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On a continuity of characteristic exponents of linear discrete time-varying systems

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In this paper we present a sufficient condition for continuity of Lyapunov exponents of discrete time-varying linear system. Basing on this result we show that Lyapunov exponents of time-invariant systems depend continuously on the time-varying perturbations.

Key words: time-varying discrete linear systems, Lyapunov exponents, perturbation theory, characteristic exponents

1. Introduction

Consider the linear discrete time-varying system

$$x(n+1) = A(n)x(n), n \geqslant 0 \tag{1}$$

where $(A(n))_{n\in\mathbb{N}}$ is a bounded sequence of invertible s-by-s real matrices such that sequence $(A^{-1}(n))_{n\in\mathbb{N}}$ is bounded. By $\|\cdot\|$ we denote the Euclidean norm in $\mathbb{R}^s$ and the induced operator norm. The transition matrix is defined as

$$\mathcal{A}(m) = A(m-1)\cdot\ldots\cdot A(0)$$

for $m>0$ and $\mathcal{A}(0) = \mathbf{I}$ is the identity matrix. For an initial condition x_0 the solution of (1) is denoted by $x(n,x_0)$, so

$$x(n,x_0) = \mathcal{A}(n)x_0.$$

Let $a = (a(n))_{n\in\mathbb{N}}$ be a sequence of real numbers. The numbers (or the symbol $\pm\infty$) defined as

$$\lambda(a) = \limsup_{n\to\infty} \frac{1}{n} \ln|a(n)|$$

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is called the characteristic exponent of sequence $(a(n))_{n\in\mathbb{N}}$. For $x_0 \in \mathbb{R}^s$, $x_0 \neq \mathbf{0}$ the Lyapunov exponent $\lambda(x_0)$ of (1) is defined as characteristic exponent of $(\|x(n,x_0)\|)_{n\in\mathbb{N}}$ therefore

$$\lambda_A(x_0) = \limsup_{n\to\infty} \frac{1}{n} \ln \|x(n,x_0)\|.$$

It is well known [2] that the set of all Lyapunov exponents of system (1) contains at most s elements, say $-\infty < \lambda_1(A) < \lambda_2(A) < \ldots < \lambda_r(A) < \infty, r \leqslant s$ and the set $\{\lambda_1(A), \lambda_2(A), \ldots, \lambda_r(A)\}$ is called the spectrum of (1). For each $\lambda_i, i = 1, \ldots, r$ we consider the following subspace of $\mathbb{R}^s$

$$E_i = \{v \in \mathbb{R}^s : \lambda(v) \leqslant \lambda_i\}$$

and we set $E_0 = \{0\}$. The multiplicity n_i of Lyapunov exponent λ_i is defined as $\dim E_i - \dim E_{i-1}$. For a base $V = \{v_1, \ldots, v_s\}$ of $\mathbb{R}^s$ we define the sum σ_V of Lyapunov exponents

$$\sigma_V = \sum_{i=1}^{s} \lambda(v_i).$$

It is known (see [11]) that if $v_1, \ldots, v_s$ is a basis of $\mathbb{R}^s$ then the following Lyapunov inequality holds:

$$\limsup_{n\to\infty} \frac{1}{n} \ln |\det \mathcal{A}(n)| \leqslant \sum_{l=1}^{s} \lambda(v_l). \tag{2}$$

The basis $v_1, \ldots, v_s$ is called normal if for each $i = 1, \ldots, s$ there exists a basis of E_i composed of vectors $\{v_1, \ldots, v_s\}$. Formally, we should say that a basis is normal with respect to family $E_i, i = 1, \ldots, s$. It can be shown (see [3], remark after Theorem 1.2.5) that there always exist normal bases $v_1, \ldots, v_s$ and $w_1, \ldots, w_s$ (respectively of families E_i and F_i) which are dual. It can be also shown (see [3], Theorem 1.2.3) that for normal basis the sum σ_V of Lyapunov exponents is minimal and then, according to Lyapunov inequality (see [11]), equal to

$$\limsup_{n\to\infty} \frac{1}{n} \ln |\det \mathcal{A}(n)|.$$

For a basis $v_1, \ldots, v_s$ of $\mathbb{R}^s$ matrix $\mathcal{V}(n), n \in \mathbb{N}$ whose columns are $x(n,v_1), \ldots, x(n,v_s)$ is called fundamental matrix of (1). For a fundamental matrix the kernel $\mathcal{G}(n,m) = \mathcal{V}(n)\mathcal{V}^{-1}(m), n,m \in \mathbb{N}$ is called Green's matrix of (1). If the base is normal, then the fundamental and Green's matrices are called normal.

Consider the values

$$\lambda_1'(A) \leqslant \lambda_2'(A) \leqslant \ldots \leqslant \lambda_s'(A) \tag{3}$$

of the Lyapunov exponents of (1), counted with their multiplicities. Together with (1) we consider the following disturbed system:

$$y(n+1) = (A(n) + \Delta(n))y(n), \tag{4}$$

where $(\Delta(n))_{n\in\mathbb{N}}$ is bounded sequence of s-by-s real matrices. Denote by $\lambda'_1(A+\Delta) \leqslant \lambda'_2(A+\Delta) \leqslant \ldots \leqslant \lambda'_s(A+\Delta)$ the Lyapunov exponents of (4) counted with their multiplicities.

