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A new approach to the realization problem for fractional discrete-time linear systems

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Abstract. A new approach to the realization problem for fractional discrete-time linear systems is proposed. A procedure for computation of fractional realizations of given transfer matrices is presented and illustrated by numerical examples.

Key words: fractional, linear, discrete-time, system, computation procedure, realization, transfer matrix.

1. Introduction

Determination of the state space equations for given transfer matrices is a classical problem, called the realization problem, which has been addressed in many papers and books [1–8]. An overview of the positive realization problem is given in [1, 2, 6, 9]. The realization problem for positive continuous-time and discrete-time linear system has been considered in [6, 7, 10–22] and for linear systems with delays in [6, 10, 15, 21–24]. The realization problem for fractional linear systems has been analyzed in [6, 7, 25–30] for positive 2D hybrid linear systems in [24, 31, 32] and for fractional systems with delays in [33, 34]. A new modified state variable diagram method for determination of positive realizations with reduced number of delays for given proper transfer matrices has been proposed in [35].

In this paper a new approach to the realization problem for fractional discrete-time linear systems will be proposed. The paper is organized as follows. Some preliminaries and problem formulation are given in Sec. 2. In Sec. 3 the solution to the realization problem for fractional discrete-time linear systems is presented and illustrated by numerical examples. Concluding remarks are given in Sec. 4.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re^{n\times m}(w)$ - the set of $n\times m$ rational matrices in w with real coefficients, Z_+ – the set of nonnegative integers, I_n – the $n\times n$ identity matrix.

2. Preliminaries and problem formulation

Consider the fractional discrete-time linear system

$$\Delta^\alpha x_i = Ax_i + Bu_i, \qquad i\in Z_+ = \{0,1,...\}, \tag{1a}$$

$$y_i = Cx_i + Du_i, \tag{1b}$$

where

$$\Delta^\alpha x_i = \sum_{j=0}^{i} c_j x_{i-j},$$

$$c_j = (-1)^j \begin{pmatrix}\alpha\\ j\end{pmatrix} \tag{1c}$$

$$= \begin{cases} 1 & \text{for} \quad j=0 \\ (-1)^j \frac{\alpha(\alpha-1)...(\alpha-j+1)}{j!} & \text{for} \quad j=1,2,... \end{cases},$$

$x_i\in\Re^n$, $u_i\in\Re^m$, $y_i\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

Using the Z-transformation to (1a) and (1b) for zero initial conditions we obtain [6]

$$Z[\Delta^\alpha x_i] = wX(z) = AX(z) + BU(z), \quad i\in Z_+ = \{0,1,...\} \tag{2a}$$

$$Y(z) = CX(z) + DU(z), \tag{2b}$$

where

$$\begin{gathered} Z[\Delta^\alpha x_i] = (1-z^{-1})^\alpha X(z) = w(z)X(z) = wX(z), \\ w = w(z) = (1-z^{-1})^\alpha = \sum_{i=0}^{\infty} c_i z^{-i}, \\ X(z) = Z[x_i] = \sum_{i=0}^{\infty} x_i z^{-i}, \\ U(z) = Z[u_i], \qquad Y(z) = Z[y_i]. \end{gathered} \tag{2c}$$

From (2) we have the transfer matrix

$$T(w) = C[I_n w - A]^{-1}B + D. \tag{3}$$

The transfer matrix $T(z)$ is called proper if and only if

$$\lim_{w\to\infty} T(w) = D \in \Re^{p\times m} \tag{4}$$

and it is called strictly proper if and only if $D=0$.

From (3) we have

$$\lim_{w\to\infty} T(w) = D \tag{5}$$

since $\lim_{w\to\infty} [I_n w - A]^{-1} = 0$.

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Definition 1. The matrices A, B, C, D are called a fractional realization of a given transfer matrix $T(w)$ if they satisfy the equality (3).

A fractional realization A, B, C, D is called minimal if the dimension of the matrix A is minimal among all realizations of $T(w)$.

The fractional realization problem can be stated as follows. Given a proper transfer matrix $T(w) \in \Re^{p\times m}(w)$ find a fractional realization A, B, C, $D$of the matrix $T(w)$.

