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# Descriptor standard and positive discrete-time nonlinear systems

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A method of analysis of descriptor nonlinear discrete-time systems with regular pencils of linear part is proposed. The method is based on the Weierstrass-Kronecker decomposition of the pencils. Necessary and sufficient conditions for the positivity of the nonlinear systems are established. A procedure for computing the solution to the equations describing the nonlinear systems are proposed and demonstrated on numerical examples.

**Key words:** descriptor, Weierstrass-Kronecker decomposition, non-linear system, positivity.

## 1. Introduction

Descriptor (singular) linear systems have been considered in many papers and books [1-8, 16- 23]. The eigenvalues and invariants assignment by state and output feedbacks have been investigated in [4, 11, 20] and the minimum energy control of descriptor linear systems in [12, 14]. The computation of Kronecker's canonical form of singular pencil has been analyzed in [22]. The positive linear systems with different fractional orders have been addressed in [13]. Selected problems in theory of fractional linear systems has been given in monograph [19].

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in [15]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

Descriptor standard positive linear systems by the use of Drazin inverse has been addressed in [1-4, 10, 19, 20]. The shuffle algorithm has been applied to checking the positivity of descriptor linear systems in [10]. The stability of positive descriptor systems has

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been investigated in [23]. Reduction and decomposition of descriptor fractional discrete-time linear systems have been considered in [17]. A new class of descriptor fractional linear discrete-time systems has been introduced in [18].

In this paper a method of analysis of descriptor standard and positive nonlinear discrete-time systems with regular pencils will be proposed. The method is based on the Weierstrass-Kronecker decomposition of the pencil of the linear part of the equation describing the nonlinear system.

The paper is organized as follows. In section 2 the Weierstrass-Kronecker decomposition is applied to analysis of the descriptor nonlinear systems. Necessary and sufficient conditions for the positivity of the nonlinear systems is established in section 3. In section 4 the proposed procedure of finding the solution to the equations describing the nonlinear system is illustrated by numerical example. Concluding remarks are given in section 5.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $Z_+$ – the set of nonnegative integers, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re^{n\times 1}$, $I_n$ – the $n\times n$ identity matrix.

## 2. Descriptor nonlinear systems

Consider the descriptor discrete-time nonlinear system

$$Ex_{i+1} = Ax_i + f(x_i,u_i), \quad i\in Z_+ = \{0,1,\dots\}, \tag{1}$$

$$y_i = g(x_i,u_i), \tag{2}$$

where $x_i\in\Re^n$, $u_i\in\Re^m$, $y_i\in\Re^p$, $i\in Z_+$ are the state, input and output vectors, $f(x_i,u_i)\in\Re^n$, $g(x_i,u_i)\in\Re^p$ are continuous vector functions of $x_i$ and $u_i$ satisfying the conditions $f(0,0)=0$, $g(0,0)=0$ and $E,A\in\Re^{n\times n}$. It is assumed that $\det E=0$ and the

$$\det[Ez-A]\neq 0 \;\text{ for some }\; z\in C \quad \text{(the field of complex numbers)}. \tag{3}$$

It is well-known [20] that if (3) holds then there exist nonsingular matrices $P,Q\in\Re^{n\times n}$ such that

$$P[Ez-A]Q = \begin{bmatrix} I_{n_1}z-A_1 & 0 \\ 0 & Nz-I_{n_2}\end{bmatrix}, \; A_1\in\Re^{n_1\times n_1}, \;\; N\in\Re^{n_2\times n_2} \tag{4}$$

where $n_1=\deg\{\det[Ez-A]\}$, $n_2=n-n_1$ and $N$ is the nilpotent matrix with the index $\mu$, i.e. $N^{\mu-1}\neq 0$, $N^\mu=0$

The matrices $P$ and $Q$ can be computed using procedures given in [20, 22]. Premultiplying (1) by the matrix $P$ and introducing the new state vector we obtain

$$\bar{x}_i = \begin{bmatrix} \bar{x}_{1,i} \\ \bar{x}_{2,i}\end{bmatrix} = Q^{-1}x_i, \;\; \bar{x}_{1,i}\in\Re^{n_1}, \;\; \bar{x}_{2,i}\in\Re^{n_2}. \tag{5}$$

