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Positive electrical circuits and their reachability*

TADEUSZ KACZOREK

Faculty of Electrical Engineering, Bialystok University of Technology
Wiejska 45D, 15-351 Białystok

e-mail: kaczorek@isep.pw.edu.pl

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Abstract: Conditions for the positivity of linear electrical circuits composed of resistances, coils, capacitors and voltage (current) sources are established. It is shown that: 1) the electrical circuit composed of resistors, coils and voltage source is positive for any values of their resistances, inductances and source voltages if and only if the number of coils is less or equal to the number of its linearly independent meshes, 2) the electrical circuit is not positive for any values of its resistances, capacitances and source voltages if each its branch contains resistor, capacitor and voltage source, 3) the positive *n*-meshes electrical circuit with only one inductance in each linearly independent mesh is reachable if all resistances of branches belonging to two linearly independent meshes are zero.

Key words: positivity, electrical linear circuit, conditions, reachability

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs [2, 5]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

The notion of controllability and observability and the decomposition of linear systems have been introduced by Kalman [8, 9]. These notions are the basic concepts of the modern control theory [1, 4, 7, 10, 11]. They have been also extended to positive linear systems [2, 5]. The decomposition of the pair (A, B) and (A, C) of the positive discrete-time linear system has been addressed in [3].

The reachability of linear systems is closely related to the controllability of the systems. Specially for positive linear systems the conditions for the controllability are much stronger

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than for the reachability [5]. Tests for the reachability and controllability of standard and positive linear systems are given in [5, 6].

In this paper the necessary and sufficient conditions for the positivity of linear electrical circuits composed of resistances, coils, capacitors and voltage (current) sources will be established. The conditions for the reachability of the positive electrical circuits will be also given.

The paper is organized as follows. In Section 2 the conditions for the positivity of *R*, *L*, *e* type electrical circuits are established. Similar problem for *R*, *C*, *e* type electrical circuits is discussed in Section 3. The reachability of the positive electrical circuits is analyzed in Section 4. Concluding remarks are given in Section 5.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. Positive *R*, *L*, *e* electrical circuits

Consider the linear continuous-time system described by the state equations

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t),\end{aligned} \tag{2.1}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

Definition 2.1. [2, 5] The system (2.1.) is called (internally) positive if for any $x(0) = x_0\in\Re_+^n$ and every $u(t)\in\Re_+^m$, $t\ge 0$ we have $x(t)\in\Re_+^n$ and $y(t)\in\Re_+^p$, $t\ge 0$.

Theorem 2.1. [2, 5] The system (1) is positive if and only if

$$A\in M_n,\quad B\in\Re_+^{n\times m},\quad C\in\Re_+^{p\times n},\quad D\in\Re_+^{p\times m}. \tag{2.2}$$

It is well-known [1, 4, 7, 10, 11] that any linear electrical circuit composed of resistances, inductances, capacitances and voltage (current) sources can be described by the state equations (2.1). Usually as the state variables $x_1(t),\ldots,x_n(t)$ (the components of the state vector $x(t)$) the currents in the coils and voltages on the capacitances are chosen.

Definition 2.2. An electrical circuit is called (internally) positive if its matrices *A*, *B*, *C*, *D* satisfy the conditions (2.2).

In this section the necessary and sufficient conditions will be established under which an electrical circuit is a positive one.

Example 2.1. Consider the electrical circuit shown on Figure 1 with given resistances R_k, $k=1,\ldots,8$, inductances L_1, L_2 and source voltages e_3, e_4, e_8.

Denote by i_k, $k=1,\ldots,4$ the mesh currents and by e_{kk}, $k=1,\ldots,4$ the mesh source voltages. Using the mesh method we obtain the following equations

$$L_1 \frac{di_1}{dt} = -R_{11} i_1 + R_{13} i_3 + R_{14} i_4 + e_{11}$$
$$L_2 \frac{di_2}{dt} = -R_{22} i_2 + R_{23} i_3 + R_{24} i_4, \tag{2.3a}$$

$$0 = R_{31} i_1 + R_{32} i_2 - R_{33} i_3 + e_{33}$$
$$0 = R_{41} i_1 + R_{42} i_2 - R_{44} i_4 + e_{44}\ , \tag{2.3b}$$

where

$$R_{11} = R_1 + R_3 + R_7,\quad R_{13} = R_{31} = R_7,\quad R_{14} = R_{41} = R_3,$$
$$R_{22} = R_2 + R_5 + R_6,\quad R_{23} = R_{32} = R_6,\quad R_{24} = R_{42} = R_5,$$
$$R_{33} = R_6 + R_7 + R_8,\quad R_{44} = R_3 + R_4 + R_5, \tag{2.3c}$$
$$e_{11} = e_3,\quad e_{33} = e_8,\quad e_{44} = e_4 - e_3.$$

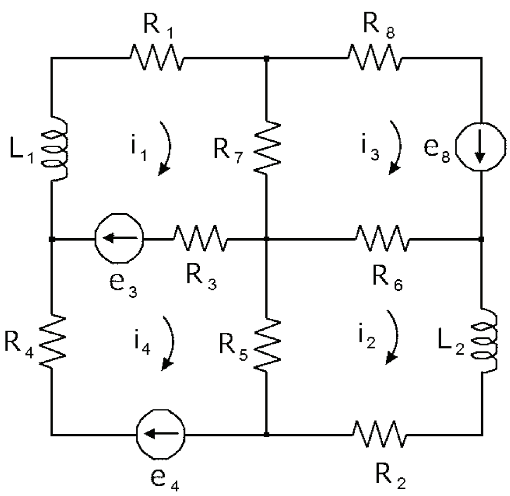

Fig. 1. Electrical circuit

From (2.3b) we have

$$i_3 = \frac{1}{R_{33}}(R_{31} i_1 + R_{32} i_2 + e_{33}),\quad i_4 = \frac{1}{R_{44}}(R_{41} i_1 + R_{42} i_2 + e_{44}). \tag{2.4}$$