3. Problem solution

3.1. Single-input single-output systems. First the essence of the proposed method is presented for single-input single-output (SISO) fractional discrete-time linear systems with the transfer function

$$T(w) = \frac{b_n w^n + b_{n-1}w^{n-1} + ... + b_1 w + b_0}{w^n + a_{n-1}w^{n-1} + ... + a_1 w + a_0}. \tag{6}$$

Using (4) for (6) we obtain

$$D = \lim_{w\to\infty} T(w) = b_n \tag{7}$$

and

$$T_{sp}(w) = T(w) - D = \frac{\overline{b}_{n-1}w^{n-1} + ... + \overline{b}_1 w + \overline{b}_0}{w^n + a_{n-1}w^{n-1} + ... + a_1 w + a_0}, \tag{8a}$$

where

$$\overline{b}_k = b_k - a_k b_n, \qquad k = 0, 1, ..., n-1. \tag{8b}$$

Therefore, the realization problem has been reduced to finding matrices $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$ for given strictly proper transfer function (8a).

Multiplying the numerator and the denominator of (8a) by w^{-n} we obtain

$$T_{sp}(w) = \frac{Y}{U} = \frac{\overline{b}_{n-1}w^{-1} + ... + \overline{b}_1 w^{1-n} + \overline{b}_0 w^{-n}}{1 + a_{n-1}w^{-1} + ... + a_1 w^{1-n} + a_0 w^{-n}}, \tag{9}$$

where Y and U are the Z-transforms of y_i and u_i, respectively.

Define

$$E = \frac{U}{1 + a_{n-1}w^{-1} + ... + a_1 w^{1-n} + a_0 w^{-n}}. \tag{10}$$

From (9) and (10) we have

$$E = U - (a_{n-1}w^{-1} + ... + a_1 w^{1-n} + a_0 w^{-n})E, \tag{11a}$$

$$Y = (\overline{b}_{n-1}w^{-1} + ... + \overline{b}_1 w^{1-n} + \overline{b}_0 w^{-n})E. \tag{11b}$$

From (11) follows the block diagram shown in Fig. 1.

Assuming as the state variables $x_{1,i}$, $x_{2,i}$,..., $x_{n,i}$ the outputs of the delay elements we may write the equations

$$\begin{aligned}
\Delta^\alpha x_{1,i} &= x_{2,i}, \\
\Delta^\alpha x_{2,i} &= x_{3,i}, \\
&\vdots \\
\Delta^\alpha x_{n-1,i} &= x_{n,i}, \\
\Delta^\alpha x_{n,i} &= -a_0 x_{1,i} - a_1 x_{2,i} - ... - a_{n-1}x_{n,i} + u_i
\end{aligned} \tag{12a}$$

and

$$y_i = \overline{b}_0 x_{1,i} + \overline{b}_1 x_{2,i} + ... + \overline{b}_{n-1}x_{n,i}. \tag{12b}$$

The Eq. (12) can be written in the form

$$\Delta^\alpha x_i = Ax_i + Bu_i, \tag{13a}$$

$$y_i = Cx_i, \tag{13b}$$

where

$$\begin{gathered}
x_i = [\ x_{1,i} \quad x_{2,i} \quad \dots \quad x_{n,i}\ ]^T, \qquad i \in Z_+, \\
A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \\
B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad C = [\ \overline{b}_0 \quad \overline{b}_1 \quad \cdots \quad \overline{b}_{n-1}\ ].
\end{gathered} \tag{14}$$