Under the influence of the perturbation $(\Delta(n))_{n\in\mathbb{N}}$, the characteristic exponents of (1) vary, in general, discontinuously. It is possible, that $\|\Delta(n)\| \underset{n\to\infty}{\to} 0$ and the spectra of systems (1) and (4) are different (see Example with calculations presented in [9, Section V]). In this paper we will propose certain conditions that guarantee the continuity of the spectrum with respect to the coefficients of (1).

This problem for continuous-time case is known as the problem of stability of characteristic exponents and it is completely solved. Necessary and sufficient conditions for the stability of Lyapunov exponents were published by Bylov and Isobov (joint papers [5] and [6]) and Milionschikov [14].

2. Main results

We start with the following definition.

Definition 2 *The Lyapunov exponents of system* (1) *are called stable if for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that inequality*

$$\sup_{n\in\mathbb{N}} \|\Delta(n)\| < \delta \tag{5}$$

implies the inequality

$$\left|\lambda'_i(A) - \lambda'_i(A+\Delta)\right| < \varepsilon \quad, i = 1,\ldots,s.$$

Taking into consideration the definition shown above, we can interpret it as a definition of continuity of Lyapunov exponents as functions of coefficients with the matrix norm as a metric. Indeed, if we take $\mathcal{L} : D \to \mathbb{C}^n$, where D is a family of matrices sequences $(A(n))_{n\in\mathbb{N}}$, we can say, that $\mathcal{L}$ is a continuous if, and only if, for all $A \in D$ and $\varepsilon > 0$ there exists $\delta > 0$, such that for all $\bar{A} \in D$ we have

$$\|A - \bar{A}\| \leqslant \delta \quad \Rightarrow \quad \|\mathcal{L}(A) - \mathcal{L}(\bar{A})\| < \varepsilon,$$

where for Lyapunov exponents we use metric

$$d(\lambda,\mu) = \min_{\sigma} \max_{1\leqslant j\leqslant n} |\lambda_j - \mu_{\sigma(j)}|, \qquad \lambda = (\lambda_1,\ldots,\lambda_n),\ \mu = (\mu_1,\ldots,\mu_n).$$

To formulate our main results for a Green's matrix of (1) denote by $x_i(m,n)$ the i-th column of it and by μ_i characteristic exponent of the sequence $(\|x_i(m,n)\|)_{m\in\mathbb{N}}$, $i = 1,\ldots,s$. The next theorem [10] constitutes discrete-time version of Malkin's (see [13]) sufficient condition for continuity of Lyapunov exponents.

Theorem 3 *Suppose that for certain Green's $\mathcal{G}(m,n)$ matrix of* (1) *and any $\gamma > 0$ there exists $d > 0$ such that*

$$\|x_i(m,n)\| \leqslant d \exp[(\mu_i + \gamma)(m-n)] \quad \text{for } m,n \in \mathbb{N}, m \geqslant n, i = 1,\dots,s \tag{6}$$

$$\|x_i(m,n)\| \leqslant d \exp[(\mu_i - \gamma)(m-n)] \quad \text{for } m,n \in \mathbb{N}, n \geqslant m, i = 1,\dots,s. \tag{7}$$

Then the Lyapunov exponents of system (1) *are stable.*

Proof The proof of the theorem consists of the following three parts:

1. the shift of the characteristic exponents to the right is small,
2. there exists limit $\lim_{n\to\infty} \frac{1}{n} \ln|\det \mathcal{A}(n)| = \sum_{i=1}^{s} \mu_i$,
3. the shift of the characteristic exponents to the left is small.

1. Let $n_0 \in \mathbb{N}$. According to the variation of constant formula [1] any solution $y(n,y_0)$ of (4) satisfying $y(n_0,y_0) = y_0$ satisfies the equation

$$y(n,y_0) = \begin{cases} \mathcal{G}(n,n_0)y_0 + \sum_{l=n_0+1}^{n} \mathcal{G}(n,l)\Delta(l-1)y(l-1,y_0) & \text{for } n \geqslant n_0 \\ \mathcal{G}(n,n_0)y_0 - \sum_{l=n+1}^{n_0} \mathcal{G}(n,l)\Delta(l-1)y(l-1,y_0) & \text{for } n < n_0. \end{cases} \tag{8}$$

