrix (5) given by

$$\mathbf{W}(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{31}$$

which has full rank for all $\mathbf{x}(t) \in \Re^2$. Therefore the system (26) is controllable with the SDC parameterization (29).

Using symbolic software package we obtain solution of Riccati equation (23) as following gains functions:

$$\mathbf{K}_1(x_1, x_2) = \begin{bmatrix} k_{1,11}(x_1, x_2) & k_{1,12}(x_1, x_2) \\ k_{1,21}(x_1, x_2) & k_{1,22}(x_1, x_2) \end{bmatrix} \tag{32}$$

where

$$\begin{aligned}
k_{1,11}(x_1,x_2) &= \frac{\sqrt{2x_1^2 r^2 + b_{21}^2 q_{11} r}}{b_{21}^2 a_{12} r} \times \\
&\sqrt{b_{21}^2 q_{22} r + 4x_1 a_{12} r^2 + 2\sqrt{\begin{matrix} b_{21}^2 x_1^2 a_{12}^2 r^4 + \\ x_1^2 a_{12}^2 r^4 + b_{21}^2 q_{11} a_{12}^2 r^3 \end{matrix}}} \\
k_{1,12}(x_1,x_2) &= \frac{1}{2a_{12}r}\left(\begin{matrix} 2x_1 a_{12} r^2 \left(1 + \dfrac{1}{b_{21}^2}\right) + \\ \dfrac{2}{b_{21}^2}\sqrt{2x_1^2 a_{12}^2 r^4 + b_{21}^2 q_{11} a_{12}^2 r^3} + 2q_{22} r \end{matrix} \right) \\
k_{1,21}(x_1,x_2) &= \frac{1}{2a_{12}r}\left(\begin{matrix} 2x_1 a_{12} r^2 \left(1 + \dfrac{1}{b_{21}^2}\right) + \\ \dfrac{2}{b_{21}^2}\sqrt{2x_1^2 a_{12}^2 r^4 + b_{21}^2 q_{11} a_{12}^2 r^3} + 2q_{22} r \end{matrix} \right) \\
k_{1,22}(x_1,x_2) &= \frac{1}{b_{21}^2}\sqrt{\begin{matrix} b_{21}^2 q_{22} r + 4x_1 a_{12} r^2 + \\ 2\sqrt{\begin{matrix} b_{21}^2 x_1^2 a_{12}^2 r^4 + \\ x_1^2 a_{12}^2 r^4 + b_{21}^2 q_{11} a_{12}^2 r^3 \end{matrix}} \end{matrix}}.
\end{aligned} \tag{33}$$

The gain matrix (18) is computed by using pseudoinverse operation and yields

$$\mathbf{K}_2(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}. \tag{34}$$

As an example, the closed-loop system with nonlinear plant (26) was computed with feedback compensator gains (32) and (34).

We first consider the system behavior when the initial condition is $\mathbf{x}_0 = [1 \quad 2]^T$. Figure 1 depicts the state dynamics of the controlled system. Figure 2 presents SDRE formulated control and Fig. 3 shows the value of the cost functional integrand over time span $0 \leq t \leq 10$ seconds.