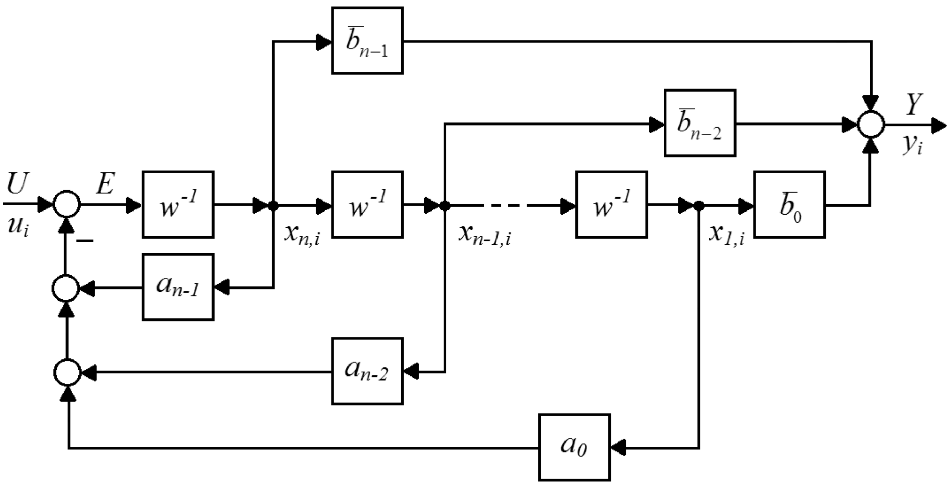

Fig. 1. State diagram for transfer function (9)

Remark 1. If we choose the state variables so that $x_k = x_{n-k+1}$ for $k = 1, ..., n$ then the realization of (8) has the form

$$A_1 = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad C_1 = [\ \overline{b}_{n-1} \quad \overline{b}_{n-2} \quad \cdots \quad \overline{b}_0\ ]. \tag{15}$$

Remark 2. Note that the transposition (denoted by T) of the transfer function does change it, i.e. $[T_{sp}(w)]^T = T_{sp}(w) = [C[I_n w - A]^{-1}B]^T = B^T[I_n w - A^T]^{-1}C^T$ and the matrices

$$A_2 = A^T = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}, \quad B_2 = C^T = \begin{bmatrix} \overline{b}_0 \\ \overline{b}_1 \\ \vdots \\ \overline{b}_{n-1} \end{bmatrix}, \quad C_2 = B^T = [\ 0 \quad \cdots \quad 0 \quad 1\ ] \tag{16}$$

and

$$A_3 = A_1^T = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B_3 = C_1^T = \begin{bmatrix} \overline{b}_{n-1} \\ \overline{b}_{n-2} \\ \vdots \\ \overline{b}_0 \end{bmatrix}, \quad C_3 = B_1^T = [\ 1 \quad 0 \quad \cdots \quad 0\ ] \tag{17}$$

are also the realizations of the transfer function (8).

Example 1. Find the fractional realization of the transfer function

$$T(w) = \frac{2w^2 + 11w + 10}{w^2 + 3w + 4}. \tag{18}$$

Using (7) we obtain

$$D = \lim_{w\to\infty} T(w) = 2 \tag{19}$$

and

$$T_{sp}(w) = T(w) - D = \frac{5w + 2}{w^2 + 3w + 4} = \frac{5w^{-1} + 2w^{-2}}{1 + 3w^{-1} + 4w^{-2}}. \tag{20}$$

In this case we have

$$E = \frac{U}{1 + 3w^{-1} + 4w^{-2}} \tag{21}$$

and

$$E = U - (3w^{-1} + 4w^{-2})E, \tag{22a}$$

$$Y = (5w^{-1} + 2w^{-2})E. \tag{22b}$$

The block diagram corresponding to (22) is shown in Fig. 2.

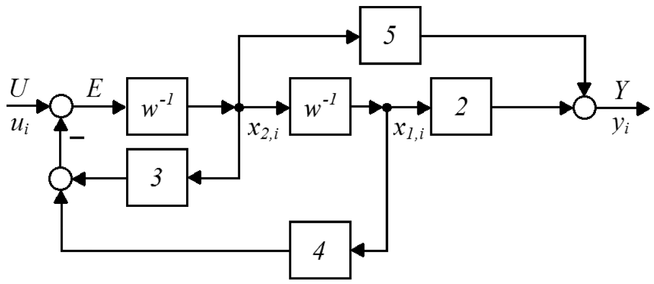

Fig. 2. State diagram for transfer function (20)

