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Reduced order hybrid function projective combination synchronization of three Josephson junctions

KAYODE. S. OJO, ABDULAHI N. NJAH, SAMUEL T. OGUNJO and OLASUNKANMI I. OLUSOLA

In this paper, we examine reduced order hybrid function projective combination synchronization of three chaotic systems consisting of: (i) two third chaotic Josephson junctions as drives and one second order chaotic Josephson junction as response system; (ii) one third order chaotic Josephson junction as the drive and two second order chaotic Josephson junctions as the slaves using active backstepping technique. The analytic results confirm the realization of reduced order hybrid function projective combination synchronization using active backstepping technique. Numerical simulations are performed to validate the analytical results.

Key words: function projective synchronization, Josephson junction, backstepping control, chaos, reduced order synchronization, combination synchronization

1. Introduction

Chaotic systems exhibit sensitive dependence on initial conditions, hence, the convergence of the trajectory of two similar or different chaotic systems originating from different initial conditions seems improbable. Pecora and Caroll [11] pioneered the studies of synchronization of two chaotic systems in 1990. Since then many types of synchronization methods including lag synchronization [3], complete synchronization [11], phase synchronization [1], projective synchronization [7] have been proposed while many techniques have been by employed to design suitable control functions for synchronization of chaotic systems. These techniques include optimal control, active control, sliding mode control, impulsive control, backstepping control, etc.

Backstepping scheme has been efficient in the design technique for stabilization, tracking and synchronization of chaotic systems. According to Tian et al., [13], some of the advantages of the method include: applicability to a variety of chaotic systems irrespective of whether they contain external excitation or not; needs only one controller to realize synchronization of chaotic systems and finally there is no derivative in the

K.S. Ojo and O.I. Olusola are with Department of Physics, University of Agriculture, Abeokuta, Nigeria. K.S. Ojo and A.N. Njach is with Department of Physics, University of Lagos, Lagos, Nigeria. S.T. Ogunjo is with Department of Physics, Federal University of Technology, Akure, Ondo State, Nigeria. Corresponding Author is K.S. Ojo, e-mail: kaystephe@yahoo.com

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controller. Backstepping technique offers faster and better transient error dynamics convergence and synchronization time than the active control technique [8].

The idea of projective synchronization was proposed by Mainieri and Rehacek [7]. Projective Synchronization (PS)refers to the dynamical behavior in which the responses of two identical systems synchronize up to a constant scaling factor $\alpha \in \Re$ [14]. When $\alpha = 1$ we have complete synchronization and $\alpha = -1$ gives antisynchronization of the systems. Chen and Li [17] designed a new form of projective synchronization between two identical systems using a scaling function factor. This new scheme was extended to two different systems by Du et al [2]. Function projective synchronization in addition to faster communication, enhances security in communication due to the unpredictability in scaling function [16].

Hybrid function projective synchronization (HFPS) extends function projective synchronization by using scaling functions which consists of different time varying functions [4, 10]. When the scaling function is a matrix, a new form of function projective synchronization called generalized function projective synchronization is achieved [16]. This scheme has been implemented for chaotic systems with unknown parameters [5], fractional order [14], and other systems.

All of the synchronization schemes mentioned above and many others [8, 9] are between two similar or different chaotic systems. However, Runzi and Yinglan [12] have proposed a communication scheme (combination synchronization) whereby the signal to be transmitted is split between two drive systems so it can be transmitted at different intervals. If this is effective, the transmitted signal may have stronger anti-attack ability and antitranslated capability than that transmitted by the usual transmission model. Furthermore, in a communication network, there are many users (slave) but one control (master) which connects different users to one another. There is the need to implement a synchronization scheme whereby many users can be connected to and routed through a single master securely. Combination synchronization of three classic systems was achieved by Luo et al [6].

Although, reduced order and increased order synchronization have been implemented between two systems, research on reduced order combination synchronization are few. Increased order and reduced order combination synchronization of three different nonlinear systems was implemented using active backstepping design [15] but the reduced order hybrid function projective combination synchronization has not been implemented. This scheme has great prospect in secure communication and other allied fields. Hence, in this paper, we intend to design controllers to achieve reduced order hybrid function projective synchronization among three chaotic Josephson junction under two conditions: i) two third order chaotic Josephson junctions as drives and one second order chaotic Josephson junction as response system; (ii) one third order chaotic Josephson junction as the drive and two second order chaotic Josephson junctions as the slaves using active backstepping technique.