Substitution of (2.4) into (2.3a) yields

$$L_1 \frac{di_1}{dt} = -R'_{11} i_1 + R'_{12} i_2 + e'_{11}$$
$$L_2 \frac{di_2}{dt} = R'_{21} i_1 - R'_{22} i_2 + e'_{22}\ , \tag{2.5a}$$

where

$$R'_{11} = R_{11} - \frac{R_{13}R_{31}}{R_{33}} - \frac{R_{14}R_{41}}{R_{44}}, \quad R'_{12} = R'_{21} = \frac{R_{13}R_{32}}{R_{33}} + \frac{R_{14}R_{42}}{R_{44}},$$

$$e'_{11} = e_{11} + \frac{R_{13}}{R_{33}}e_{33} + \frac{R_{14}}{R_{44}}e_{44}, \tag{2.5b}$$

$$R'_{22} = R_{22} - \frac{R_{23}R_{32}}{R_{33}} - \frac{R_{24}R_{42}}{R_{44}}, \quad e'_{22} = \frac{R_{23}}{R_{33}}e_{33} + \frac{R_{24}}{R_{44}}e_{44}.$$

If the mesh currents i_1, i_2 are chosen as the state vectors $x_1 = i_1$, $x_2 = i_2$ then the equations (2.5a) can be written in the form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B\begin{bmatrix} e'_{11} \\ e'_{22} \end{bmatrix}, \tag{2.6a}$$

where

$$A = \begin{bmatrix} -\frac{R'_{11}}{L_1} & \frac{R'_{12}}{L_1} \\ \frac{R'_{21}}{L_2} & -\frac{R'_{22}}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}. \tag{2.6b}$$

From (2.6b) and (2.5b) it follows that A is the Metzler matrix. If we choose

$$y_1 = R_1 i_1, \quad y_2 = R_8 i_3 - \frac{R_8}{R_{33}}e_{33} \tag{2.7}$$

as the output then

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} R_1 & 0 \\ \frac{R_8 R_{31}}{R_{33}} & \frac{R_8 R_{32}}{R_{33}} \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{2.8}$$

and

$$C = \begin{bmatrix} R_1 & 0 \\ \frac{R_8 R_{31}}{R_{33}} & \frac{R_8 R_{32}}{R_{33}} \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.9}$$

The matrices (2.6b) and (2.9) satisfy the condition (2.2) and by Definition 2.2 the electrical circuit is positive.

In general case let us consider the n-mesh electrical circuit with some given resistances R_k, $k = 1, ..., q$, inductances $L_1, ..., L_r$ and m-mesh source voltages e_{jj}, $j = 1, ..., m$. Let $i_1, ..., i_n$ be the mesh currents of the electrical circuit. Using the mesh method in a similar way as for the electrical circuit shown on Figure 1 we obtain the equation

$$\begin{bmatrix} \dot{x}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \tag{2.10a}$$

where

$$x_1 = \begin{bmatrix} i_1 \\ \vdots \\ i_r \end{bmatrix}, \quad x_2 = \begin{bmatrix} i_{r+1} \\ \vdots \\ i_n \end{bmatrix}, \quad u = \begin{bmatrix} e_{11} \\ \vdots \\ e_{mm} \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} -\frac{R_{11}}{L_1} & \frac{R_{12}}{L_1} & \cdots & \frac{R_{1,r}}{L_1} \\ \frac{R_{21}}{L_2} & -\frac{R_{22}}{L_2} & \cdots & \frac{R_{2,r}}{L_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{R_{r,1}}{L_r} & \frac{R_{r,2}}{L_r} & \cdots & -\frac{R_{rr}}{L_r} \end{bmatrix}, \quad A_{12} = \begin{bmatrix} \frac{R_{1,r+1}}{L_1} & \cdots & \frac{R_{1,n}}{L_1} \\ \vdots & \cdots & \vdots \\ \frac{R_{r,r+1}}{L_r} & \cdots & \frac{R_{rr}}{L_r} \end{bmatrix}, \tag{2.10b}$$

$$A_{21} = \begin{bmatrix} R_{r+1,1} & \cdots & R_{r+1,r} \\ \vdots & \cdots & \vdots \\ R_{n,1} & \cdots & R_{nr} \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -R_{r+1,r+1} & R_{r+1,r+2} & \cdots & R_{r+1,n} \\ R_{r+2,r+1} & -R_{r+2,r+2} & \cdots & R_{r+2,n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n,r+1} & R_{n,r+2} & \cdots & -R_{nn} \end{bmatrix},$$

$$R_{ji} = R_{ij} = \begin{cases} > 0 & \text{for} \quad i = j \\ \geq 0 & \text{for} \quad i \neq j \end{cases}. \tag{2.10c}$$

It is well-known [5] that $(-A_{22})^{-1} \in \Re_+^{(n-r)\times(n-r)}$ and from (2.10a) we have

$$x_2 = (-A_{22})^{-1}(A_{21}x_1 + B_2 u) \in \Re_+^{n-r} \quad \text{for} \quad x_1 \in \Re_+^r \quad \text{and} \quad u \in \Re_+^m. \tag{2.11}$$

