PRZEGLĄD STATYSTYCZNY
R. LXIV – ZESZYT 3 – 2017

ZBIGNIEW ŚWITALSKI[1]

# STABILITY AND PRICE EQUILIBRIA IN A MANY-TO-MANY GALE-SHAPLEY MARKET MODEL[2]

## 1. INTRODUCTION

Much research on markets with indivisible goods have their sources in the classical models of Gale (model of buying *n* houses by *n* buyers, 1960, Ch. V, § 6), Gale, Shapley (college admissions model, 1962) and Shapley, Shubik ("assignment game" with quasi-linear utility, 1971/72). Different variants of these models have many applications in the theory of recruitment systems, auctions markets, labor markets and so on (see, e.g., Roth, Sotomayor, 1992; Crawford, Knoer, 1981; Andersson, Erlanson, 2013; Biro, Kiselgof, 2013, many interesting applications are presented in the survey of Sönmez, Ünver, 2011, the role of such models in modern economics is explained in Roth, 2002).

One of the main topics, considered in the literature related to such models and their generalizations, is the problem of relationships between the concept of stability (or the core outcome) and the concept of competitive equilibrium. Stable outcomes are often defined as allocations of goods among agents such that no coalition of agents can reallocate the goods in such a way that the situation of all members of the coalition will be improved. Competitive equilibrium is in most cases defined as an allocation of goods and a price vector such that each agent obtains the most preferred goods from the set of all feasible goods (for this agent). The two notions are defined differently, but for many models, for which prices of the goods can be defined, it can be proved that equilibria allocations are stable and stable outcomes are equilibria allocations, associated with some price vectors (see, e.g., Shapley, Shubik, 1971/72; Camina, 2006; Sotomayor, 2007; Hatfield et al., 2013; Herings, 2015).

Studying relationships between stability and competitive equilibria is very important. First, we can find in this way some new characterization of stable outcomes and this may have much importance for stability theory, and second, we can characterize competitive equilibria allocations as stable outcomes, which gives possibility of proving results on existence of equilibria (because existence of stable outcomes can be, in many cases, proved with relatively small effort).

[1] University of Zielona Góra, Faculty of Mathematics, Computer Science and Econometrics, 4a Prof. Z. Szafrana St., 65-516 Zielona Góra, Poland, e-mail: z.switalski@wmie.uz.zgora.pl.

[2] The research was financially supported by the Polish National Science Centre (NCN) – grant DEC-2011/01/B/HS4/00812.

Most of the research in the theory of "matching markets" concerning relationships between stability and competitive equilibria are based on the Shapley-Shubik type models and relatively few are devoted to such relationships for the Gale-Shapley model (see Świtalski, 2008, 2010, 2015, 2016; Azevedo, Leshno, 2011).

In the paper of Świtalski (2016) a variant of many-to-many Gale-Shapley market model was presented, for which exact relationships between stable matchings and a kind of generalized equilibria (called there order equilibria) were proved (see theorems 1 and 2 below). Stability in the paper of Świtalski (2016) is understood as pairwise stability in the sense of Echenique, Oviedo (2006) and is equivalent to the concept of stability used by Alkan, Gale (2003). Order equilibria generalize classical price equilibria, and so the results of the paper of Świtalski (2016) can be used to study relationships between stability and price equilibria for the GS-models.

The presented paper may be treated as a continuation of the paper of Świtalski (2016). Using the results of our previous paper, we study in detail relationships between (pairwise) stability and price equilibria for a generalized many-to-many Gale-Shapley market model with choice functions representing preferences of the buyers, reservation prices of the buyers and weak orders representing preferences of the sellers. In our model we assume that preferences of the sellers are determined by (or at least are closely related to) reservation prices of the buyers (see definition 13). The model is a generalization of the standard one-to-one GS-model, but can also be treated as a many-to-many generalization of the Chen, Deng and Ghosh's model (Chen et al., 2014) of matching markets with budgets (we use the term "reservation price" instead of "budget", see the comments at the beginning of section 3).

The main result of our paper is theorem 5 which shows that, under the assumption of path independence of choice functions of the buyers, strongly stable matchings for the generalized GS-model are identical with price equilibria allocations for this model. Using this result and some results of Alkan, Gale (2003) we prove theorems on existence of price equilibria for many-to-many GS-model or for many-to-many version of Chen, Deng and Ghosh's model (Chen et al., 2014) (theorems 7 and 8).

Our results can be treated as a far-reaching generalization of "supply and demand lemma" of Azevedo, Leshno (2011, p. 18; see also introduction in Świtalski, 2016). First, we consider many-to-many model with choice functions at one side of the market and weak orders at the second side (Azevedo and Leshno study many-to-one model with strict linear orders at both sides of the market). Secondly, we consider different conditions of compatibility of reservation prices (= scores in Azevedo, Leshno) with preferences of the buyers (colleges) and two different conditions of stability (see definition 7). We also study in detail how implications relating stable matchings and equilibria allocations (i.e. stable matchings ⇒ equilibria allocations and equilibria allocations ⇒ stable matchings) depend on particular properties of choice functions (outcast and heritage properties, see theorem 3 and 4). We consequently use the terminology of equilibrium theory (e.g. prices and equilibria allocations) and reformulate Azevedo and Leshno's result (which can be treated as a special case of our results) using this terminology.

It is worth noting that the many-to-many model we use cannot be treated as a special case of typical many-to-many models (e.g. presented by Echenique, Oviedo, 2006 or used in the contract theory, see e.g. Klaus, Walz, 2009; Kominers, 2012; Hatfield, Kominers, 2016). The reason is that we use choice functions which are not necessarily generated by strict linear orders on the families of subsets of feasible contracts (such assumption is commonly used in "many-to-many" papers). A similar approach with general choice functions for many-to-many models was used by Alkan, Gale (2003).

Alkan, Gale (2003, pp. 290, 291) argue that the model with choice functions may be more suitable for markets in which sellers (or buyers), e.g. colleges, want to satisfy requirements concerned with diversity (e.g. racial or ethnical) of chosen groups (e.g. groups of students). What's more, assumption about linear ordering of subsets of agents, in many cases, is not necessary because all relevant information about preferences of the agent is included in the choice function alone. In our paper we follow the Alkan and Gale's approach and show that there can be proved interesting results about relationships between stability and equilibria without the assumption about linear ordering of subsets of agents (or contracts).

In our model we use standard pairwise stability condition which can also be defined without the assumption about preference ordering of subsets of contracts (our definition agree with the definition of Echenique, Oviedo, 2006 and the one used by Alkan, Gale, 2003) and do not refer to other stability concepts (for example setwise stability or other stabilities defined by Echenique, Oviedo, 2006). Pairwise stability gives possibility of using the result of Alkan, Gale (2003) for proving existence of price equilibria in our model (see theorem 7).