2. Reduced order hybrid function projective combination synchronization of two third order and one second order Josephson junctions

2.1. Design of controller via active backstepping technique

In this section, two third order Josephson junction in (1) and (2) are taken as the drive system while one second order non-autonomous Josephson junction (3) is taken as the response system in order to achieve generalized reduced order combination synchronization among the three chaotic Josephson junctions.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3) \\ \dot{x}_3 &= \frac{1}{\beta_L}(x_2 - x_3) \end{aligned} \tag{1}$$

the second drive system is

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= \frac{1}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3) \\ \dot{x}_3 &= \frac{1}{\beta_L}(y_2 - y_3) \end{aligned} \tag{2}$$

while the response system is given as

$$\begin{aligned} \dot{z}_1 &= z_2 + u_1 \\ \dot{z}_2 &= -\alpha z_2 - \sin z_1 + a + b\sin \omega t + u_2 \end{aligned} \tag{3}$$

where $u_i(t)$, $i = 1,2$ are the controllers to be designed. We define the error systems as follows

$$\begin{aligned} e_1 &= z_1 - (\alpha_1(t)x_1 + \beta_1(t)y_1 + \alpha_3(t)x_3 + \beta_3(t)y_3) \\ e_2 &= z_2 + (\alpha_2(t)x_2 + \beta_2(t)y_2). \end{aligned} \tag{4}$$

Using the error systems defined in (4) with systems defined in (1), (2) and (3) yields the following error dynamics

$$\dot{e}_1 = z_2 - \alpha_1(t)x_2 - \beta_1(t)y_2 - \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) - \frac{\beta_3(t)}{\beta_L}(y_2 - y_3)$$

$$-\dot{\alpha}_1(t)x_1 - \dot{\beta}_1(t)y_1 - \dot{\alpha}_3(t)x_3 - \dot{\beta}_3(t)y_3 + u_1$$

$$= e_2 - \alpha_2(t)x_2 - \beta_2(t)y_2 - \alpha_1(t)x_2 - \beta_1(t)y_2 - \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) - \frac{\beta_3(t)}{\beta_L}(y_2 - y_3) + u_1$$

$$-\dot{\alpha}_1(t)x_1 - \dot{\beta}_1(t)y_1 - \dot{\alpha}_3(t)x_3 - \dot{\beta}_3(t)y_3$$

$$\dot{e}_2 = -\alpha z_2 - \sin z_1 + a + b\sin\omega t + u_2 + \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3)$$

$$+\frac{\beta_2(t)}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3) + \dot{\alpha}_2(t)x_2 + \dot{\beta}_2(t)$$

$$= -\alpha(e_2 - \alpha_2(t)x_2 - \beta_2(t)y_2) + \dot{\alpha}_2(t)x_2 + \dot{\beta}_2(t) - \sin z_1 + a + b\sin\omega t + u_2$$

$$+\frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3) + \frac{\beta_2(t)}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3).$$

Thus,the error dynamics of the system can be written as:

$$\dot{e}_1 = e_2 + u_1 + A_1 \tag{5}$$

$$\dot{e}_2 = -\alpha e_2 + u_2 + A_2 \tag{6}$$

where

$$A_1 = -\alpha_2(t)x_2 - \beta_2(t)y_2 - \alpha_1(t)x_2 - \beta_1(t)y_2 - \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) - \frac{\beta_3(t)}{\beta_L}(y_2 - y_3)$$

$$-\dot{\alpha}_1(t)x_1 - \dot{\beta}_1(t)y_1 - \dot{\alpha}_3(t)x_3 - \dot{\beta}_3(t)y_3$$

$$A_2 = \alpha(\alpha_2(t)x_2 + \beta_2(t)y_2) - \sin z_1 + a + b\sin\omega t + \dot{\alpha}_2(t)x_2 + \dot{\beta}_2(t)$$

$$+\frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3) + \frac{\beta_2(t)}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3).$$

Our goal is to find the control functions which will enable the systems (1), (2) and (3) realize reduced order hybrid function projective combination synchronization by active backstepping technique. The design procedure includes three steps as shown below:

Step 1

Let $q_1 = e_1$, its time derivative is

$$\dot{q}_1 = \dot{e}_1 = e_2 + u_1 + A_1 \tag{7}$$

where $e_2 = \alpha_1(q_1)$ can be regarded as virtual controller. In order to stabilize q_1-subsystem, we choose the Lyapunov function $v_1 = \frac{1}{2}q_1^2$. The time derivative of v_1 is

$$\dot{v}_1 = q_1\dot{q}_1 = q_1(\alpha_1(q_1) + u_1 + A_1). \tag{8}$$

Suppose $\alpha_1(q_1) = 0$ and the control function u_1 is chosen as

$$u_1 = -(A_1 + kq_1) \tag{9}$$

then $\dot{v}_1 = -kq_1^2 < 0$ where k is positive constant which represent the feedback gain. Then, $\dot{v}_1$ is negative definite and the subsystem q_1 is asymptotically stable. Since, the virtual controller $\alpha_1(q_1)$ is estimative, the error between e_2 and $\alpha_1(q_1)$ can be denoted by $q_2 = e_2 - \alpha_1(q_1)$. Thus, we have the (q_1, q_2)-subsystems

$$\begin{aligned} \dot{q}_1 &= q_2 - kq_1 \\ \dot{q}_2 &= -\alpha q_2 + u_2 + A_2. \end{aligned} \tag{10}$$

Step 2
In order to stabilize subsystem (10), a Lyapunov function is chosen as $v_2 = v_1 + \frac{1}{2}q_2^2$. Its time derivative of v_2 is

$$\dot{v}_2 = -q_1^2 + q_2(q_1 - \alpha q_2 + u_2 + A_2). \tag{11}$$

If the control function u_2 is chosen as

$$u_2 = \alpha q_2 - q_1 - A_2 - kq_2 \tag{12}$$

then $\dot{v}_2 = -kq_1^2 - kq_2^2 < 0$ where k is a positive constant which represents the feedback gain. Hence, $\dot{v}_2$ is negative definite and the subsystem (q_1, q_2) in (10) is asymptotically stable. This implies that reduced order hybrid function projective combination synchronization of the drive systems (1) and (2) and the response system (3) is achieved. Finally, we have the subsystems

$$\begin{aligned} \dot{q}_1 &= q_2 - kq_1 \\ \dot{q}_2 &= -q_1 - kq_2. \end{aligned} \tag{13}$$

Let $\alpha_1 = \alpha_2 = \alpha_3 = 0$ then, we have Corollary 1.

Corollary 1 *If the controllers are chosen as*

$$\begin{aligned} u_1 &= (\beta_1(t) + \beta_2(t))y_2 + \frac{\beta_3(t)}{\beta_L}(y_2 - y_3) + \dot{\beta}_1(t)y_1 + \dot{\beta}_3(t)y_3 - kq_1 \\ u_2 &= (\alpha - k)q_2 - q_1 - \alpha\beta_2(t)y_2 + \sin z_1 - a - b\sin\omega t \\ &\quad - \frac{\beta_2(t)}{\beta_C}(i - g(y_2)y_2 - \sin y_1 - y_3) - \dot{\beta}_2(t)y_2 \end{aligned} \tag{14}$$

where $q_1 = z_1 - \beta_1(t)y_1 - \beta_3(t)y_3$, $q_2 = z_2 + \beta_2(t)y_2$ *then the drive system* (2) *achieve reduced order modified hybrid function projective synchronization with the response system* (3).

Let $\beta_1(t) = \beta_2(t) = \beta_3(t) = 0$, then we obtain Corollary 2.

Corollary 2 *If the controllers are chosen as*

$$\begin{aligned} u_1 &= (\alpha_1(t)+\alpha_2)x_2 + \frac{\alpha_3(t)}{\beta_L}(x_2-x_3) - kq_1 + \dot{\alpha}_1(t)x_1 + \dot{\alpha}_3(t)x_3 \\ u_2 &= (\alpha-k)q_2 - q_1 - \alpha\alpha_2(t)x_2 + \sin z_1 - a - b\sin\omega t \\ &\quad -\frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3) - \dot{\alpha}_2(t)x_2 \end{aligned} \tag{15}$$

where $q_1 = z_1 - \alpha_1(t)x_1 - \alpha_3(t)x_3$, $q_2 = z_2 + \alpha_2(t)x_2$ *then the drive system* (1) *achieves reduced order modified hybrid function projective synchronization with the response system* (3).

Suppose $\alpha_1(t) = \alpha_2(t) = \alpha_3(t) = \beta_1(t) = \beta_2(t) = \beta_3(t) = 0$, then we obtain Corollary 3.