For the choice of the state variables shown in Fig. 2 we obtain the equations

$$\begin{aligned} \Delta^\alpha x_{1,i} &= x_{2,i}, \\ \Delta^\alpha x_{2,i} &= -4x_{1,i} - 3x_{2,i} + u, \end{aligned} \tag{23a}$$

$$y_i = 2x_{1,i} + 5x_{2,i} \tag{23b}$$

and the realization

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [\ 2 \quad 5\ ]. \tag{24}$$

3.2. Multi-input multi-output systems. Consider a strictly proper transfer matrix $T_{sp}(w) \in \Re^{p\times m}(w)$. Let

$$D_i(w) = w^{d_i} - (a_{i,d_i-1}w^{d_i-1} + ... + a_{i,1}w + a_{i,0}), \quad i = 1, ..., m \tag{25}$$

be the least common denominator of all entries of the i-th column of $T_{sp}(w)$.

Using (25) we may write $T_{sp}(w)$ in the form

$$T_{sp}(w) = \begin{bmatrix} \frac{N_{11}(w)}{D_1(w)} & \cdots & \frac{N_{1m}(w)}{D_m(w)} \\ \vdots & \ddots & \vdots \\ \frac{N_{p1}(w)}{D_1(w)} & \cdots & \frac{N_{pm}(w)}{D_m(w)} \end{bmatrix} = N(w)D^{-1}(w), \tag{26a}$$

where

$$N(w) = \begin{bmatrix} N_{11}(w) & \cdots & N_{1m}(w) \\ \vdots & \ddots & \vdots \\ N_{p1}(w) & \cdots & N_{pm}(w) \end{bmatrix}, \quad D(w) = \text{diag}[\ D_1(w) \quad \cdots \quad D_m(w)\ ]. \tag{26b}$$

From (25) it follows that

$$D(w) = \mathrm{diag}[\ w^{d_i} \quad \cdots \quad w^{d_m}\ ] - \overline{A}_m W, \tag{27a}$$

where

$$\overline{A}_m = \mathrm{blockdiag}[\ a_1 \quad \cdots \quad a_m\ ], \quad a_i = [\ a_{i,0} \quad \cdots \quad a_{i,d_i-1}\ ], \tag{27b}$$

$$W = \mathrm{blockdiag}[\ W_1 \quad \cdots \quad W_m\ ], \quad W_i = [\ 1 \quad w \quad \cdots \quad w^{d_i-1}\ ]. \tag{27c}$$

Note that if

$$N_{ij}(w) = c_{ij}^{d_j-1} w^{d_j-1} + \ldots + c_{ij}^1 w + c_{ij}^0, \tag{28a}$$

then

$$N(w) = CW, \tag{28b}$$

where

$$C = \begin{bmatrix} c_{11}^0 & c_{11}^1 & \cdots & c_{11}^{d_1-1} & \cdots & c_{1m}^0 & c_{1m}^1 & \cdots & c_{1m}^{d_m-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ c_{p1}^0 & c_{p1}^1 & \cdots & c_{p1}^{d_1-1} & \cdots & c_{pm}^0 & c_{pm}^1 & \cdots & c_{pm}^{d_m-1} \end{bmatrix}. \tag{28c}$$

We shall show that the matrices

$$A = \mathrm{blockdiag}[\ A_1 \quad \cdots \quad A_m\ ],$$

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_{i,0} & a_{i,1} & a_{i,2} & \cdots & a_{i,d_i-1} \end{bmatrix}, \quad i = 1, \ldots, m, \tag{29}$$

$$B = \mathrm{blockdiag}[\ b_1 \quad \cdots \quad b_m\ ],$$

$$b_i = [0 \quad \cdots \quad 0 \quad 1]^T \in \Re^{d_i}, \qquad i = 1, \ldots, m$$

and (28c) are the desired realization of (26).

Using (27) and (28) it is easy to verify that

$$b_i D_i(w) = [I_n w - A_i] \begin{bmatrix} 1 \\ w \\ \vdots \\ w^{d_i-1} \end{bmatrix} \tag{30}$$

and

$$BD(w) = [I_n w - A]W. \tag{31}$$

Premultiplication of (31) by $C[I_n w - A]^{-1}$ and postmultiplication by $D^{-1}(w)$ yields

$$C[I_n w - A]^{-1} B = CWD^{-1}(w) = N(w)D^{-1}(w) = T_{sp}(w). \tag{32}$$

Therefore, we have the following procedure for finding a fractional realization of a given proper transfer matrix $T(w)$.