The paper is organized as follows. In section 2 we present the model of market which is taken from Świtalski (2016) and define generalized (order) equilibria for such a model. In section 3 we study the problem of relationships between price equilibria and order equilibria. In section 4 we formulate and prove results on relationships between price equilibria and stable matchings and on existence of price equilibria.

## 2. THE MODEL

We present here a generalized many-to-many Gale-Shapley market model described in the paper of Świtalski (2016), where the reader can also find the motivation for studying such a model. The model is based on a many-to-many version of the Gale-Shapley (1962) college admissions model (many-to-many models of the GS type are studied, e.g., in the paper of Alkan, Gale, 2003 or Echenique, Oviedo, 2006, in both papers preferences of agents are represented by choice functions, but in the last paper with the additional assumption that the choice functions are generated by strict linear orders on the families of subsets of contracts).

In some cases (for preferences satisfying additional requirements) many-to-many model can be easily reduced to many-to-one or one-to-many model by assuming that, say, buyers, matched to multiple sellers, are represented by sets of buyers with identical preferences and matched to only one seller. We do not follow this way of reasoning. We think that studying many-to-many model in full generality is more elegant and with obvious real-life interpretations. Moreover, for the presented below many-to-many model we can directly use the results of our previous paper (Świtalski, 2016).

In our model we have a finite set of buyers – $B$, a finite set of sellers – $S$, preferences of buyers over sellers and preferences of sellers over buyers. As an example (see Echenique, Oviedo, 2006), let $B$ be a set of firms and $S$ – a set of consultants. Each firm wants to hire a set of consultants and each consultant wants to work for a set of firms. Firms rank consultants according to their competences and consultants rank firms according to their subjective preferences.

There are many other examples of many-to-many markets, mainly related to labor markets. For example, Echenique, Oviedo (2006) mention markets for medical interns in the U.K. or teacher (university professor) markets in some countries where teachers (or professors) can work in more than one school (university). Another example from Echenique, Oviedo (2006) is a model of contracting between down-stream firms and up-stream providers.

In what follows we use basic notation and definitions from the paper of Świtalski (2016).

We consider Cartesian product $B \times S$ of the sets $B$ and $S$. For any relation $u \subset B \times S$ and for any $b \in B$, $s \in S$, we define the sets:

$$u(b) = \{s \in S: \ (b, s) \in u\}, \tag{1}$$

$$u(s) = \{b \in B: \ (b, s) \in u\}. \tag{2}$$

We assume that a non-empty set of acceptable (feasible) pairs $F \subset B \times S$ is defined. We interpret an acceptable pair $(b, s) \in F$ as a possible transaction which can be realized in the market or (from the point of view of contract theory of Hatfield et al., 2013), as a possible contract which can be signed by $b$ and $s$ (for example in the "consultants" market, $(b, s) \in F$ means that it is possible for a firm $b$ to hire a consultant $s$ and it is possible for $s$ to work for $b$). According to (1) and (2), the sets $F(b)$ and $F(s)$ for any $b \in B$ and $s \in S$ can be defined. The set $F(b)$ can be interpreted as the set of sellers which can sign a contract with $b$, and $F(s)$ – as the set of buyers which can sign a contract with $s$.

We assume that $b$ can sign many contracts with different sellers, but only one contract with a given seller $s$, and $s$ can sign many contracts with different buyers, but only one contract with a given buyer $b$ (hence we assume that the "unitarity" assumption is satisfied (see, e.g., Kominers, 2012).

In our model we introduce quotas for buyers and sellers. Let $q(b) \geq 1$ be the quota for $b$, which is interpreted as maximal number of contracts which $b$ can sign with different sellers and $q(s) \geq 1$ – the quota for $s$, which is interpreted as maximal number of contracts which $s$ can sign with different buyers. We assume that $\# F(b) \geq q(b)$ and $\# F(s) \geq q(s)$ ($\# A$ denotes cardinality of $A$).

Preferences of sellers are represented by weak orders. Namely, we assume that on every set $F(s)$, a weak order (transitive and complete relation) $\geq_s$ is defined (i.e. some buyers may be indifferent for the seller $s$). The symbols $>_s$ and $\approx_s$ denote the respective strict order and indifference relation. Hence the notation $b >_s c$ means that the buyer $b$ is better than the buyer $c$ for the seller $s$, and $b \approx_s c$ means that $b$ and $c$ are indifferent for $s$.

Preferences of buyers are represented by choice functions (standard theory of choice functions is described, e.g., by Aizerman, Aleskerov, 1995 or by Aleskerov, Monjardet, 2002, applications for matching markets can be found, e.g., in Echenique, 2007; Klaus, Walzl, 2009; Hatfield et al., 2013).

We assume that a choice function is defined for every feasible set $F(b)$ (for a given buyer $b$). This means that for every buyer $b$ and every set of feasible sellers $X \subset F(b)$, a set $C(b, X) \subset X$ is defined. The set $C(b, X)$ is interpreted in the following way. Assume that $b$ considers some set of feasible sellers $X$. Then his decision will be to choose the set $C(b, X)$ as the set of sellers, with which he will sign a contract. We consider only the so-called quota-filling choice functions (Alkan, Gale, 2003), i.e. we assume that:

$$\text{(i)} \qquad C(b, X) = X, \qquad \text{if} \quad \# X < q(b),$$

$$\text{(ii)} \quad \# C(b, X) = q(b), \qquad \text{if} \quad \# X \geq q(b).$$

We do not assume that the choice function of a buyer $b$ is generated by a linear order over the subsets of the set $F(b)$ (hence we follow the model of Alkan and Gale and do not follow standard approaches on many-to-many markets or contract theory as in, e.g., Echenique, Oviedo, 2006; Klaus,Walz, 2009; Kominers, 2012).

It is worth noting that any linear order $>_s$ on $F(s)$ (when there are no indifferences between different buyers) generates, in an obvious way, a quota-filling choice function on $F(s)$ (we take, for any $X \subset F(s)$, the set of $q(s)$ best buyers in $X$, or the set $X$, if $\# X < q(s)$).