Corollary 3 *If the controllers are chosen as*

$$\begin{aligned} u_1 &= -kq_1 \\ u_2 &= (\alpha-k)q_2 - q_1 + \sin z_1 - a - b\sin\omega t \end{aligned} \tag{16}$$

where $q_1 = z_1$, $q_2 = z_2$ *then the equilibrium point* $(0,0,0)$ *of the response system* (3) *is asymptotically stable.*

Suppose $\beta_1(t) = \beta_2(t) = \beta_3(t) = \alpha_1(t) = \alpha_2(t) = \alpha_3(t) = \gamma$. We obtain Corollary 4.

Corollary 4 *If the controllers are chosen as*

$$\begin{aligned} u_1 &= \frac{\gamma(t)}{\beta_L}(x_2 - x_3 + y_2 - y_3) + 2\alpha_1(t)(x_2+y_2) + \dot{\gamma}(t)(x_1+y_1+x_3+y_3) - kq1 \\ u_2 &= (\alpha-k)q_2 - q_1 - \alpha\gamma(t)(x_2+y_2) + \sin z_1 - a - b\sin\omega t - \dot{\gamma}(t)(x_2+y_2) \\ &\quad -\frac{\gamma(t)}{\beta_C}(2i - g(x_2)x_2 - g(y_2)y_2 - \sin x_1 - \sin y_1 - x_3 - y_3) \end{aligned} \tag{17}$$

where $q_1 = z_1 - \gamma(t)(x_1+y_1+x_3+y_3)$, $q_2 = z_2 + \gamma(t)(x_2+y_2)$ *then the drive systems* (1) *and* (2) *achieve reduced order hybrid function projective combination synchronization with the response system* (3).

Let all the scaling functions be $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$, $\beta_1(t)$, $\beta_2(t)$ and $\beta_3(t)$, then we obtain Corollary 5.

Corollary 5 *If the controllers are chosen as*

$$\begin{aligned} u_1 &= \alpha_2(t)x_2+\beta_2(t)y_2+\alpha_1(t)x_2+\beta_1(t)y_2+\frac{\alpha_3(t)}{\beta_L}(x_2-x_3)+\frac{\beta_3(t)}{\beta_L}(y_2-y_3) \\ &\quad +\dot{\alpha}_1(t)x_1+\dot{\beta}_1(t)y_1+\dot{\alpha}_3(t)x_3+\dot{\beta}_3(t)y_3-kq_1 \\ u_2 &= -\alpha(\alpha_2(t)x_2+\beta_2(t)y_2)+\sin z_1-a-b\sin\omega t-\dot{\alpha}_2(t)x_2-\dot{\beta}_2(t)+(\alpha-k)q_2-q_1 \\ &\quad -\frac{\alpha_2(t)}{\beta_C}(i-g(x_2)x_2-\sin x_1-x_3)-\frac{\beta_2(t)}{\beta_C}(i-g(y_2)y_2-\sin y_1-y_3) \end{aligned} \tag{18}$$

$q_1 = z_1-(\alpha_1(t)x_1+\beta_1(t)y_1+\alpha_3(t)x_3+\beta_3(t)y_3)$, $q_2 = z_2+(\alpha_2(t)x_2+\beta_2(t)y_2)$ *then the drive system* (1) *and* (2) *achieve reduced order modified hybrid function projective combination synchronization with the response system* (3).

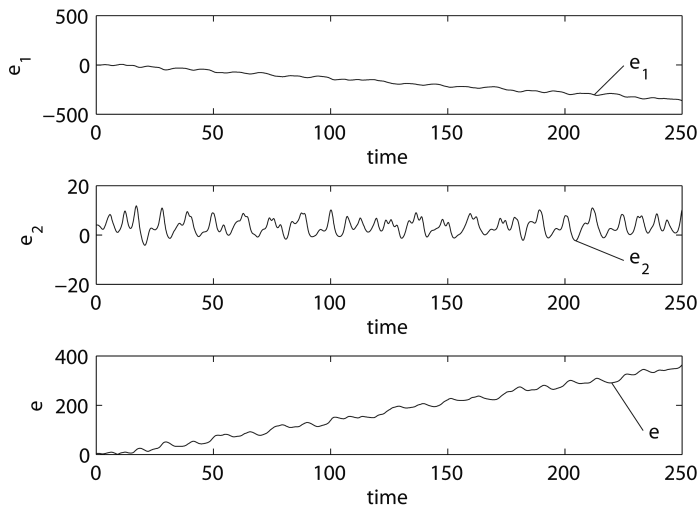

Figure 1. Error with control function deactivated.