Substituting (2.11) into

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1 u \tag{2.12}$$

we obtain

$$\dot{x}_1 = [A_{11} + A_{12}(-A_{22})^{-1}A_{21}]x_1 + [B_1 + A_{12}(-A_{22})^{-1}B_2]u = A'_{11}\,x_1 + B'_1\,u, \tag{2.13}$$

where

$$A'_{11} = A_{11} + A_{12}(-A_{22})^{-1}A_{21} \in M_n, \quad B'_1 = B_1 + A_{12}(-A_{22})^{-1}B_2 \in \Re_+^{n\times m}. \tag{2.14}$$

In what follows it is assumed that by suitable choice of the outputs of the electrical circuit the matrices C and D have nonnegative entries, i.e.

$$C \in \Re_+^{p\times n}, \quad D \in \Re_+^{p\times m}. \tag{2.15}$$

Therefore, the following theorem has been proved.

Theorem 2.2. The linear electrical circuit composed of resistors, coils and voltage sources is positive for any values of the resistances, inductances and source voltages if the number of coils is less or equal to the number of its linearly independent meshes and the direction of the mesh currents are consistent with the directions of the mesh source voltages.

Example 2.2. Consider the electrical circuit shown on Figure 2 with given resistances R_1, R_2, R_3, inductances L_1, L_2, L_3 and source voltages e_1, e_2.

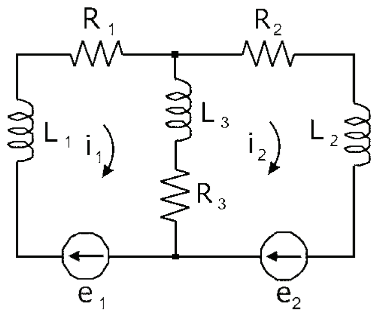

Fig. 2. Electrical circuit

Note that in this case the number of coils is greater than the number of linearly independent meshes. Using the mesh method we obtain the following equations

$$\begin{aligned} e_1 &= (R_1 + R_3)i_1 - R_3 i_2 + (L_1 + L_3)\frac{di_1}{dt} - L_3\frac{di_2}{dt} \\ e_2 &= -R_3 i_1 + (R_2 + R_3)i_2 + (L_2 + L_3)\frac{di_2}{dt} - L_3\frac{di_1}{dt}. \end{aligned} \tag{2.16}$$

The equations (2.16) can be written in the form

$$\begin{bmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{bmatrix} \frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -R_{11} & R_{12} \\ R_{21} & -R_{22} \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{2.17a}$$

where

$$\begin{aligned} &R_{11} = R_1 + R_3, \quad R_{12} = R_{21} = R_3, \quad R_{22} = R_2 + R_3, \\ &L_{11} = L_1 + L_3, \quad L_{12} = L_{21} = L_3, \quad L_{22} = L_2 + L_3. \end{aligned} \tag{2.17b}$$

Note that the inverse matrix

$$L^{-1} = \begin{bmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{bmatrix}^{-1} = \frac{1}{L_1(L_2 + L_3) + L_2 L_3}\begin{bmatrix} L_{22} & L_{12} \\ L_{21} & L_{11} \end{bmatrix} \tag{2.18}$$

has positive entries.
From (2.17) we have

$$\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{2.19}$$

where

$$A = L^{-1}\begin{bmatrix} -R_{11} & R_{12} \\ R_{21} & -R_{22} \end{bmatrix} =$$

$$= \frac{1}{L_1(L_2+L_3)+L_2L_3}\begin{bmatrix} -L_2(R_1+R_3)-L_3R_1 & L_2R_3-L_3R_2 \\ L_1R_3-L_3R_1 & -L_1(R_2+R_3)-L_3R_2 \end{bmatrix} \tag{2.20}$$

$$B = L^{-1} \in \Re_+^{2\times 2}.$$

Note that the matrix $A \in M_2$ if and only if

$$L_2R_3 \geq L_3R_2 \text{ and } L_1R_3 \geq L_3R_1 \tag{2.21}$$

Therefore, the electrical circuit is positive if and only if $A \in M_2$ i.e. the condition (2.21) is met.

In general case let us consider the n-mesh electrical circuit with given resistances R_k, $k = 1,\ldots,q$, inductances $L_1, \ldots, L_r$ for $r \geq n$ and $m \leq n$ mesh source voltages e_{jj}, $j = 1, \ldots, m$. Denote by $i_1, \ldots, i_n$ the mesh currents. In a similar way as for the electrical circuit shown on Figure 2 using the mesh method we obtain the equation

$$L\frac{d}{dt}\begin{bmatrix} i_1 \\ \vdots \\ i_n \end{bmatrix} = A'\begin{bmatrix} i_1 \\ \vdots \\ i_n \end{bmatrix} + \begin{bmatrix} e_{11} \\ \vdots \\ e_{mm} \end{bmatrix}, \tag{2.22a}$$

where

$$L = \begin{bmatrix} L_{11} & -L_{12} & \ldots & -L_{1,n} \\ -L_{21} & L_{22} & \ldots & -L_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ -L_{n,1} & -L_{n,2} & \ldots & L_{nn} \end{bmatrix}, \quad A' = \begin{bmatrix} -R_{11} & R_{12} & \ldots & R_{1,n} \\ R_{21} & -R_{22} & \ldots & R_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n,1} & R_{n,2} & \ldots & -R_{nn} \end{bmatrix}. \tag{2.22b}$$

Note that $-L \in M_n$, $A' \in M_n$ and $L^{-1} \in \Re_+^{n\times n}$.
Premultiplying (2.22a) by L^{-1} we obtain

$$\frac{d}{dt}\begin{bmatrix} i_1 \\ \vdots \\ i_n \end{bmatrix} = A\begin{bmatrix} i_1 \\ \vdots \\ i_n \end{bmatrix} + B\begin{bmatrix} e_{11} \\ \vdots \\ e_{mm} \end{bmatrix}, \tag{2.23a}$$

where

$$A = L^{-1}A', \quad B = L^{-1} \in \Re_+^{n\times n}, \tag{2.23b}$$

The electrical circuit is positive if and only if the matrix $L^{-1}A'$ is a Metzler matrix, i.e.

$$L^{-1}A' \in M_n. \tag{2.24}$$

By suitable choice of the outputs of the electrical circuit we have (2.15). Therefore, the following theorem has been proved.