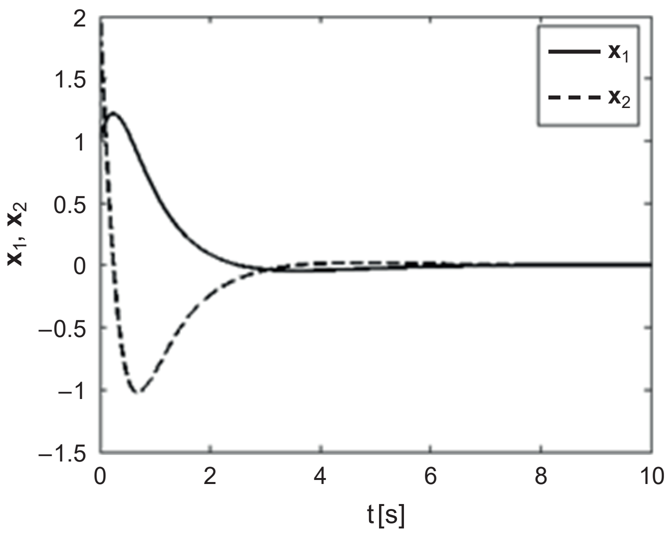

Fig. 1. Closed-loop state dynamics of the first system

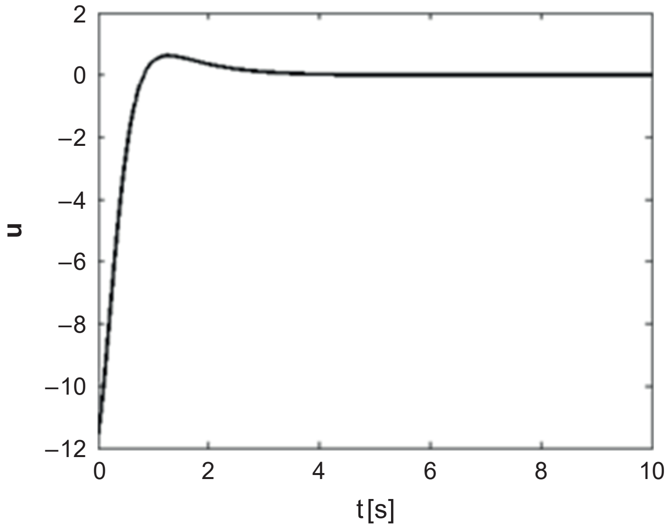

Fig. 2. SDRE closed-loop control of the first system

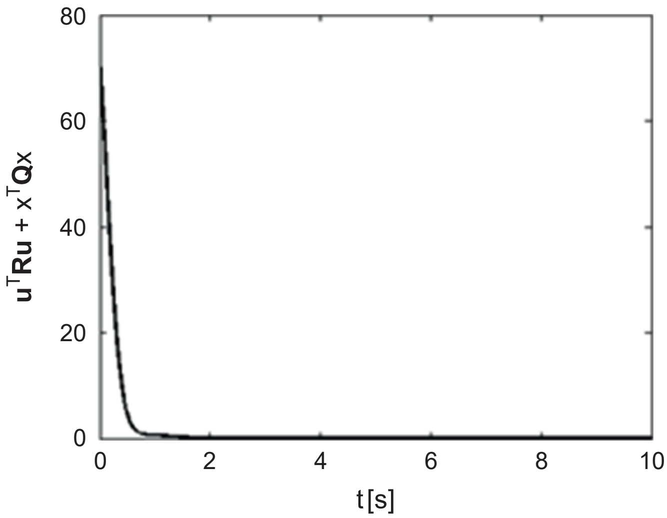

Fig. 3 Cost functional integrand

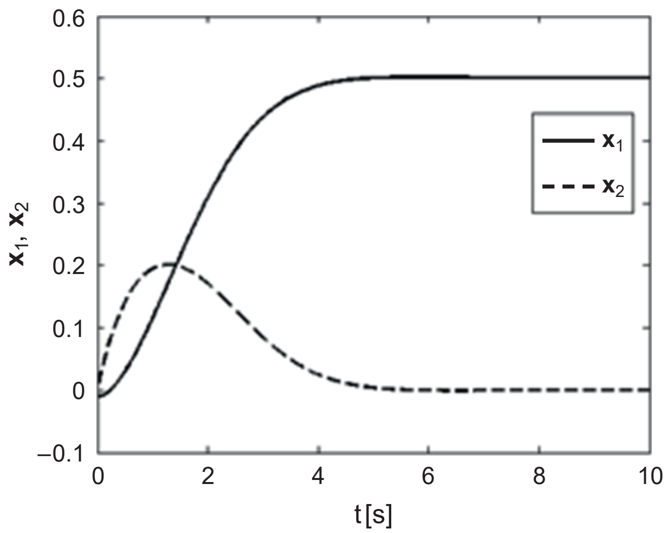

Fig. 4 Closed-loop state dynamics of the second system

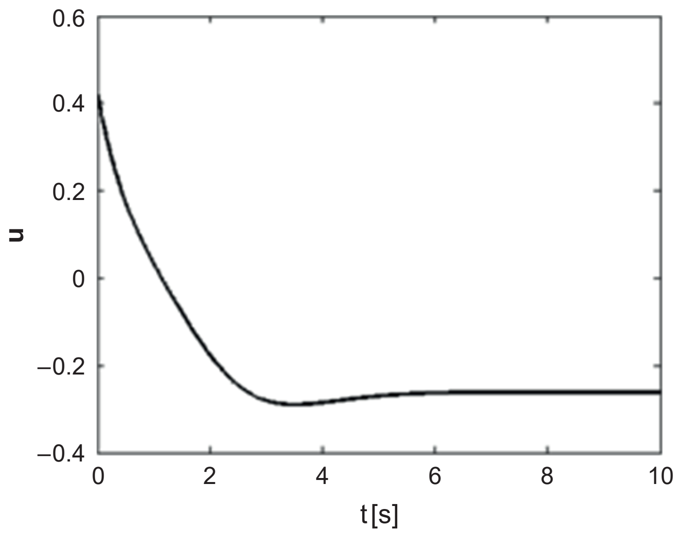

Fig. 5 SDRE closed-loop control of the second system

Next, we consider control problem for reference state $\mathbf{x}_{ref} = [0.5 \quad 0]^T$, in other we transfer the system (26) from initial state $x_{1,0} = 0$ to state $x_{1,\infty} = 0.5$ in infinite time. For the problem solution, the control law has been modified to

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^T(\mathbf{x})\big[\mathbf{K}_1(\mathbf{x}) + \mathbf{K}_2(\mathbf{x})\mathbf{\Psi}(\mathbf{x})\big]\big(\mathbf{x} - \mathbf{x}_{ref}\big). \tag{35}$$

In order to demonstrate the usefulness of the control method, closed-loop state dynamics and SDRE control are presented in Figs. 4 and 5, respectively.

During this process both states are positive, while the second state reaches the zero value in steady state.

Numerical simulations prove that the proposed method is useful for control of nonlinear locally positive systems. The methodology can be successfully applied to find suboptimal control of nonlinear systems described by state equation (1).

5. Conclusions

The infinite time control problem for nonlinear locally positive continuous-time systems with nonlinear feedback compensator was formulated and solved. The method for computation of suboptimal control input that minimizes performance index was proposed. The effectiveness of presented technique was demonstrated on numerical examples. The presented method can be extended to finite time problem and discrete-time systems.

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