2.2. Numerical simulation results

The designed controllers are verified in our numerical simulation using the ode45 fourth order Runge-Kutta algorithm in Matlab. In the numerical simulation procedure we used the systems parameters within the chaotic region and controllers are chosen in accordance with Corollary 4. The initial conditions of the drive systems and response system are given as $(x_1,x_2,x_3) = (0,0,0)$, $(y_1,y_2,y_3) = (111)$, $(z_1,z_2) = (0,1)$, $\gamma(t) = 2.0+0.01\sin(0.05t)$ and $k = 1$. Corresponding numerical results are as follows: Fig. 1 depicts the error dynamics of the systems when controllers were deactivated. The errors did not converge to zero. Fig. 2 shows that reduced order hybrid function projective combination synchronization among systems (1), (2) and (3) is achieved as indicated

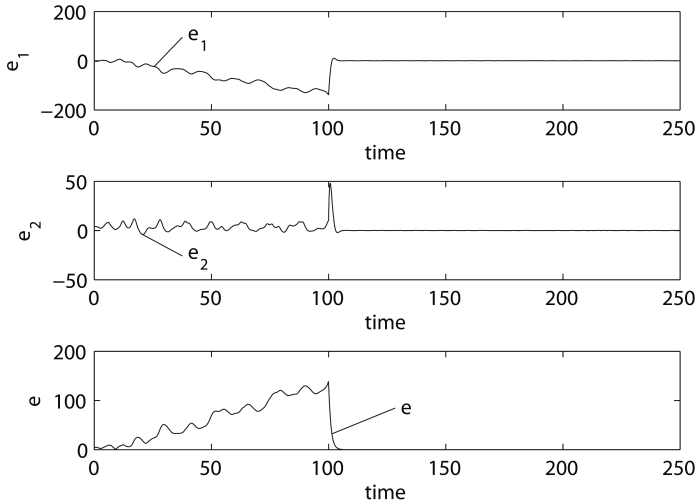

Figure 2. Error with control function activated at $t \geqslant 100$.

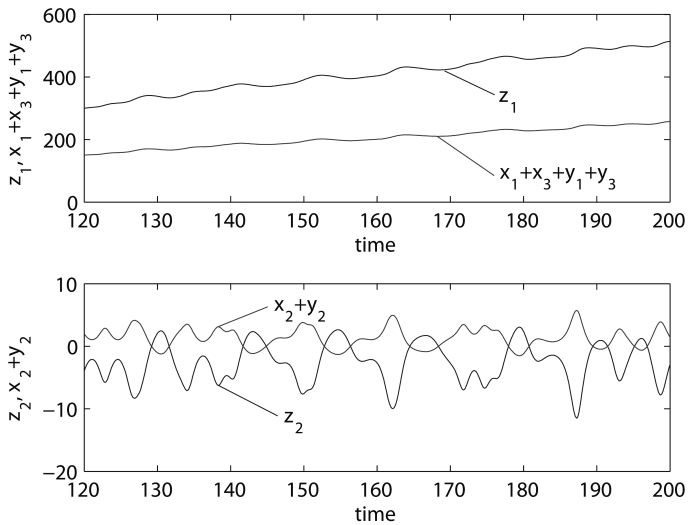

Figure 3. Dynamics of the state variables.

by the convergence of the error state variables to zero as soon as the controllers are switched on for $t \geqslant 100$. Fig. 3 shows the projection of the drive state variables on the projection of the response state variables when the controllers are activated for $t \geqslant 100$ which also confirms reduced order combination synchronization among systems (1),

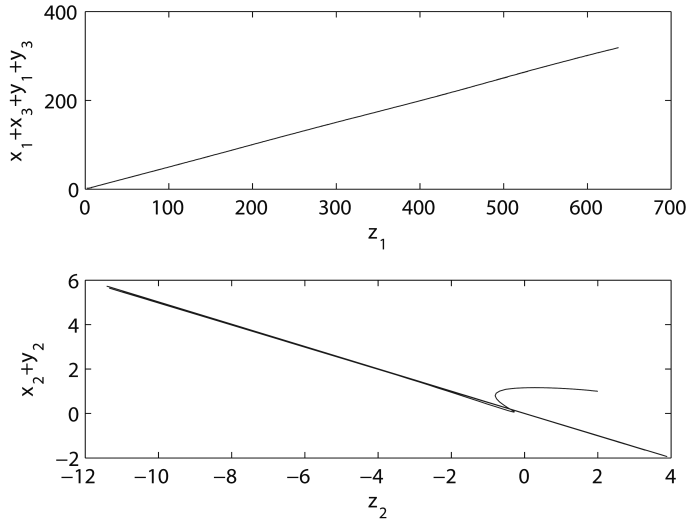

Figure 4. Evidence of hybrid projective synchronization.