Theorem 2.3. The linear electrical circuit composed of resistors, coils and voltage sources is positive for $r > n$ if its resistances and inductances satisfy the condition (2.24).

Remark 2.1. In the case $r = n$ if it is possible to choose the n linearly independent meshes so that the each mesh belongs only one coil. Then the matrix $L = \mathrm{diag}[L_1 \ \ \dots \ \ L_n]$ and the condition (2.24) is met for any values of the resistances and inductances of the electrical circuit.

Remark 2.2. Note that it is impossible to choose the n linearly independent meshes so that to each mesh belongs only one coil if all branches belonging to the same node contain the coils. In this case we can eliminate one of the branch currents using the fact that the sum of the currents in the coils is equal to zero.

From Theorem 2.2 and 2.3 and Remark 2.1 we have the following important theorem.

Theorem 2.4. The linear electrical circuit composed of resistors, coils and voltage sources is positive for any values of the resistances, inductances and source voltages if and only if the number of coils is less or equal to the number of its linearly independent meshes and the directions of the mesh currents are consistent with the directions of the mesh source voltages.

3. Positive *R*, *C*, *e* electrical circuits

Now let us consider the linear electrical circuits composed of resistances (conductances), capacitances and source voltages.

Example 3.1. Consider the electrical circuit shown on Figure 3 with given resistances R_1, R_2, R_3 (conductances $G_k = R_k^{-1}$, $k = 1, 2, 3$), capacitances C_1, C_2, C_3 and source voltages e_1, e_2, e_3.

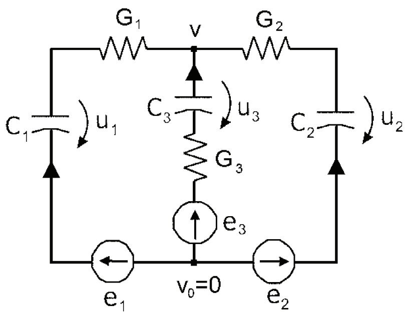

Fig. 3. Electrical circuit

Using the Kirchhoff's laws we may write the equations

$$C_1 \frac{du_1}{dt} = G_1(e_1 - u_1 - v)$$

$$C_2 \frac{du_2}{dt} = G_2(e_2 - u_2 - v) \tag{3.1a}$$

$$C_3 \frac{du_3}{dt} = G_3(e_3 - u_3 - v)$$

and

$$G_3(e_3 - u_3 - v) + G_1(e_1 - u_1 - v) + G_2(e_2 - u_2 - v) = 0. \tag{3.1b}$$

From (3.1b) we have

$$v = \frac{G_1(e_1 - u_1) + G_2(e_2 - u_2) + G_3(e_3 - u_3)}{G_1 + G_2 + G_3}. \tag{3.2}$$

Substitution of (3.2) into (3.1a) yields

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = A_3 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + B_3 \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \tag{3.3a}$$

where

$$A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B_3 = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \tag{3.3b}$$

and

$$\begin{aligned} a_{11} &= -\frac{G_1(G_2 + G_3)}{C_1(G_1 + G_2 + G_3)}, \quad a_{12} = \frac{G_1 G_2}{C_1(G_1 + G_2 + G_3)}, \quad a_{13} = \frac{G_1 G_3}{C_1(G_1 + G_2 + G_3)}, \\ a_{21} &= \frac{G_1 G_2}{C_2(G_1 + G_2 + G_3)}, \quad a_{22} = -\frac{G_2(G_1 + G_3)}{C_2(G_1 + G_2 + G_3)}, \quad a_{23} = \frac{G_2 G_3}{C_2(G_1 + G_2 + G_3)}, \\ a_{31} &= \frac{G_1 G_3}{C_3(G_1 + G_2 + G_3)}, \quad a_{32} = \frac{G_2 G_3}{C_3(G_1 + G_2 + G_3)}, \quad a_{33} = -\frac{G_3(G_1 + G_2)}{C_3(G_1 + G_2 + G_3)}, \end{aligned} \tag{3.4a}$$

$$\begin{aligned} b_{11} &= \frac{G_1(G_2 + G_3)}{C_1(G_1 + G_2 + G_3)}, \quad b_{12} = -\frac{G_1 G_2}{C_1(G_1 + G_2 + G_3)}, \quad b_{13} = -\frac{G_1 G_3}{C_1(G_1 + G_2 + G_3)}, \\ b_{21} &= -\frac{G_1 G_2}{C_2(G_1 + G_2 + G_3)}, \quad b_{22} = \frac{G_2(G_1 + G_3)}{C_2(G_1 + G_2 + G_3)}, \quad b_{23} = -\frac{G_2 G_3}{C_2(G_1 + G_2 + G_3)}, \\ b_{31} &= -\frac{G_1 G_3}{C_3(G_1 + G_2 + G_3)}, \quad b_{32} = -\frac{G_2 G_3}{C_3(G_1 + G_2 + G_3)}, \quad b_{33} = \frac{G_3(G_1 + G_2)}{C_3(G_1 + G_2 + G_3)}, \end{aligned} \tag{3.4b}$$

Remark 3.1. From (3.4) we have

$$1_3^T(\mathrm{diag}[C_1 \quad C_2 \quad C_3])A_3 = 0 \text{ and } 1_3^T(\mathrm{diag}[C_1 \quad C_2 \quad C_3])B_3 = 0, \tag{3.5}$$

where $1_3^T = [1 \quad 1 \quad 1]$.