We need the following properties of the function $C$ (interpretation and comments on these properties can be found in the paper of Świtalski, 2016):

**Definition 1.** A choice function $C$ satisfies the outcast property if for every $b \in B$ and $X, Y \subset F(b)$ we have

$$Y \subset X \setminus C(b, X) \quad \Rightarrow \quad C(b, X \setminus Y) = C(b, X). \tag{3}$$

**Definition 2.** A choice function $C$ satisfies the heritage property if for every $b \in B$ and $X, Y \subset F(b)$ we have

$$Y \subset X \quad \Rightarrow \quad Y \cap C(b, X) \subset C(b, Y). \tag{4}$$

We use here the classical terms (outcast and heritage) taken from the literature on choice theory (Aizerman, Aleskerov, 1995; Aleskerov, Monjardet, 2002). In the matching literature outcast property is known also under the name independence (see, e.g., Echenique, 2007) or consistency (Alkan, Gale, 2003) and heritage property under the name substitutability (Echenique, Oviedo, 2006) or persistency (Alkan, Gale, 2003).

Choice functions satisfying the outcast and heritage properties are called path independent (or Plott) choice functions (see Danilov, Koshevoy, 2005). Example of Plott choice function is the choice determined by some linear order (then $C(b, X)$ is the set of $q(b)$ best sellers in $X$, if $\# X \geq q(b)$, and $C(b, X) = X$ otherwise).

The next examples show that there exist quota-filling Plott choice functions with real-life interpretations which are not generated by any linear order.

**Example 1.** We fix a buyer $b \in B$. Let $q(b) = 2q$ be a fixed even number ($q \geq 1$). Assume that the set of sellers $S$ is divided into two disjoint subsets $Y$ and $Z$ (i.e. $S = Y \cup Z$ and $Y \cap Z = \varnothing$). For example $Y$ may be a set of men and $Y$ a set of women, or $Y$ – a set of statisticians and $Z$ – a set of computer scientists (in the set of consultants $S$). Assume that there is a (strict) linear order $M$ on $S$ and for any $X \subset F(b)$ and any $n \geq 1$ define

$$M(X, n) = \left| \begin{array}{l} \text{the set of } n \text{ best elements in } X \text{ with respect to } M, \quad \text{if} \quad \#X \geq n, \\ X, \quad \text{if} \quad \#X < n. \end{array} \right.$$

We want to take into account, when choosing the agents from $X$, the gender quotas (or quotas for consultants of specific professions). To this end we can define the following choice function:

$$C(b, X) = \left| \begin{array}{lll} M(X \cap Y, q) \cup M(X \cap Z, q), & \text{if } \# X \cap Y \geq q, & \# X \cap Z \geq q, \\ (X \cap Y) \cup M(X \cap Z, 2q - m), & \text{if } \# X \cap Y = m < q, & \# X \cap Z \geq q, \\ M(X \cap Y, 2q - m) \cup (X \cap Z), & \text{if } \# X \cap Y \geq q, & \# X \cap Z = m < q, \end{array} \right.$$

and $C(b, X) = X$ otherwise.

Hence we choose the $q$ best consultants from the set $X \cap Y$ and the $q$ best consultants from the set $X \cap Z$, or $m$ consultants from one of the sets and $2q - m$ (or less) best consultants from the second set (if $m < q$ and the first set contains only $m$ consultants) or the whole set of consultants if both sets ($X \cap Y$ and $X \cap Z$) contain less than $q$ consultants.

It is not difficult to prove that the defined above choice function satisfies both the outcast and heritage properties and hence is a Plott choice function.

**Example 2.** We assume now that there are two different linear orders $K$ and $L$ defined on the set $X$ (e.g. we order the set of consultants according to two criteria: creativity ($K$) and experience ($L$)). We want to have, when choosing the specialists from the set $X$, a balance between the number of creative members of the group and the number of experienced members. We define the following choice function (the symbols $K(X, q)$ and $L(X, q)$ are defined similarly as $M(X, n)$ in the previous example):

$$C(b, X) = \begin{cases} X, & \text{if} \quad \#X < 2q, \\ K(X, q) \cup L(X \setminus K(X, q), q), & \text{if} \quad \#X \geq 2q. \end{cases}$$

Hence, if we have at our disposal at least $2q$ consultants, we choose first the $q$ most creative ones and then the $q$ most experienced from the rest of the group. It is also easy to show that such choice function satisfies the Plott conditions (outcast and heritage).

We define a generalized GS-model as a 6-tuple $(B, S, F, C, P, q)$, where $F$ is the set of acceptable pairs, $C$ is the family of choice functions (defined for all $b \in B$), $P$ is the family of weak orders (defined for all $s \in S$), and $q$ is the vector of quotas (defined for all $b \in B$ and all $s \in S$).

In the next definitions we define matchings and (strongly) stable matchings for a generalized GS-model $(B, S, F, C, P, q)$ (see Świtalski, 2016). Our definition of stability is equivalent to the definition of Alkan, Gale (2003) and is often called pairwise stability in the matching literature (see, e.g., Echenique, Oviedo, 2006).

**Definition 3.** A relation $u \subset B \times S$ is a matching if

(i) $u \subset F$,

(ii) $\# u(b) \leq q(b), \ \forall b \in B$,

(iii) $\# u(s) \leq q(s), \ \forall s \in S$.

A matching $u$ can be interpreted as a set of actual transactions realized in the market (contrary to $F$ which can be treated as the set of potential (possible) transactions in the market).

**Definition 4.** Let $u \subset B \times S$ be a matching. We say that a seller $s \in F(b)$ improves the situation of a buyer $b \in F(s)$ (we write $s >_b u(b)$) if $s \in C\ (b, u(b) \cup \{s\})$.

**Definition 5.** Let $u \subset B \times S$ be a matching. We say that a buyer $b \in F(s)$ improves the situation (weakly improves the situation) of a seller $s \in F(b)$ (we write $b >_s u(s)$ or $b \geq_s u(s)$ respectively) if at least one of the following conditions hold:

(i) $\# \, u(s) < q(s)$,

(ii) $\exists \, c \in u(s), \; b >_s c \qquad (b \geq_s c)$.

**Definition 6.** A pair $(b, s) \in B \times S$ is a blocking pair (weakly blocking pair) for a matching $u \subset B \times S$ if

(i) $(b, s) \in F \setminus u$,

(ii) $s >_b u(b)$,

(iii) $b >_s u(s) \qquad (b \geq_s u(s))$.

**Definition 7.** A matching $u \subset B \times S$ is stable (strongly stable) if there are no blocking pairs (weakly blocking pairs) for $u$.