(2) and (9). Fig 4 shows the evidence of realization of reduced order hybrid function projective combination synchronization.

3. Reduced order hybrid function projective combination synchronization of one third order and two second order Josephson junctions

3.1. Design of controller via active backstepping technique

In this section, one third order Josephson junction in (1) is taken as the drive system while two second order non-autonomous Josephson junction in (19) and (20) are taken as the response systems in order to achieve reduced order hybrid function projective combination synchronization among the three chaotic Josephson junctions .

$$\begin{aligned} \dot{y}_1 &= y_2 + u_1 \\ \dot{y}_2 &= -\alpha y_2 - \sin y_1 + a + b\sin \omega t + u_2 \end{aligned} \tag{19}$$

$$\begin{aligned} \dot{z}_1 &= z_2 + u_3 \\ \dot{z}_2 &= -\alpha z_2 - \sin z_1 + a + b\sin \omega t + u_4 \end{aligned} \tag{20}$$

where u_1, u_2, u_3 and u_4 are the controllers to be designed. We define the error systems as follows

$$e_1 = z_1 + y_1 - (\alpha_1(t)x_1 + \alpha_3(t)x_3)$$

$$e_2 = z_2 + y_2 + \alpha_2 x_2. \tag{21}$$

Using the error systems defined in (21) with systems defined in (1), (19) and (20) yields the following error dynamics

$$\dot{e}_1 = z_2 + y_2 + u_3 - \alpha_1(t)x_2 + u_1 - \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) - \dot{\alpha}_1(t)x_1 - \dot{\alpha}_3(t)x_3$$

$$= e_2 - (\alpha_2(t) + \alpha_1(t))x_2 - \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) - \dot{\alpha}_1(t)x_1 - \dot{\alpha}_3(t)x_3 u_1 + u_3$$

$$\dot{e}_2 = -\alpha z_2 - \sin z_1 + a + b\sin\omega t + u_4 - \alpha y_2 - \sin y_1 + a + b\sin\omega t + u_2$$

$$+\frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3) + \dot{\alpha}_2(t)x_2$$

$$= -\alpha(e_2 + \alpha_2(t)x_2) - \sin z_1 + 2a + 2b\sin\omega t - \sin y_1 + \dot{\alpha}_2(t)x_2$$

$$+\frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3) + u_2 + u_4.$$

Thus,the error dynamics of the system can be written as:

$$\dot{e}_1 = e_2 + U_1 + B_1 \tag{22}$$

$$\dot{e}_2 = -\alpha e_2 + U_2 + B_2 \tag{23}$$

where

$$B_1 = -(\alpha_2(t) + \alpha_1(t))x_2 - \frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) - \dot{\alpha}_1(t)x_1 - \dot{\alpha}_3(t)x_3$$

$$B_2 = \alpha\alpha_2(t)x_2) - \sin z_1 + 2a + 2b\sin\omega t - \sin y_1 + \dot{\alpha}_2(t)x_2$$

$$+\frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3)$$

$$U_1 = u_1 + u_3$$

$$U_2 = u_2 + u_4.$$

Our goal is to find the control functions which will enable the systems (1), (19) and (20) realize generalized reduced order combination synchronization by active backstepping technique. The design procedures includes three steps as shown below:

Step 1
Let $q_1 = e_1$, its time derivative is

$$\dot{q}_1 = \dot{e}_1 = e_2 + U_1 + B_1 \tag{24}$$

where $e_2 = \alpha_1(q_1)$ can be regarded as virtual controller. In order to stabilize q_1-subsystem, we chose a Lyapunov function $v_1 = \frac{1}{2}q_1^2$. Time derivative of v_1 is

$$\dot{v}_1 = q_1\dot{q}_1 = q_1(\alpha_1(q_1) + U_1 + B_1). \tag{25}$$

Suppose $\alpha_1(q_1) = 0$ and the control function U_1 is chosen as

$$U_1 = -(B_1 + kq_1) \tag{26}$$

then $\dot{v}_1 = -kq_1^2 < 0$ where k is a positive constant which represent the feedback gain. Then, $\dot{v}_1$ is negative definite and the subsystem q_1 is asymptotically stable. Since, the virtual controller $\alpha_1(q_1)$ is estimative, the error between e_2 and $\alpha_1(q_1)$ can be denoted by $q_2 = e_2 - \alpha_1(q_1)$. Thus, we have the (q_1, q_2)-subsystems

$$\begin{aligned} \dot{q}_1 &= q_2 - kq_1 \\ \dot{q}_2 &= -\alpha q_2 + U_2 + B_2. \end{aligned} \tag{27}$$