The equations (3.5) follow from the first Kirchhoff's law

$$C_1 \frac{du_1}{dt} + C_2 \frac{du_2}{dt} + C_3 \frac{du_3}{dt} = 0. \tag{3.6}$$

From (3.3) and (3.4) it follows that the electrical circuit shown on Figure 3 is not positive for any nonzero its parameters C_k, G_k, $k = 1, 2, 3$ and source voltages e_k, $k = 1, 2, 3$.
The electrical circuit is positive if $C_3 = 0$ and $e_1 = e_2 = 0$. In this case we have

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A_2 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + B_2 e_3, \tag{3.7a}$$

where

$$A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_2 \quad \text{and} \quad B_2 = \begin{bmatrix} b_{13} \\ b_{23} \end{bmatrix}. \tag{3.7b}$$

In general case for the linear electrical circuits with n branches we have the following theorem.

Theorem 3.1. The linear electrical circuit is not positive for any values of its resistances, capacitances and source voltages if each its branch contains resistance, capacitance and source voltage.

Proof. From the first Kirchhoff's law we have (Remark 3.1)

$$1_n^T(\operatorname{diag}[C_1 \quad \dots \quad C_n])B = 0, \tag{3.8}$$

where C_i is the capacitance of ith ($i = 1, \dots, n$) branch. The equality (3.8) implies that some entries of the matrix B are negative. Therefore, the electrical circuit is not positive.

Consider the electrical circuit shown on Figure 4 with given resistances R_k, $k = 0, 1, \dots, n$, capacitances C_j, $j = 1, \dots, n$ and source voltages e.

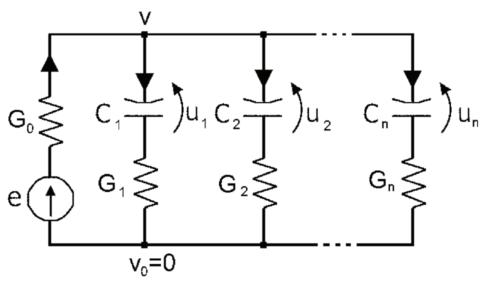

Fig. 4. Electrical circuit

Using the Kirchhoff's laws we may write the equations

$$G_0(e - v) = \sum_{j=1}^{n} G_j(v - u_j) \tag{3.9a}$$

and

$$C_k \frac{du_k}{dt} = G_k(v - u_k), \quad k = 1, \dots, n. \tag{3.9b}$$

From (3.9a) we have

$$v = \frac{1}{G}\left(G_0 e + \sum_{j=1}^{n} G_j u_j\right), \quad G = \sum_{i=0}^{n} G_i . \tag{3.10}$$

Substitution of (3.10) into (3.9b) yields

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = A\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + Be, \tag{3.11a}$$

where

$$A = \begin{bmatrix} -\frac{G_1 G - G_1^2}{C_1 G} & \frac{G_1 G_2}{C_1 G} & \cdots & \frac{G_1 G_n}{C_1 G} \\ \frac{G_2 G_1}{C_2 G} & -\frac{G_2 G - G_2^2}{C_2 G} & \cdots & \frac{G_2 G_n}{C_2 G} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{G_n G_1}{C_n G} & \frac{G_n G_2}{C_n G} & \cdots & -\frac{G_n G - G_n^2}{C_n G} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{G_0 G_1}{C_1 G} \\ \vdots \\ \frac{G_0 G_n}{C_n G} \end{bmatrix}. \tag{3.11b}$$

From (3.11) it follows that $A \in M_n$ and $B \in \Re_+^n$. Therefore, the electrical circuit is positive for all values of the resistances R_k, $k = 0, 1, ..., n$, all values of capacitances C_j, $j = 1, ..., n$ and source voltage e. The following theorem has been proved.

Theorem 3.2. The electrical circuit shown on Figure 4 is positive for any values of the resistances R_k, $k = 0, 1, ..., n$, capacitances C_j, $j = 1, ..., n$ and source voltage e.

Example 3.2. Consider the electrical circuit shown on Figure 5 with given conductances G_k, $k = 1, ..., 6$, capacitances C_1, C_2 and source voltages e_2, e_4.