Now we start describing the notion of generalized equilibrium in our model. Generalized equilibria for the model $(B, S, F, C, P, q)$ are defined with the help of families of sets $W(s) \subset F(s)$ where the sets $W(s)$ are such that being in the set $W(s)$ is for $b \in F(s)$ a necessary condition to sign a contract with $s$ (for example a consultant $s$ determine some minimal conditions under which she can work for a firm $b$, the set of all firms satisfying such conditions will be denoted by $W(s)$). The feasible sets $F(s)$ are fixed, but the sets $W(s)$ can vary for the given model $(B, S, F, C, P, q)$. Special case of conditions $W(s)$ are the price conditions of the form $W(s) = \{b \in F(s){:}\ r(b, s) \geq p(s)\}$, where $p(s)$ is a price of a good offered by $s$ and $r(b, s)$ – maximal price at which $b$ can buy the good offered by $s$ (see Świtalski, 2016).

A family $W = \{W(s)\}$ $(s \in S)$ will be called a system of conditions. The set of feasible sellers for a buyer $b$ under the system $W = \{W(s)\}$ is defined as:

$$F(W, b) = \{s \in S{:}\ \ b \in W(s)\}.$$

$F(W, b)$ is the set of sellers $s$, such that $b$ can sign a contract with $s$ ($b$ satisfies the conditions $W(s)$ stated by $s$). Obviously, $F(W, b) \subset F(b)$, i.e., each seller feasible for $b$ under $W$ is acceptable for $b$.

We define also the set of "best" sellers (contracts) for $b$ under the system $W$ as

$$M(W, b) = C(b, F(W, b)),$$

and the demand set for the seller $s$ (under the conditions $W$) as

$$D(W, s) = \{b \in F(s){:}\ \ s \in M(W, b)\}.$$

Demand set $D(W, s)$ is the set of all buyers for which $s$ is among the "best" sellers.

Now we define generalized equilibrium for the model $(B, S, F, C, P, q)$ (see Świtalski, 2016).

**Definition 8.** A system of conditions $W = \{W(s)\}$ is an *equilibrium system* for the model $(B, S, F, C, P, q)$ if

(i) $\# D(W, s) \leq q(s)$, for all $s \in S$.
(ii) $W(s) = F(s)$, for all $s \in S$ such that $\# D(W, s) < q(s)$.

For an equilibrium system $W = \{W(s)\}$ we define a matching associated with $W$ as:

$$u(W) = \{(b, s) \in F\colon\ b \in D(W, s)\}.$$

**Definition 9.** An *equilibrium* (for a generalized GS-model $(B, S, F, C, P, q)$) is a pair $(u, W)$ such that $W$ is an equilibrium system and $u = u(W)$.

If $(u, W)$ is an equilibrium we will say that $u = u(W)$ is an equilibrium allocation associated with $W$.

In the paper of Świtalski (2016) some special kind of generalized equilibria have been defined, namely the so-called order equilibria. These are equilibria for which the conditions $W(s)$ are "order" conditions in the sense that if a buyer $b$ satisfies $W(s)$, then all the buyers better (not worse) than $b$ also satisfy $W(s)$ (so there is some kind of "compatibility" of $W(s)$ with the preferences $P$).

Formal definitions are the following:

**Definition 10.** A system of conditions $W = \{W(s)$ is compatible (strongly compatible) with the sellers' preferences if

$$b \in W(s) \ \wedge\ c >_s b \ \Rightarrow\ c \in W(s), \quad \text{for all } s \in S,$$

$$(b \in W(s) \ \wedge\ c \geq_s b \ \Rightarrow\ c \in W(s), \quad \text{for all } s \in S).$$

**Definition 11.** An equilibrium $(u, W)$ for a generalized GS-model $(B, S, F, C, P, q)$ is an *order equilibrium* (strongly order equilibrium) if $W$ is compatible (strongly compatible) with the sellers' preferences.

Let $(B, S, F, C, P, q)$ be a generalized GS-model. We will say that the family $C$ satisfies the outcast (heritage) property if all the choice functions in the family $C$ satisfy this property.

In Świtalski (2016) some relationships between stability and order equilibria were proved. Namely, it was proved that under the outcast property of $C$, order (strongly order) equilibria allocations are stable (strongly stable) and that under the heritage property, stable (strongly stable) matchings are order (strongly order) equilibria allocations. The exact formulations (see lemmas 1 and 2 in Świtalski, 2016) are the following:

**Theorem 1.** Let $M = (B, S, F, C, P, q)$ be a generalized GS-model such that $C$ satisfies the outcast property. Let $(u, W)$ be an order (strongly order) equilibrium for the model $M$. Then the matching $u$ is stable (strongly stable).

**Theorem 2.** Let $u$ be a stable (strongly stable) matching in a generalized GS-model $M = (B, S, F, C, P, q)$ such that $C$ satisfies the heritage property. Then there exists a system of conditions $W$ compatible (strongly compatible) with $P$ such that $(u, W)$ is an order (strongly order) equilibrium.

In section 4 we use theorems 1 and 2 to characterize price equilibria for the generalized GS-models with reservation prices (= maximal prices, at which buyers are ready to sign contracts with the sellers).

Models with reservation prices, price equilibria and relationships between price equilibria and order equilibria are described in the next section.

## 3. PRICE EQUILIBRIA AND ORDER EQUILIBRIA

In most market models competitive equilibria are defined as price equilibria. In Świtalski (2016, section 2) price equilibria were defined for the simplest (one-to-one) version of the Gale-Shapley model. To define such equilibria we have introduced reservation prices $r(b, s)$, where the number $r(b, s)$ is interpreted as maximal price at which buyer $b$ is willing to enter into the transaction with the seller $s$ (to sign a contract with $s$). Such equilibrium model is analogous to the model of equilibrium for matching markets with budgets described by Chen et al. (2014) (authors interpret the number $r(b, s)$ as budget which is at the disposal of $b$ when signing the contract with $s$).

If we assume that in the one-to-one Gale-Shapley model sellers' preferences are represented by linear orders and are determined by reservation prices, i.e., if the following condition is satisfied (for any $b, c \in B$ and $s \in S$):

$$b >_s c \quad \Leftrightarrow \quad r(b, s) > r(c, s), \tag{5}$$

then it can be proved that the respective price equilibria allocations are stable (in the sense of Gale, Shapley, 1962) and vice versa (see, Świtalski, 2016, theorem 1). Similar result, with the college admissions interpretation, was proved by Azevedo, Leshno (2011, "supply and demand lemma", p. 18).

We generalize (see theorem 6) this result to many-to-many GS-models described in section 2. We start now with describing many-to-many models with reservation prices and with defining price equilibria for such models.

Consider a generalized GS-model $(B, S, F, C, P, q)$. Assume that for every buyer $b$ and every seller $s$ such that $(b, s) \in F$, similarly as in the one-to-one model, a reservation price $r(b, s)$ is defined such that $b$ is ready to pay no more than $r(b, s)$, when signing the contract $(b, s)$.