Step 2
In order to stabilize system (8), the following Lyapunov function can be chosen as $v_2 = v_1 + \frac{1}{2}q_2^2$. Its time derivative of v_2 is

$$\dot{v}_2 = -q_1^2 + q_2(q_1 - \alpha q_2 + U_2 + B_2). \tag{28}$$

If the control function u_2 is chosen as

$$U_2 = -B_2 - kq_2 + \alpha q_2 - q_1 \tag{29}$$

then $\dot{v}_2 = -kq_1^2 - kq_2^2 < 0$ where k is a positive constant which represent the feedback gain. Then, $\dot{v}_2$ is negative definite and the subsystem (q_1, q_2) in (27) is asymptotically stable. this implies that the drive system (1) and the response systems (19) and (20) achieve reduced order function projective combination synchronization. Finally, we have the following subsystems

$$\begin{aligned} \dot{q}_1 &= q_2 - kq_1 \\ \dot{q}_2 &= -q_1 - kq_2. \end{aligned} \tag{30}$$

Here we limit our results to only two major Corollaries which result into hybrid function projective combination synchronization.

Let $\alpha_1 = \alpha_2 = \alpha_3$, $u_1 = u_3$ and $u_2 = u_4$. Then, we have Corollary 6.

Corollary 6 *If the controllers are chosen as*

$$\begin{aligned} u_1 &= u_3 = \frac{1}{2}\left(\frac{\alpha_1(t)}{\beta_L}(x_2 - x_3) + \dot{\alpha}_1(t)(x_1 + x_3) - kq_1 + 2\alpha_1(t)\right) \\ u_2 &= u_4 = \frac{1}{2}\left((\alpha - k)q_2 - q_1 - (\alpha\alpha_1(t) + \dot{\alpha}_1(t))x_2 + \sin z_1 + \sin y_1 - 2a - 2b\sin\omega t\right. \\ &\quad \left. - \frac{\alpha_1(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3)\right) \end{aligned} \tag{31}$$

where $e_1 = z_1 - \alpha_1(x_1 + x_3)$, $e_2 = z_2 + \alpha_1 x_2$ *then the drive system* (1) *achieve reduced order hybrid function projective combination synchronization with the response systems* (19) *and* (20).

Let all the scaling functions be $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$ with $u_1 = u_3$ and $u_2 = u_4$. Then, we have Corollary 7.

Corollary 7 *If the controllers are chosen as*

$$\begin{aligned} u_1 &= u_3 = \frac{1}{2}\left((\alpha_1(t) + \alpha_2(t))x_2 + \left(\frac{\alpha_3(t)}{\beta_L}(x_2 - x_3) + \dot{\alpha}_1(t)x_1 + \dot{\alpha}_3(t)x_3 - kq_1\right)\right. \\ u_2 &= u_4 = \frac{1}{2}\left((\alpha - k)q_2 - q_1 - (\alpha\alpha_2(t) + \dot{\alpha}_2(t))x_2 + \sin z_1 + \sin y_1 - 2a - 2b\sin\omega t\right. \\ &\quad \left. - \frac{\alpha_2(t)}{\beta_C}(i - g(x_2)x_2 - \sin x_1 - x_3)\right) \end{aligned} \tag{32}$$

where $e_1 = z_1 + y_1 - (\alpha_1(t)x_1 + \alpha_3(t))$, $e_2 = z_2 + y_2 + \alpha_2(t)x_2$ *then reduced order modified hybrid function projective combination synchronization is achieved between the drive systems* (1) *and the response systems* (19) *and* (20).