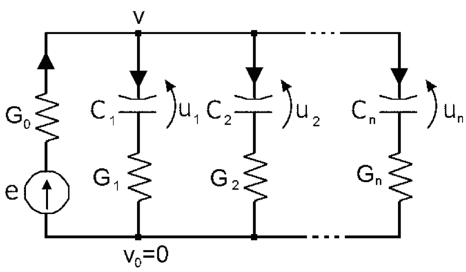

Fig. 5. Electrical circuit

Using the Kirchhoff's laws we may write the equations

$$\begin{aligned} C_1 \frac{du_1}{dt} &= G_1(v_1 - u_1) \\ C_2 \frac{du_2}{dt} &= G_2(e_2 - u_2 - v_3) \end{aligned} \tag{3.12a}$$

and

$$G_1u_1-(G_1+G_3+G_6)v_1+G_6v_2+G_3v_3=0$$
$$G_6v_1-(G_4+G_5+G_6)v_2+G_4v_3+G_4e_4=0 \qquad (3.12b)$$
$$-G_2u_2+G_3v_1+G_4v_2-(G_2+G_3+G_4)v_3+G_2e_2+G_4e_4=0.$$

The equations (3.12) can be rewritten in the form

$$\frac{d}{dt}\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}-\frac{G_1}{C_1} & 0\\ 0 & -\frac{G_2}{C_2}\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}+\begin{bmatrix}\frac{G_1}{C_1} & 0 & 0\\ 0 & 0 & -\frac{G_2}{C_2}\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}+\begin{bmatrix}0 & 0\\ \frac{G_2}{C_2} & 0\end{bmatrix}\begin{bmatrix}e_2\\e_4\end{bmatrix} \qquad (3.13a)$$

and

$$G\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=-\begin{bmatrix}G_1 & 0\\ 0 & 0\\ 0 & G_2\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}-\begin{bmatrix}0 & 0\\ 0 & G_4\\ G_2 & G_4\end{bmatrix}\begin{bmatrix}e_2\\e_4\end{bmatrix}, \qquad (3.13b)$$

where

$$G=\begin{bmatrix}-(G_1+G_3+G_6) & G_6 & G_3\\ G_6 & -(G_4+G_5+G_6) & G_4\\ G_3 & G_4 & -(G_2+G_3+G_4)\end{bmatrix}. \qquad (3.14)$$

The matrix G is the Metzler matrix and $-G^{-1}\in\Re_+^{3\times 3}$. From (3.13b) we have

$$\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=-G^{-1}\begin{bmatrix}G_1 & 0\\ 0 & 0\\ 0 & G_2\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}-G^{-1}\begin{bmatrix}0 & 0\\ 0 & G_4\\ G_2 & G_4\end{bmatrix}\begin{bmatrix}e_2\\e_4\end{bmatrix}. \qquad (3.15)$$

Substituting (3.15 into 3.13a we obtain)

$$\frac{d}{dt}\begin{bmatrix}u_1\\u_2\end{bmatrix}=A\begin{bmatrix}u_1\\u_2\end{bmatrix}+B\begin{bmatrix}e_2\\e_4\end{bmatrix}, \qquad (3.16)$$

where

$$A=\begin{bmatrix}-\frac{G_1}{C_1} & 0\\ 0 & -\frac{G_2}{C_2}\end{bmatrix}-\begin{bmatrix}\frac{G_1}{C_1} & 0 & 0\\ 0 & 0 & -\frac{G_2}{C_2}\end{bmatrix}G^{-1}\begin{bmatrix}G_1 & 0\\ 0 & 0\\ 0 & -G_2\end{bmatrix},$$
$$B=\begin{bmatrix}0 & 0\\ \frac{G_2}{C_2} & 0\end{bmatrix}-\begin{bmatrix}\frac{G_1}{C_1} & 0 & 0\\ 0 & 0 & -\frac{G_2}{C_2}\end{bmatrix}G^{-1}\begin{bmatrix}0 & 0\\ 0 & G_4\\ G_2 & G_4\end{bmatrix}. \qquad (3.17)$$

The electrical circuit is positive if and only if the matrix A is the Metzler matrix and the matrix B has nonnegative entries.

In general case let us consider the electrical circuit composed of q resistances R_k, $k = 1, ..., q$, (conductances G_k), r capacitances C_i, $i = 1, ..., r$ and m source voltages e_j, $j = 1, ..., m$. Let n be the number of linearly independent nodes of the electrical circuit and $n > r$.

In a similar way as in Example 3.2. we may write the equation

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} = A_r \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} + A_n \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + B_m \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}, \tag{3.18}$$

where u_i is the i-th ($i = 1, \ldots, m$) voltage on the capacitor, v_j is the voltage of the j-th node ($j = 1, \ldots, n$), $A_r \in \Re^{r\times r}$ is the diagonal Metzler matrix, $A_n \in \Re^{r\times n}$ and $B_m \in \Re^{r\times m}$.

Using the node method we obtain

$$G\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = -F\begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} - H\begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}, \tag{3.19}$$

where $G \in \Re^{n\times n}$ is a Metzler matrix $F \in \Re^{n\times r}$ and $H \in \Re^{n\times m}$.
Taking into account that $-G^{-1} \in \Re_+^{n\times n}$ from (3.19) we obtain

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = -G^{-1}F\begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} - G^{-1}H\begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}. \tag{3.20}$$

Substitution of (3.20) into (3.18) yields

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} = A\begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} + B\begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix}, \tag{3.21}$$

where

$$A = A_r - A_n G^{-1} F, \quad B = B_m - A_n G^{-1} H. \tag{3.22}$$

The electrical circuit described by the equation (3.16) is positive if and only if (under the assumption $C \in \Re_+^{p\times r}$, $D \in \Re_+^{p\times r}$) the matrix A is a Metzler matrix and the matrix B has nonnegative entries. Therefore, the following theorem has been proved.

Theorem 3.3. The linear electrical circuit composed of q resistors, r capacitors and m source voltages is positive if and only if $r < n$ and

$$A_r - A_n G^{-1} F \in M_r, \quad B_m - A_n G^{-1} H \in \Re_+^{r\times m} \tag{3.23}$$

4. Reachability of positive electrical circuits

Consider the positive electrical circuit described by the equations (2.1).