From (1) and (5) it follows

$$PEQQ^{-1}x_{i+1} = PAQQ^{-1}x_i + Pf(Q\bar{x}_i, u_i), \tag{6}$$

and

$$\bar{x}_{1,i+1} = A_1\bar{x}_{1,i} + \bar{f}_1(\bar{x}_i, u_i), \tag{7}$$

$$N\bar{x}_{2,i+1} = \bar{x}_{2,i} + \bar{f}_2(\bar{x}_i, u_i), \tag{8}$$

where

$$\begin{bmatrix} \bar{f}_1(\bar{x}_i, u_i) \\ \bar{f}_2(\bar{x}_i, u_i) \end{bmatrix} = Pf(Q\bar{x}_i, u_i). \tag{9}$$

To simplify the notation it is assumed that the nilpotent matrix contains only one block, i.e.

$$N = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \Re^{n_2 \times n_2}. \tag{10}$$

In this case the solution to the equation (1-2) for given initial conditions $x_0 \in \Re^n$ and input $u_i \in \Re^m$ for $i = 0, 1, \dots$ can be computed iteratively as follows.

From (8) and (10) for $i = 0$ we have

$$\begin{aligned} \bar{x}_{22,1} &= \bar{x}_{21,0} + f_{21}(\bar{x}_0, u_0) \\ \bar{x}_{23,1} &= \bar{x}_{22,0} + f_{22}(\bar{x}_0, u_0) \\ &\vdots \\ \bar{x}_{2n_2,1} &= \bar{x}_{2n_2-1,0} + f_{2n_2-1}(\bar{x}_0, u_0) \end{aligned} \tag{11}$$

$$\bar{x}_{2n_2,0} = -f_{2n_2}(\bar{x}_0, u_0) \tag{12}$$

where

$$\bar{x}_{2,i} = \begin{bmatrix} \bar{x}_{21,i} & \bar{x}_{22,i} & \dots & \bar{x}_{2n_2,i} \end{bmatrix}^T \tag{13}$$

$$f_2(\bar{x}_0, u_0) = \begin{bmatrix} f_{21}(\bar{x}_0, u_0) & f_{22}(\bar{x}_0, u_0) & \dots & f_{2n_2}(\bar{x}_0, u_0) \end{bmatrix}^T. \tag{14}$$

From (13) it follows that $\bar{x}_{21,1}$ can be chosen arbitrary and $\bar{x}_{2n_2,0}$ should satisfy the condition (12).

Next using (7) for $i = 0$ we have

$$\bar{x}_{1,1} = \begin{bmatrix} \bar{x}_{11,1} \\ \bar{x}_{12,1} \\ \vdots \\ \bar{x}_{1n_1,1} \end{bmatrix} = A_1\bar{x}_{1,0} + f_1(\bar{x}_0, u_0). \tag{15}$$

Knowing $\bar{x}_1$ we can compute from (8)

$$\begin{aligned} \bar{x}_{22,2} &= \bar{x}_{21,1} + f_{21}(\bar{x}_1, u_1) \\ \bar{x}_{23,2} &= \bar{x}_{22,1} + f_{22}(\bar{x}_1, u_1) \\ &\vdots \\ \bar{x}_{2n_2,2} &= \bar{x}_{2n_2-1,1} + f_{2n_2-1}(\bar{x}_1, u_1) \end{aligned} \tag{16}$$

$$\bar{x}_{2n_2,1} = -f_{2n_2}(\bar{x}_1, u_1) \tag{17}$$

and next from (7)

$$\bar{x}_{1,2} = \begin{bmatrix} \bar{x}_{21,1} \\ \bar{x}_{22,1} \\ \vdots \\ \bar{x}_{2n_1,1} \end{bmatrix} = A_1\bar{x}_{1,1} + f_1(\bar{x}_1, u_1). \tag{18}$$

Repeating the procedure we may compute the state vector $\bar{x}_i$ for $i = 0, 1, \ldots$ and next from the equality

$$x_i = Q\bar{x}_i \tag{19}$$

the desired solution $x_i$ of the equation (1).

## 3. Positive descriptor nonlinear systems

Consider the descriptor discrete-time nonlinear system (1-2).

**Definition 1** *The descriptor discrete-time nonlinear system described by the equations* (1-2) *is called positive if* $x_i \in \Re^n_+$, $y_i \in \Re^p_+$, $i \in Z_+$ *for any consistent initial conditions* $x_0 \in X_0 \in \Re^n_+$ *and all admissible inputs* $u_i \in U_a \in \Re^m_+$.