A generalized GS-model with prices is defined as a 7-tuple $(B, S, F, C, P, q, r)$, where $r$ denotes the vector of reservation prices (for all $(b, s) \in F$).

Assume that every seller $s \in S$ announces some price $p(s) \geq 0$ interpreted as minimal price at which she is ready to sign contracts with the buyers. We define a price vector $p$ as a sequence $p = (p(s))$ $(s \in S)$ of prices announced by sellers.

We can define price conditions $W(p)(s)$ in the following way:

$$W(p)(s) = \{b \in F(s): \ \ r(b, s) \geq p(s)\}.$$

Inequality $r(b, s) \geq p(s)$ can be interpreted as some kind of "budget constraint" (similarly as in the neoclassical model of consumer choice). The set $W(p)(s)$ can be interpreted as the set of buyers, with which $s$ can sign a contract when prices in the market are $p$. Having defined the conditions $W(p)(s)$ we can easily define price equilibria for the model $(B, S, F, C, P, q, r)$ namely:

**Definition 12.** Let $M = (B, S, F, C, P, q, r)$ be a generalized GS-model with prices and $(u, W)$ an equilibrium for the model $(B, S, F, C, P, q)$ (according to definition 9). We say that $(u, W)$ is a price equilibrium for the model $M$ if there exists a price vector $p$ such that $W = W(p)$.

If $(u, W)$ is a price equilibrium and $W = W(p)$, we also use the notation $(u, W) = (u, W(p)) = (u, p)$.

To state the results on relationships between stability and price equilibria in many-to-many case (see theorems 3 and 4 below) we use three different conditions of compatibility of reservation prices $r$ with preferences $P$, one of which is equivalent to (5) and two others are weaker versions of (5).

A triple $(b, c, s)$ is called acceptable if $(b, s) \in F$ and $(c, s) \in F$.

**Definition 13.** We say that the prices $r$ are

(i) compatible with the preferences $P$ (shortly – COMP) if for all acceptable triples $(b, c, s)$ we have

$$b >_s c \quad \Rightarrow \quad r(b, s) \geq r(c, s), \tag{6}$$

(ii) strongly compatible with the preferences $P$ (shortly – SCOMP) if for all acceptable triples $(b, c, s)$ we have

$$b \geq_s c \quad \Rightarrow \quad r(b, s) \; \geq \; r(c, s), \tag{7}$$

(iii) very strongly compatible with the preferences $P$ (shortly – VSCOMP) if for all acceptable triples $(b, c, s)$ we have

$$b \geq_s c \quad \Leftrightarrow \quad r(b, s) \; \geq \; r(c, s). \tag{8}$$

It is easy to see that (8) is the strongest condition and (7) is stronger than (6). In other words, implications (8) ⇒ (7) ⇒ (6) are valid, although none of these implications can be reversed. It is also easy to check that (8) is equivalent to (5).

Conditions (6)–(8) mean that there is some relationship between reservation prices and the sellers' preferences. In the case of the strongest condition VSCOMP = (8) it means that preferences are determined by reservation prices similarly as in the one-to-one case (see (5)). Observe that the conditions (6)–(8) do not restrict the domain of possible preference orderings of the sellers, because for any weak order $\geq_s$ we can obviously find reservation prices $r$ satisfying VSCOMP (and hence COMP and SCOMP).

Using the conditions (6)–(8) we study now relationships between price equilibria and order (or strongly order) equilibria defined in section 2 (such relationships are necessary for transforming theorems 1 and 2 into theorems about price equilibria).

The following proposition shows that under the conditions of (strong) compatibility price equilibria are order (or strongly order) equilibria

**Proposition 1.** If prices $r$ in a model $M = (B, S, F, C, P, q, r)$ are (strongly) compatible with the preferences $P$, then any price equilibrium for $M$ is a (strongly) order equilibrium for $(B, S, F, C, P, q)$.

**Proof.** Assume that the prices $r$ are compatible with $P$. Then, for any acceptable triple $(b, c, s)$ we have:

$$b >_s c \quad \Rightarrow \quad r(b, s) \; \geq \; r(c, s). \tag{9}$$

We want to prove (by definition 10) that for any price vector $p = (p(s))$ and any acceptable triple $(b, c, s)$ we have:

$$r(c, s) \; \geq \; p(s) \quad \wedge \quad b >_s c \quad \Rightarrow \quad r(b, s) \; \geq \; p(s). \tag{10}$$

It is easy to see that (9) implies (10). The proof for strong compatibility and strong equilibrium is quite analogous (we change $b >_s c$ by $b \geq_s c$). □

Of course we can also ask the reverse question: whether every (strongly) order equilibrium is a price equilibrium. The following proposition shows that this is true under the condition of very strong compatibility (taking into account the previous proposition we can prove even "if and only if" statement in this case).

**Proposition 2.** If prices $r$ in a model $M = (B, S, F, C, P, q, r)$ are very strongly compatible with the preferences $P$, then an equilibrium $(u, W)$ is a price equilibrium for $M$ if and only if it is a strongly order equilibrium for $(B, S, F, C, P, q)$.

**Proof.** ($\Rightarrow$) Let $(u, W)$ be a price equilibrium for $M$. Very strong compatibility of $r$ implies strong compatibility of $r$, hence, by proposition 1, $(u, W)$ is a strongly order equilibrium for $(B, S, F, C, P, q)$.
($\Leftarrow$) Let $(u, W)$ be a strongly order equilibrium for $(B, S, F, C, P, q)$. By the definition of equilibrium system, all the sets $W(s)$ are non-empty. Let $c(s)$ be the worst buyer (one of the worst buyers) in the set $W(s)$. We define $p(s) = r(c(s), s)$. We want to prove that $(u, W)$ is a price equilibrium for $M$. It suffices to show that $W = W(p)$, where $p = (p(s))$ $(s \in S)$ or, equivalently, to show that $W(s) = W(p)(s)$ for all $s \in S$. To prove that $W(s) \subset W(p)(s)$, let $b \in W(s)$. Then $b \geq_s c(s)$ (because $c(s)$ is worst in $W(s)$). By (8), $r(b, s) \geq r(c(s), s) = p(s)$, hence $b \in W(p)(s)$.
To prove that $W(p)(s) \subset W(s)$, let $b \in W(p)(s)$. Then $b \in F(s)$ and $r(b, s) \geq p(s) = r(c(s), s)$. By (8), $b \geq_s c(s)$. We have $c(s) \in W(s)$ and $(u, W)$ is a strongly order equilibrium, hence, by definition 10, $b \in W(s)$. □

Propositions 1 and 2 will be used to prove the characterization results (theorems 3 and 4) in section 4.