Definition 4.1. The positive electrical circuit (2.1) (or the positive pair (*A*, *B*)) is called reachable if for any given final state $x_f \in \Re_+^n$ there exists an input $u(t) \in \Re_+^m$, for $t \in [0, t_f]$ that steers the state of the circuit from zero initial state $x(0) = 0$ to the state x_f, i.e. $x(t_f) = x_f$.

A real square matrix is called monomial if each its row and each its column contains only one positive entry and the remaining entries are zero.

Theorem 4.1. The positive electrical circuit (2.1) is reachable if the matrix

$$R_f = \int_0^{t_f} e^{A\tau} BB^T e^{A^T \tau} d\tau, t_f > 0 \tag{4.1}$$

is monomial. The input that steers the state of the electrical circuit in time t_f from $x(0) = 0$ to the state x_f is given by the formula

$$u(t) = B^T e^{A^T (t_f - t)} R_f^{-1} x_f \quad \text{for} \quad t \in [0, t_f]. \tag{4.2}$$

The proof is given in [5].

Example 4.1. Consider the electrical circuit shown on Figure 6 with given resistances R_1, R_2, R_3, inductances L_1, L_2 and source voltages e_1, e_2.

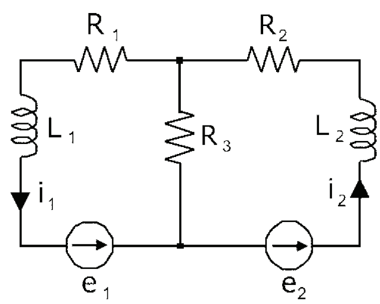

Fig. 6. Electrical circuit

Using the Kirchhoff's laws we can write the equations

$$\begin{aligned} e_1 &= R_3(i_1 - i_2) + R_1 i_1 + L_1 \frac{di_1}{dt} \\ e_2 &= R_3(i_2 - i_1) + R_2 i_2 + L_2 \frac{di_2}{dt}, \end{aligned} \tag{4.3}$$

which can be written in the form

$$\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{4.4a}$$

where

$$A = \begin{bmatrix} -\frac{R_1 + R_3}{L_1} & \frac{R_3}{L_1} \\ \frac{R_3}{L_2} & -\frac{R_2 + R_3}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}. \tag{4.4b}$$

The electrical circuit is positive since the matrix A is Metzler matrix and the matrix B has nonnegative entries.

We shall show that the circuit is reachable if $R_3 = 0$ and

$$A = \begin{bmatrix} -\frac{R_1}{L_1} & 0 \\ 0 & -\frac{R_2}{L_2} \end{bmatrix}. \tag{4.5}$$

Taking into account that

$$e^{A\tau} = \begin{bmatrix} e^{-\frac{R_1}{L_1}\tau} & 0 \\ 0 & e^{-\frac{R_2}{L_2}\tau} \end{bmatrix} \tag{4.6}$$

from (4.1) we obtain

$$R_f = \int_0^{t_f} e^{A\tau} B B^T e^{A^T \tau} d\tau = \int_0^{t_f} \begin{bmatrix} \frac{1}{L_1^2} e^{-\frac{2R_1}{L_1}\tau} & 0 \\ 0 & \frac{1}{L_2^2} e^{-\frac{2R_2}{L_2}\tau} \end{bmatrix} d\tau. \tag{4.7}$$

The matrix (4.7) is monomial and by Theorem 4.1 the electrical circuit is reachable if $R_3 = 0$.

Now let us consider the n-mesh electrical circuit with given resistances R_k, $k = 1, ..., q$, inductances L_i, $i = 1, ..., n$ and m-mesh source voltages e_{jj}, $j = 1, ..., n$. It is assumed that to each linearly independent mesh belongs only one inductance. In this case the matrix L defined by (2.22b) is diagonal one and the condition (2.24) is met.

Theorem 4.2. The positive n-meshes electrical circuit with only one inductance in each linearly independent mesh is reachable if

$$R_{ij} = 0 \quad \text{for} \quad i \neq j, \quad i, j = 1, ..., n \tag{4.8}$$

where R_{ij} are entries of the matrix A' defined by (2.22b).

Proof. If the condition (4.8) is met then the Metzler matrix A' is diagonal. The matrix L defined by (2.22b) is also diagonal since by assumption only one inductance belongs to each linearly independent mesh. In this case the matrix $A = L^{-1}A'$ is diagonal Metzler matrix and $B = L^{-1} \in \Re_+^{n\times n}$ is also diagonal. For diagonal Metzler matrix A and diagonal B the matrix

$e^{A\tau}B$ is also diagonal and the matrix R_f defined by (4.1) is monomial. By Theorem 4.1 the positive electrical circuit is reachable.

Remark 4.1. The condition (4.8) is met if the resistance of the branch belonging to two linearly independent meshes is zero. This result is consistent with the one obtained in Example 4.1.

Consider the electrical circuit shown on Figure 7 with given conductances G_k, k = 1, 2, 3; G_{12}, G_{13}, G_{23}, capacitances C_k, $k = 1, 2, 3$ and source voltages e_k, $k = 1, 2, 3$.