3.2. Numerical simulation results

The designed controllers are verified in our numerical simulation using the ode45 fourth order Runge-Kutta algorithm in Matlab. In the numerical simulation we used the systems parameters within the chaotic region and controllers are chosen in accordance with Corollary 6. The initial conditions of the drive systems and response system are given as $(x_1, x_2, x_3) = (0, 0, 0)$, $(y_1, y_2, y_3) = (111)$, $(z_1, z_2) = (0, 1)$, $\gamma(t) = 2.0 + 0.01\sin(0.05t)$ and $k = 1$. Corresponding numerical results are as follows: Fig. 5 shows the error dynamics of the drive and slave systems when the controllers were deactivated. Fig. 6 shows that reduced order hybrid function projective combination synchronization among systems (1), (19) and (20) is achieved as pointed by the convergence of the error state to zero as soon as the controllers are switch on for $t \geqslant 100$. Fig.7 shows projection of the drive state variables on the response state variables when the controllers are activated for $t \geqslant 100$ which also confirms reduced order hybrid function projective combination synchronization among systems (1), (19) and (20).

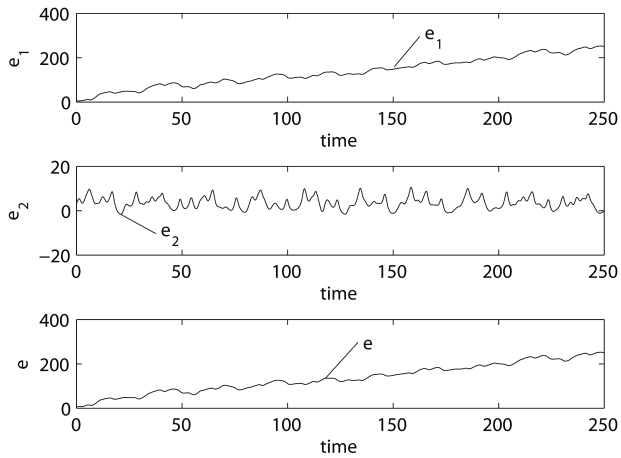

Figure 5. Error with control function deactivated.

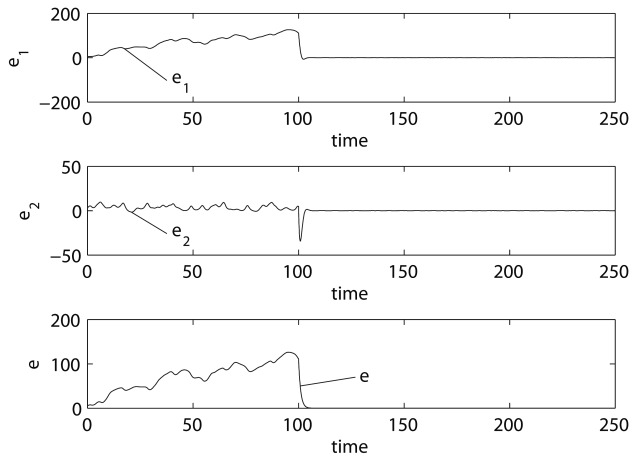

Figure 6. Error with control function activated.

4. Conclusion

Reduced order hybrid function projective combination synchronization of three chaotic systems consisting of: (i) two third order chaotic Josephson junctions as drives and one second order chaotic Josephson junction as response system; (ii) one third order chaotic Josephson junction as the drive and two second order chaotic Josephson junctions as the slaves using active backstepping technique has been realized. We showed

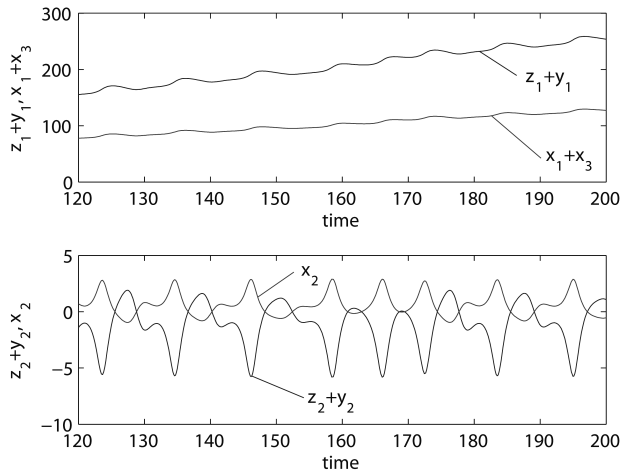

Figure 7. Evidence of hybrid projective synchronization

from the theoretical analysis that various controllers which are suitable for different type of synchronization scheme can be obtained from the general results. Moreover, reduced order hybrid function projective combination synchronization has more potential applications to secure information transmission in communication systems and information processing in biological systems.

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