Note that for positive systems (1-2) $\bar{x}_i = Q^{-1}x_i \in \Re^n_+$ if and only if the matrix $Q \in \Re^{n\times n}_+$ is monomial. In this case $Q^{-1} \in \Re^{n\times n}_+$.

Note that for positive systems (7) $\bar{x}_i = Q^{-1}x_i \in \Re^n_+$ if and only if

$$A_1 \in \Re^{n_1\times n_1}_+ \text{ and } \bar{f}_1(\bar{x}_i, u_i) \in \Re^{n_1}_+ \text{ for all } \bar{x}_i \in \Re^n_+ \text{ and } u_i \in \Re^m_+,\ i = 0, 1, \ldots \tag{20}$$

From the structure of the matrix (10) and the equation (8) it follows that $\bar{x}_{2,i} \in \Re^{n_2}_+$, $i = 0, 1, \ldots$ if and only if

$$\bar{f}_{2,i}(\bar{x}_i, u_i) \in \Re^{n_2}_+ \text{ and } -\bar{f}_{2n_2}(\bar{x}_i, u_i) \in \Re^{n_2}_+ \text{ for all } \bar{x}_i \in \Re^n_+ \text{ and } u_i \in \Re^m_+,\ i = 0, 1, \ldots \tag{21}$$

The solution of the equations (7-9) $\bar{x}_i \in \Re^n_+$ if and only if the conditions (20) and (21) are satisfied. Therefore, the following theorem of the positivity of the system (1-2) has been proved.

**Theorem 9** *The descriptor nonlinear system* (1-2) *is positive if and only if the conditions* (20) *and* (21) *are satisfied and the matrix* $Q \in \Re_+^{n\times n}$ *is monomial.*

**Remark 3** If the nilpotent matrix $N$ consists of $q$ blocks then the condition (12) should be substituted by suitable $q$ conditions of each of the block.

**Remark 4** If the nilpotent matrix $N$ consists of $q$ blocks then for each of the blocks one state variable can be chosen arbitrarily.

## 4. Example

Consider the descriptor discrete-time nonlinear system (1-2).

$$E = \begin{bmatrix} 0 & 0 & 0.5 & -0.5 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.2 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.5 & -0.5 & 0 & 0 \\ 0.8 & 0 & 0.4 & -0.2 \\ 0.5 & 0.5 & 0 & 0 \\ 0.4 & 0 & 0.2 & 0.4 \end{bmatrix},$$

$$f(x_i, u_i) = \begin{bmatrix} 0.5x_{3,i}^2 - x_{2,i}^2 + e^{-i} - 0.5 \\ 0.2x_{1,i}^2 + 0.2e^{-i} + 0.4(1+i^2) \\ x_{2,i}^2 + 0.5x_{3,i}^2 + 0.5 \\ 0.2(1+i^2) - 0.4x_{1,i}^2 - 0.4e^{-i} \end{bmatrix}, \tag{22}$$

with the initial conditions

$$x_0 = \begin{bmatrix} x_{1,0} \\ x_{2,0} \\ x_{3,0} \\ x_{4,0} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}. \tag{23}$$

The assumption (3) is satisfied since

$$\det E = \begin{vmatrix} 0 & 0 & 0.5 & -0.5 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.2 & 0 & 0 & 0 \end{vmatrix} = 0 \tag{24}$$

and

$$\det[Ez-A] = \begin{vmatrix} -0.5 & 0.5 & 0.5z & -0.5z \\ 0.4z-0.8 & 0 & -0.4 & 0.2 \\ -0.5 & -0.5 & 0.5z & 0.5z \\ 0.2z-0.4 & 0 & -0.2 & -0.4 \end{vmatrix} = 0.1z^2 - 0.2z - 0.1 \neq 0. \quad (25)$$

In this case

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (26)$$

Using (4), (7-9) and (26) we obtain

$$P[Ez-A]Q = \begin{bmatrix} I_{n_1}z - A_1 & 0 \\ 0 & Nz - I_{n_2} \end{bmatrix}, \quad n_1 = n_2 = 2,$$
$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad (27)$$

$$\bar{x}_i = \begin{bmatrix} \bar{x}_{1,i} \\ \bar{x}_{2,i} \\ \bar{x}_{3,i} \\ \bar{x}_{4,i} \end{bmatrix} = Q^{-1}x_i = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ x_{3,i} \\ x_{4,i} \end{bmatrix} = \begin{bmatrix} x_{3,i} \\ x_{1,i} \\ x_{2,i} \\ x_{4,i} \end{bmatrix}, \quad (28)$$