Denote by PE – the condition of being a price equilibrium, by OE – the condition of being an order equilibrium, by SOE – the condition of being a strongly order equilibrium.

Taking into account that SOE $\Rightarrow$ OE, propositions 1 and 2 imply the following statements:

$$\text{COMP} \quad \Rightarrow \quad (\text{PE} \;\Rightarrow\; \text{OE}), \tag{11}$$

$$\text{SCOMP} \quad \Rightarrow \quad (\text{PE} \;\Rightarrow\; \text{SOE} \;\Rightarrow\; \text{OE}), \tag{12}$$

$$\text{VSCOMP} \quad \Rightarrow \quad (\text{PE} \;\Leftrightarrow\; \text{SOE} \;\Rightarrow\; \text{OE}). \tag{13}$$

The next examples show that the implications at the right side of the implications (11)–(13) cannot be reversed or that they cannot be made stronger (example 1 shows that under the assumptions COMP, SCOMP or VSCOMP, OE $\Rightarrow$ PE may be not true, example 2 shows that under COMP or SCOMP, SOE $\Rightarrow$ PE may be not true and example 3 shows that under COMP, PE $\Rightarrow$ SOE may be not true).

**Example 1.** Let $B = \{b, c\}$, $S = \{s\}$, $F = \{(b, s), (c, s)\}$, $C(b, \{s\}) = \{s\}$, $C(c, \{s\}) = \{s\}$, $b$ and $c$ are indifferent for $s$, all quotas are equal to 1, $r(b, s) = r(c, s) = 1$, $W(s) = \{b\}$. Then VSCOMP = (8) is satisfied, and hence COMP = (6) and also SCOMP = (7) (because VSCOMP $\Rightarrow$ SCOMP $\Rightarrow$ COMP). It is easy to see that $(u, W)$ is an order (but not strongly order) equilibrium with allocation $u = \{(b, s)\}$ and $(u, W)$ is not a price equilibrium (for any equilibrium prices $p$ we should have $W(p)(s) \neq \emptyset$, hence $W(p)(s) = \{b, c\} \neq W(s)$, because $r(b, s) = r(c, s) = 1$). Hence $(u, W)$ satisfies OE but not PE and so the implication OE $\Rightarrow$ PE is not true.

**Example 2.** Let $B = \{a, b, c\}$, $S = \{s, t, v\}$, $F = B \times S$ and all quotas are equal to 1. Preferences of the buyers and sellers are the following:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $a$: | $s$ | $t$ | $v$ | $s$: | $[a\ b]$ | $c$ | |
| $b$: | $t$ | $s$ | $v$ | $t$: | $a$ | $b$ | $c$ |
| $c$: | $s$ | $v$ | $t$ | $v$: | $c$ | $a$ | $b$ |

All preference orderings are strict (linear orders) except preferences for the seller $s$, for which the buyers $a$ and $b$ are indifferent (it is denoted by $[a\ b]$). The choice functions of the buyers are determined by the preference orderings in an obvious way (we choose the best seller from any non-empty set of sellers).

We consider the matching $u = \{(a, s), (b, t), (c, v)\}$. It is easy to see that $u$ is strongly stable (and hence stable). The only weakly blocking pair could be $(c, s)$, but this is impossible, because $a >_s c$. Hence, by theorem 2, there exists a strongly order equilibrium $(u, W)$ and, by the proof of theorem 2 (see Świtalski, 2016, proof of lemma 2), we can take $W(s) = \{a, b\}$, $W(t) = \{a, b\}$, $W(v) = \{c\}$. Assume that all reservation prices for $s$ are equal to 1. Hence the conditions COMP and SCOMP are satisfied. If $(u, p)$ would be a price equilibrium, then $W(p)(s) = \{a, b, c\}$ (because $W(p)(s) \neq \emptyset$). Hence $a, c \in D(W(p), s)$, and so $\# D(W(p), s) \geq 2$. Thus the system of conditions $W(p)$ would not be an equilibrium system (because $q(s) = 1$). Hence $(u, W)$ satisfies SOE, but not PE (there is no price equilibrium at all in this case) and the implication SOE $\Rightarrow$ PE is not valid.

**Example 3.** Consider once more example 1 but with reservation prices $r(b, s) = 2$, $r(c, s) = 1$. Hence the condition COMP is satisfied. Take $p(s) = 2$. Then $W(p)(s) = \{b\}$ and it is easy to see that $(u, W(p))$ is a price equilibrium with $u = \{(b, s)\}$. On the other hand it does not satisfy SOE (because $b \in W(p)(s)$, $c \notin W(p)(s)$ and $c \geq_s b$, by indifference of $b$ and $c$). Hence PE $\Rightarrow$ SOE is not true in this case.

## 4. PRICE EQUILIBRIA AND STABLE MATCHINGS

Now we state the main results of our paper, namely the results on relationships between stable matchings and price equilibria allocations for the models $(B, S, F, C, P, q, r)$. Proofs of these results will be based on theorems 1 and 2 and propositions 1 and 2. Firstly, using theorem 1, we prove that, under the assumptions of compatibility COMP (SCOMP), any price equilibrium allocation is stable (strongly stable).

**Theorem 3.** Let $M = (B, S, F, C, P, q, r)$ be a generalized GS-model with prices in which choice functions $C$ satisfy the outcast property and prices $r$ are compatible (strongly compatible) with the preferences $P$. Let $(u, p)$ be a price equilibrium for the model $M$. Then $u$ is stable (strongly stable).

**Proof.** If $(u, p)$ is a price equilibrium, then $(u, p) = (u, W(p))$. Prices are compatible (strongly compatible) with the preferences $P$, hence, by proposition 1, equilibrium $(u, W(p))$ is an order (strongly order) equilibrium. Hence, by theorem 1, $u$ is stable (strongly stable). □

Theorem 3 implies that under any compatibility condition (COMP, SCOMP, VSCOMP), in the generalized models with outcast choice functions, price equilibria allocations are stable or even (under SCOMP or VSCOMP) strongly stable.

Now we want to reverse theorem 3. Namely we want to state conditions under which any stable (or strongly stable) matching in a model $(B, S, F, C, P, q, r)$ is a price equilibrium allocation.

To this end, we could use theorem 2 which says that if $C$ satisfies the heritage property, then any stable (strongly stable) matching is an order (strongly order) equilibrium allocation. Unfortunately, neither COMP nor SCOMP nor VSCOMP conditions do not guarantee that the obtained order equilibrium allocation would be a price equilibrium allocation (as it is shown by example 1) and neither COMP nor SCOMP conditions do not guarantee that the obtained strongly order equilibrium allocation would be a price equilibrium allocation (example 2).