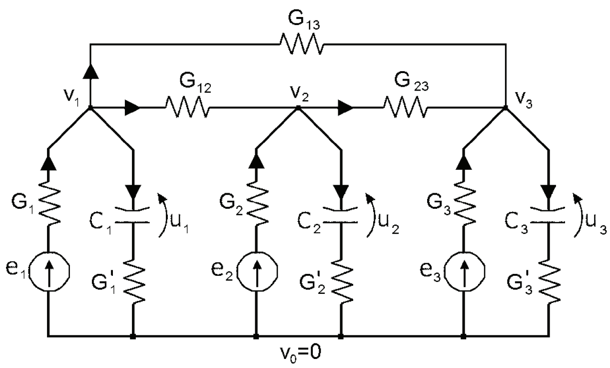

Fig. 7. Electrical circuit

Using the Kirchhoff's laws we may write the equations

$$C_k \frac{du_k}{dt} = G'_k (v_k - u_k), \quad k = 1, 2, 3 \tag{4.9}$$

and

$$G\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = -\begin{bmatrix} G'_1 & 0 & 0 \\ 0 & G'_2 & 0 \\ 0 & 0 & G'_3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & G_3 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \tag{4.10a}$$

where

$$G = \begin{bmatrix} -(G_1 + G'_1 + G_{12} + G_{13}) & G_{12} & G_{13} \\ G_{12} & -(G_2 + G'_2 + G_{12} + G_{23}) & G_{23} \\ G_{13} & G_{23} & -(G_3 + G'_3 + G_{13} + G_{23}) \end{bmatrix} \tag{4.10b}$$

is a asymptotically stable Metzler matrix.

Taking into account that $-G^{-1} \in \Re_+^{3\times 3}$ from (4.10a) we obtain

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = -G^{-1}\begin{bmatrix} G'_1 & 0 & 0 \\ 0 & G'_2 & 0 \\ 0 & 0 & G'_3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - G^{-1}\begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & G_3 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}. \tag{4.11}$$

From (4.9) we have

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -\frac{G'_1}{C_1} & 0 & 0 \\ 0 & -\frac{G'_2}{C_2} & 0 \\ 0 & 0 & -\frac{G'_3}{C_3} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} \frac{G'_1}{C_1} & 0 & 0 \\ 0 & \frac{G'_2}{C_2} & 0 \\ 0 & 0 & \frac{G'_3}{C_3} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}. \tag{4.12}$$

Substitution of (4.11) into (4.12) yields

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = A\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \tag{4.13}$$

where

$$A = \begin{bmatrix} -\frac{G'_1}{C_1} & 0 & 0 \\ 0 & -\frac{G'_2}{C_2} & 0 \\ 0 & 0 & -\frac{G'_3}{C_3} \end{bmatrix} - \begin{bmatrix} \frac{G'_1}{C_1} & 0 & 0 \\ 0 & \frac{G'_2}{C_2} & 0 \\ 0 & 0 & \frac{G'_3}{C_3} \end{bmatrix} G^{-1}\begin{bmatrix} G'_1 & 0 & 0 \\ 0 & G'_2 & 0 \\ 0 & 0 & G'_3 \end{bmatrix},$$

$$B = -\begin{bmatrix} \frac{G'_1}{C_1} & 0 & 0 \\ 0 & \frac{G'_2}{C_2} & 0 \\ 0 & 0 & \frac{G'_3}{C_3} \end{bmatrix} G^{-1}\begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & G_3 \end{bmatrix}. \tag{4.14}$$

From (4.14) it follows that the matrix A is a Metzler matrix and the matrix B has non-negative entries. Therefore, the electrical circuit is positive for all values of the conductances, capacitances and source voltages.

We shall show that the positive electrical circuit shown on Figure 7 is reachable if and only if

$$G_{12} = G_{13} = G_{23} = 0. \tag{4.15}$$

Note that the matrix G defined by (4.10b) is diagonal if and only if the equation (4.15) is met. In this case we have

$$-G^{-1} = \operatorname{diag}[(G_1 + G'_1)^{-1}, (G_2 + G'_2)^{-1}, (G_3 + G'_3)^{-1}] \tag{4.16}$$

and

$$\frac{du_k}{dt} = \left(\frac{G'^2_k}{C_k(G_k + G'_k)} - \frac{G'_k}{C_k}\right)u_k + \frac{G'_k G_k}{C_k(G_k + G'_k)}e_k, \quad k = 1, 2, 3. \tag{4.17}$$

Note that the subsystem (4.17) is reachable. Therefore, the positive electrical circuit is reachable if and only if the condition (4.15) is satisfied.

These considerations can be easily extended to the *n*-nodes electrical circuits. The considerations can be also extended for the controllability and observability of the electrical circuits.

5. Concluding remarks

The conditions for the positivity of linear electrical circuits composed of resistances, coils, capacitors and voltage (current) sources have been established. It has been shown that:

1) The electrical circuits composed of resistors coils and voltage source (shortly called *R*, *L*, *e* type) are positive for any values of their resistances, inductances and source voltages if and only if the number of coils is less or equal to the number of its linearly independent meshes (Theorem 2.4).
2) The electrical circuits composed of resistors, capacitors and voltage source (shortly called *R*, *C*, *e* type) are not positive for any values of its resistances, capacitances and voltage sources if each their branch contains resistor capacitor and voltage source (Theorem 3.1).
3) The positive *n*-meshes electrical circuits with only one coil in each linearly independent mesh are reachable if all resistances of branches belonging to two linearly independent meshes are zero.
4) The positive electrical circuit shown on Figure 7 are reachable if and only if the conductances between their nodes are zero (Condition (4.15)).

The consideration have been illustrated by examples of linear electrical circuits.

Some of these results can be also extended for the controllability and observability of the linear electrical circuit. Open problem are extension of these considerations for the following classes of the systems:

1) disturbed parameters linear systems,
2) nonlinear electrical circuits.

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