Procedure 1.

Step 1. Using (4) find the matrix D and the strictly proper transfer matrix $T_{sp}(w)$.

Step 2. Find the least common denominators $D_1(w), \ldots, D_m(w)$ and write $T_{sp}(w)$ in the form (26).

Step 3. Knowing $D(w)$ find the indices $d_1, \ldots, d_m$ and the matrices W and $\overline{A}_m$.

Step 4. Knowing $N(w)$ find the matrix C defined by (28c).

Step 5. Using (29) find the matrices A and B.

Remark 3. Similar results can be obtained for the least common denominator of all entries of the j-th row of $T_{sp}(w)$.

Example 2. Find the fractional realization of the transfer matrix

$$T(w) = \begin{bmatrix} \dfrac{2w+1}{w} & \dfrac{w+3}{w+1} \\ \dfrac{3w+8}{w+2} & \dfrac{2w+5}{w+2} \end{bmatrix}. \tag{33}$$

Using Procedure 1 and (33) we obtain the following:

Step 1. Using (4) and (33) we obtain

$$D = \lim_{w\to\infty} T(w) = \lim_{w\to\infty} \begin{bmatrix} \dfrac{2w+1}{w} & \dfrac{w+3}{w+1} \\ \dfrac{3w+8}{w+2} & \dfrac{2w+5}{w+2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \tag{34}$$

and

$$T_{sp}(w) = T(w) - D = \begin{bmatrix} \dfrac{1}{w} & \dfrac{2}{w+1} \\ \dfrac{2}{w+2} & \dfrac{1}{w+2} \end{bmatrix}. \tag{35}$$

Step 2. From (35) we have $D_1(w) = w(w+2)$, $D_2(w) = (w+1)(w+2)$ and

$$T_{sp}(w) = N(w)D^{-1}(w), \tag{36a}$$

where

$$N(w) = \begin{bmatrix} w+2 & 2(w+2) \\ 2w & w+1 \end{bmatrix}, \quad D(w) = \begin{bmatrix} w(w+2) & 0 \\ 0 & (w+1)(w+2) \end{bmatrix}. \tag{36b}$$

Step 3. From (36b) we have $d_1 = d_2 = 2$ and

$$W = \begin{bmatrix} 1 & 0 \\ w & 0 \\ 0 & 1 \\ 0 & w \end{bmatrix}, \qquad \overline{A}_m = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & -3 \end{bmatrix} \tag{37}$$

since

$$D(w) = \begin{bmatrix} w(w+2) & 0 \\ 0 & (w+1)(w+2) \end{bmatrix} = \begin{bmatrix} w^2 & 0 \\ 0 & w^2 \end{bmatrix} - \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ w & 0 \\ 0 & 1 \\ 0 & w \end{bmatrix}. \tag{38}$$

Step 4. Using (36b) we obtain

$$N(w) = \begin{bmatrix} w+2 & 2w+4 \\ 2w & w+1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 & 2 \\ 0 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ w & 0 \\ 0 & 1 \\ 0 & w \end{bmatrix} = CW \tag{39a}$$

and

$$C = \begin{bmatrix} 2 & 1 & 4 & 2 \\ 0 & 2 & 1 & 1 \end{bmatrix}. \tag{39b}$$

Step 5. Using (29) and (37) we obtain

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{40}$$

The desired fractional realization of (33) is given by (40), (39b) and (34).

4. Concluding remarks

A new approach to finding fractional realizations of given transfer matrices of discrete-time linear systems has been proposed. It has been shown that for any given proper transfer matrix there exist always many fractional realizations. A procedure for computation a fractional realization of a given transfer matrix has been proposed. The effectiveness of the procedure has been demonstrated on numerical examples. The classical Gilbert method [29] can also be applied to compute the fractional realizations of the given transfer matrices of discrete-time linear systems.

The presented method can be easily extended to positive fractional linear discrete-time systems without and with delays.

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