The only result we can obtain with the help of theorem 2, which could be interpreted as reversion of theorem 3 is the following.

**Theorem 4.** Let $M = (B, S, F, C, P, q, r)$ be a generalized GS-model with prices in which choice functions $C$ satisfy the heritage property and prices $r$ are very strongly compatible with the preferences $P$. Let $u$ be a strongly stable matching. Then there exist prices $p$, such that $(u, p)$ is a price equilibrium.

**Proof.** By theorem 2, there exist conditions $W$ such that $(u, W)$ is a strongly order equilibrium. By proposition 2, $(u, W)$ is a price equilibrium for $M$, hence there exist prices $p$ such that $(u, W) = (u, W(p)) = (u, p)$ and $(u, p)$ is obviously a price equilibrium. □

Example 2 shows that the assumption about very strong compatibility of $r$ is essential in theorem 4 (in this example $u$ is strongly stable and prices $r$ are strongly compatible with $P$, yet there is no price equilibrium of the form $(u, p)$).

Also the assumption about strong stability of $u$ is essential in theorem 4 (in the example 1 we have stable matching $u$, prices $r$ are very strongly compatible with $P$, yet there are no prices $p$ such that $(u, p)$ is a price equilibrium).

Combining theorems 3 and 4 we obtain the following result on equivalence of strongly stable matchings with price equilibria allocations in the generalized GS-models with prices.

**Theorem 5.** Let $M = (B, S, F, C, P, q, r)$ be a generalized GS-model with prices in which choice functions $C$ satisfy Plott condition and reservation prices $r$ are very strongly compatible with the preferences $P$. Then a matching $u$ is strongly stable if and only if it is a price equilibrium allocation.

Unfortunately, as the example 2 shows, there is no, in general, similar equivalence between stable matchings and price equilibria allocations. Yet, if we take a model with linear preferences (i.e. one in which preferences of the sellers are linear orders), then such equivalence is obvious, because stability = strong stability in this case. So we have the following result (a many-to-many generalization of theorem 1 in Świtalski, 2016):

**Theorem 6.** Let $M = (B, S, F, C, P, q, r)$ be a generalized GS-model with prices, with choice functions $C$ satisfying Plott condition, with linear preferences $P$ and with the reservation prices $r$ satisfying VSCOMP = (8) (equivalently (5)) condition.

Then a matching $u$ is stable if and only if it is a price equilibrium allocation.

Now we consider the problem of existence of equilibrium prices for the generalized GS-models with prices. It is easy to see that in general there can be models for which there are no equilibrium prices at all (as in the example 1). Yet, theorem 6 combined with some results of Alkan, Gale (2003) implies that for the models with linear preferences of the sellers we can prove the existence theorem, namely:

**Theorem 7.** Let $M = (B, S, F, C, P, q, r)$ be a generalized GS-model with prices, with choice functions $C$ satisfying Plott condition, with linear preferences $P$ and with the reservation prices $r$ satisfying VSCOMP condition. Then there exists a price equilibrium $(u, p)$ for the model $M$.

**Proof.** Using the results of Alkan, Gale (2003) we can prove, similarly as in the proof of theorem 3 in the paper of Świtalski (2016), that there exists a stable matching $u$ for the model $(B, S, F, C, P, q)$. Obviously $u$ is also stable for the model $M = (B, S, F, C, P, q, r)$ and, by theorem 6, it is a price equilibrium allocation. Hence, there exists a price equilibrium $(u, p)$ for the model $M$. □

Using theorem 7 we can also prove some existence result for a many-to-many variant of the model of Chen, Deng and Ghosh (CDG-model, 2014). Define generalized

CDG-model as a 6-tuple $(B, S, F, C, q, r)$ ($C$ – choice functions, $r$ – reservation prices = budgets in the terminology of Chen, Deng and Ghosh). Having reservation prices $r$ we can define preference relation $P(r, s)$ for a seller $s$ as:

$$b \ P(r, s) \ c \quad \Leftrightarrow \quad r(b, s) \ \geq \ r(c, s).$$

**Definition 14.** Let $M = (B, S, F, C, q, r)$ be a generalized CDG-model. We say that $r$ are differentiated reservation prices if for any $b, c \in B$ and any $s \in S$ we have

$$b \neq c \quad \Rightarrow \quad r(b, s) \ \neq \ r(c, s)$$

(i.e. any two different buyers have different reservation prices for the contract with the same seller).

Obviously reservation prices are differentiated if and only if all preferences $P(r, s)$ are linear orders. Hence if we have a CDG-model $(B, S, F, C, q, r)$ with differentiated $r$, then we can construct a generalized GS-model $(B, S, F, C, P(r), q, r)$ with $P(r)$ – family of linear orders $P(r, s)$. By theorem 7 we obtain the following result:

**Theorem 8.** Let $M = (B, S, F, C, q, r)$ be a generalized CDG-model with prices, with Plott choice functions $C$ and differentiated reservation prices $r$. Then there exists a price equilibrium $(u, p)$ for the model $M$.

## 5. CONCLUDING REMARKS

In our paper we have studied relationships between stable (strongly stable) matchings and price equilibria for generalized many-to-many Gale-Shapley market models with choice functions representing preferences of the buyers, weak orders representing preferences of the sellers and reservation prices of the buyers. We have shown that strongly stable matchings, under the assumptions of path independency of choice functions and very strong compatibility of reservation prices with the preferences of the sellers, are identical with price equilibria allocations. Unfortunately, in general, there is no similar characterization for stable matchings (in Świtalski, 2016 it is shown that stable matchings can be characterized by order equilibria). A special case in which such characterization for stable matchings is possible is the one with linear preferences of the sellers, because stability = strong stability in this case.

We have also shown how to use characterization results to prove existence of price equilibria for many-to-many GS-models with Plott (= path independent) choice functions and linear preferences of the sellers or for many-to-many Chen-Deng-Ghosh-models with Plott choice functions and differentiated reservation prices. Simple examples show that in the cases of non-linear preferences (for the GS-models) or non-differentiated prices (for the CDG-models) price equilibria may not exist.

We have used standard pairwise stability condition and this helped us to prove the existence result by using Alkan and Gale theory (2003). There are other stability conditions for many-to-many models (see, e.g. Echenique, Oviedo, 2006), but they are defined under the assumption that choice functions are generated by strict linear orders on the families of subsets of feasible contracts. It would be interesting to study relationships between stability and price equilibria for other stability concepts (different from pairwise stability) and for the model with general choice functions. But this could be the problem for further research.