$$Pf(\bar{x}_i, u_i) = \begin{bmatrix} f_1 Q(\bar{x}_i, u_i) \\ f_2 Q(\bar{x}_i, u_i) \end{bmatrix} = \begin{bmatrix} \bar{x}_{1,i}^2 + e^{-i} \\ 1 + i^2 \\ 2\bar{x}_{3,i}^2 - e^{-i} + 1 \\ -\bar{x}_{2,i}^2 - e^{-i} \end{bmatrix}, \quad (29)$$

and

$$\begin{bmatrix} \bar{x}_{1,i+1} \\ \bar{x}_{2,i+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \bar{x}_{1,i} \\ \bar{x}_{2,i} \end{bmatrix} + \begin{bmatrix} \bar{x}_{1,i}^2 + e^{-i} \\ 1 + i^2 \end{bmatrix}, \quad (30)$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{3,i+1} \\ \bar{x}_{4,i+1} \end{bmatrix} = \begin{bmatrix} \bar{x}_{3,i} \\ \bar{x}_{4,i} \end{bmatrix} + \begin{bmatrix} 2\bar{x}_{3,i}^2 - e^{-i} + 1 \\ -\bar{x}_{2,i}^2 - e^{-i} \end{bmatrix} \quad (31)$$

with the initial conditions

$$\bar{x}_0 = Q^{-1}x_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}. \tag{32}$$

The descriptor system (1-2) with (22-23) is a positive one since the conditions (20) and (21) are satisfied and the matrix $Q$ defined by (26) is monomial.

Using the procedure presented in section 3 we obtain the following:

From (31) for $i = 0$ we have

$$\bar{x}_{4,1} = \bar{x}_{3,0} + 2\bar{x}_{2,0}^2 - e^0 + 1 = 2, \tag{33}$$

and the condition (21) is satisfied since

$$\bar{x}_{4,0} = \bar{x}_{2,0}^2 + 1 = 2. \tag{34}$$

Using (30) for $i = 0$ and (32) we obtain

$$\begin{aligned} \bar{x}_{1,1} &= \bar{x}_{2,0} + \bar{x}_{1,0}^2 + 1 = 3, \\ \bar{x}_{2,1} &= \bar{x}_{1,0} + 2\bar{x}_{2,0} + 1 = 4 \end{aligned} \tag{35}$$

and from (31) for $i = 1$

$$\begin{aligned} \bar{x}_{4,2} &= \bar{x}_{3,1} + 2\bar{x}_{3,1}^2 - e^{-1} + 1, \\ \bar{x}_{4,1} &= \bar{x}_{2,1}^2 + e^{-1} \end{aligned} \tag{36}$$

for arbitrary $\bar{x}_{3,1} \geqslant 0$.

From (31) for $i = 1$ we have

$$\begin{aligned} \bar{x}_{1,2} &= \bar{x}_{2,1} + \bar{x}_{1,1}^2 + e^{-1}, \\ \bar{x}_{2,2} &= \bar{x}_{1,1} + 2\bar{x}_{2,1} + 2 \end{aligned} \tag{37}$$

and from (31) for $i = 2$

$$\begin{aligned} \bar{x}_{4,3} &= \bar{x}_{3,2} + 2\bar{x}_{3,2}^2 - e^{-2} + 1, \\ \bar{x}_{4,2} &= \bar{x}_{2,2}^2 + e^{-2} \end{aligned} \tag{38}$$

for arbitrary $\bar{x}_{3,2} \geqslant 0$.

Continuing the procedure we may compute the solution $\bar{x}_i$ of the equations (30) and (31) and next the solution

$$x_i = Q\bar{x}_i = \begin{bmatrix} \bar{x}_{2,i} \\ \bar{x}_{3,i} \\ \bar{x}_{1,i} \\ \bar{x}_{4,i} \end{bmatrix} \tag{39}$$

of the equation (1-2) with (22-23).

## 5. Concluding remarks

A method of analysis of descriptor nonlinear discrete-time systems described by the equation (1-2) with regular pencils (3) based on the Weierstrass-Kronecker decomposition of the pencil has been proposed. Necessary and sufficient conditions for the positivity of the nonlinear systems have been established (Theorem 1). A procedure for computing the solution to the equation (1-2) with given initial conditions and input sequences has been proposed. The procedure has been illustrated by numerical example. The considerations can be extended to fractional descriptor nonlinear discrete-time systems.

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