A method of studying many-to-many model is the transforming of such model into equivalent many-to-one or one-to-many model by cloning the agents at one side of the market (under suitable assumptions). We did not follow this way, because we could directly use the results from our previous paper (Świtalski, 2016), but of course it would be interesting to study possibility of such transformation.

It would be very important for matching theory to have a model of matching market, in which we could prove relationships between stability and equilibria as general as possible. Our model cannot be embedded directly into contract theory, hence an interesting question would be the possibility of building a general model including our model and contract models (for example the model of Hatfield et al., 2013) for which similar results on stability and equilibria will hold.

## REFERENCES

Aizerman M., Aleskerov F., (1995), *Theory of Choice*, North-Holland, Amsterdam.

Aleskerov F., Monjardet B., (2002), *Utility Maximization, Choice and Preference*, Springer, Berlin.

Alkan A., Gale D., (2003), Stable Schedule Matching under Revealed Preference, *Journal of Economic Theory*, 112, 289–306.

Andersson T., Erlanson A., (2013), Multi-item Vickrey-English-Dutch Auctions, *Games and Economic Behavior*, 81, 116–129.

Azevedo E. M., Leshno J. D., (2011), A Supply and Demand Framework for Two-sided Matching Markets, *working paper*, *www.aeaweb.org* (last access: 28.07.2016).

Biro P., Kiselgof S., (2013), College Admissions with Stable Score-Limits, *discussion paper*, Institute of Economics, Research Centre for Economic and Regional Studies, Hungarian Academy of Sciences, Budapest.

Camina E., (2006), A Generalized Assignment Game. *Mathematical Social Sciences*, 52, 152–161.

Chen N., Deng X., Ghosh A., (2014), Competitive Equilibria in Matching Markets with Budgets, *www.arxiv.org* (last access: 28.07.2016).

Crawford V. P., Knoer E. M., (1981), Job Matching with Heterogeneous Firms and Workers, *Econometrica*, 49, 437–450.

Danilov V., Koshevoy G., (2005), Mathematics of Plott Choice Functions, *Mathematical Social Sciences*, 49, 245–272.

Echenique F., (2007), Counting Combinatorial Choice Rules, *Games and Economic Behavior*, 58, 231–245.

Echenique F., Oviedo J., (2006), A Theory of Stability in Many-to-Many Matching Markets, *Theoretical Economics*, 1, 233–273.

Gale D., (1960), *The Theory of Linear Economic Models*, The University of Chicago Press, Chicago.

Gale D., Shapley L. S., (1962), College Admissions and the Stability of Marriage, *American Mathematical Monthly*, 69, 9–15.

Hatfield J. W., Kominers S. D., (2016) Contract Design and Stability in Many-to-Many Matching, *Games and Economic Behavior* (in press).

Hatfield J. W., Kominers S. D., Nichifor A., Ostrovsky M., Westkamp A., (2013), Stability and Competitive Equilibrium in Trading Networks, *Journal of Political Economy*, 121 (5), 966–1005.

Herings P. J., (2015), Equilibrium and Matching under Price Control, *working paper* RM/15/001, Maastricht University, Graduate School of Business and Economics.

Klaus B., Walzl M., (2009), Stable Many-to-Many Matchings with Contracts, *Journal of Mathematical Economics*, 45, 422–434.

Kominers S. D., (2012), On the Correspondence of Contracts to Salaries in (Many-to-Many) Matching, *Games and Economic Behavior*, 75, 984–989.

Roth A. E., (2002), The Economist as Engineer: Game Theory, Experimentation, and Computation as Tools for Design Economics, *Econometrica*, 70, 1341–1378.

Roth A. E., Sotomayor M. A., (1992), *Two-Sided Matching. A Study in Game-Theoretic Modeling and Analysis*, Cambridge University Press.

Shapley L. S., Shubik M., (1971/72), The Assignment Game I: The Core, *International Journal of Game Theory*, 1, 111–130.

Sönmez T., Ünver M. U., (2011), Matching, Allocation and Exchange of Discrete Resources, in: Benhabib J., Bisin A., Jackson M. O., (eds.), *Handbook of Social Economics*, 781–852.

Sotomayor M., (2007), Connecting the Cooperative and Competitive Structures of the Multiple Partners Assignment Game, *Journal of Economic Theory*, 134, 155–174.

Świtalski Z., (2008), Stability and Equilibria in the Matching Models, *Scientific Research of the Institute of Mathematics and Computer Science, Częstochowa University of Technology*, 2 (7), 77–85.

Świtalski Z., (2010), Equilibria in Markets with Two-Sided Preferences (in Polish), *Przegląd Statystyczny*, 4 (57), 54–69.

Świtalski Z., (2015), Some Properties of Competitive Equilibria and Stable Matchings in a Gale-Shapley Market Model, *Studia Ekonomiczne*, *Scientific Papers of the University of Economics, Katowice*, 248, 222–232.

Świtalski Z., (2016), Stability and Generalized Competitive Equilibria in a Many-to-Many Gale-Shapley Market Model, *Przegląd Statystyczny*, 3 (63), 237–253.

## STABILNOŚĆ I RÓWNOWAGI CENOWE W MODELU RYNKU GALE'A-SHAPLEYA TYPU „MANY-TO-MANY"

### Streszczenie

W artykule zbadano zależności między uogólnionymi równowagami konkurencyjnymi zdefiniowanymi w pracy Świtalskiego (2016), a równowagami cenowymi dla pewnego wariantu modelu rynku Gale'a-Shapleya (typu „many-to-many"), a także między równowagami cenowymi a skojarzeniami stabilnymi dla tego modelu. Uzyskane wyniki wykorzystano do udowodnienia twierdzeń o istnieniu równowag cenowych w modelu GS typu many-to-many oraz w pewnym modelu typu many-to-many uogólniającym model zawarty w pracy Chen i inni (2014).

**Słowa kluczowe:** skojarzenie stabilne, teoria Gale'a-Shapleya, model „many-to-many", równowaga cenowa, funkcje wyboru, dyskretny model rynku

STABILITY AND PRICE EQUILIBRIA
IN A MANY-TO-MANY GALE-SHAPLEY MARKET MODEL

Abstract

In the paper we study relationships between generalized competitive equilibria defined in the paper of Świtalski (2016) and price equilibria for some variant of many-to-many market model of Gale-Shapley type and between price equilibria and stable matchings for such a model. Obtained results are used for proving theorems on existence of price equilibria in the many-to-many GS-model and in the many-to-many model generalizing the model of Chen, Deng and Ghosh (Chen et al., 2014).

**Keywords:** stable matching, Gale-Shapley theory, many-to-many model, price equilibrium, choice function, discrete market model