For system (4) consider a normal basis $v_1,\dots,v_s$ and consider s solutions $y_i(n,v_i)$, $i = 1,\dots,s$ of (4) satisfying $y(n_0,v_i) = v_i$, $i = 1,\dots,s$. Assume that the numeration of the basis is such that

$$\lambda_A(v_i) \leqslant \lambda_A(v_{i+1}), \quad i = 1,\dots,s-1.$$

From (8) we have

$$y_i(n,v_i) = \begin{cases} x_i(n,v_i) + \sum_{l=n_0+1}^{n} \mathcal{G}(n,l)\Delta(l-1)y_i(l-1,v_i) & \text{for } n \geqslant n_0 \\ x_i(n,v_i) - \sum_{l=n+1}^{n_0} \mathcal{G}(n,l)\Delta(l-1)y_i(l-1,v_i) & \text{for } n < n_0. \end{cases} \tag{9}$$

Take any $\varepsilon > 0$ such that

$$\varepsilon < (\lambda_A(v_s) - \lambda_A(v_i))/2 \tag{10}$$

for all $i = 1,\dots,s$ such that $\lambda_A(v_s) \neq \lambda_A(v_i)$. For such ε there exists positive constant c such that

$$\|x_i(n,v_i)\| \leqslant c \exp[(\lambda_A(v_i) + \varepsilon)n]$$

for all $i = 1,\dots,s$ and $n \in \mathbb{N}$. We will show that

$$\|y_i(n,v_i)\| \leqslant 2c \exp[(\lambda_A(v_i) + \varepsilon)n] \tag{11}$$

for all $i = 1,\dots,s$ and $n \in \mathbb{N}$. For $n = 0$ the inequality (11) is true. Consider first the case of i such that $\lambda_A(v_s) = \lambda_A(v_i)$. Suppose that (11) holds for $n = 0,\dots,p-1$. Let estimate $\|y_i(p,v_i)\|$. According to the first inequality in (9) with $n_0 = 0$ we have

$$\|y_i(p,v_i)\| \leqslant \|x_i(p,v_i)\| + \sum_{l=1}^{p} \|\mathcal{G}(p,l)\| \, \|\Delta(l-1)\| \, \|y_i(l-1,v_i)\|.$$

For $\gamma = \varepsilon/2$ let find d such that (6) and (7) hold. Then

$$\|y_i(p,v_i)\| \leqslant c\exp[(\lambda_A(v_i)+\varepsilon)p]+$$

$$+2cd\delta\sum_{l=1}^{p}\exp[(\lambda_A(v_i)+\frac{\varepsilon}{2})(p-l)]\exp[(\lambda_A(v_i)+\varepsilon)(l-1)] \leqslant$$

$$\leqslant c\exp[(\lambda_A(v_i)+\varepsilon)p] + \frac{2cd\delta e^{-\lambda_A(v_i)}}{e^{\frac{\varepsilon}{2}}-1}\exp[(\lambda_A(v_i)+\varepsilon)p].$$

Taking

$$\delta < \frac{e^{\frac{\varepsilon}{2}}-1}{2cde^{-\lambda_A(v_i)}}$$

we get that (11) holds for $n = p$. Consider now the case of i such that $\lambda_A(v_s) > \lambda_A(v_i)$. Let estimate $\|y_i(p,v_i)\|$. According to second equality in (9) with $n_0 = \infty$ we have

$$\|y_i(p,v_i)\| \leqslant \|x_i(p,v_i)\| + \sum_{l=p+1}^{\infty} \|\mathcal{G}(p,l)\| \, \|\Delta(l-1)\| \, \|y_i(l-1,v_i)\|.$$

By (6) and (10) we have

$$\|y_i(p,v_i)\| \leqslant c\exp[(\lambda_A(v_i)+\varepsilon)p]+$$

$$+2xd\delta\sum_{l=p+1}^{\infty}\exp[(\lambda_A(v_s)-\frac{\varepsilon}{2})(p-l)]\exp[(\lambda_A(v_i)+\varepsilon)(l-1)].$$

As in previous case we obtain that (11) is true for $n = p$ and sufficiently small δ. It is also clear, from the estimates for

$$\sum_{l=1}^{n}\mathcal{G}(n,l)\Delta(l-1)y_i(l-1,v_i),$$

that for sufficiently small δ vectors $y_i(0,v_i)$, $i = 1,\dots,s$ differ little from the vectors v_i, $i = 1,\dots,s$ and therefore they are linearly independent. Moreover, from the estimates (11) we obtain

$$\lambda_{A+\Delta}(v_i) \leqslant \lambda_A(v_i)+\varepsilon.$$

If basis $v_1,\dots,v_s$ is not normal for (4), then, by passing to normal basis, the exponents can only diminish; therefore, we have

$$\lambda_i'(A+\Delta) \leqslant \lambda_i'(A)+\varepsilon \tag{12}$$

for $i=1,\dots,s$.

2. Using Hadamard's inequality and (6) we have

$$|\det \mathcal{G}(0,n)| \leqslant d^s \prod_{i=1}^{s} \exp\left[(\mu_i-\gamma)(0-n)\right] = d^s \exp\left[-n\sum_{i=1}^{s}\mu_i\right]\exp\left[n\gamma s\right]$$

and therefore

$$\begin{aligned}
\liminf_{n\to\infty}\frac{1}{n}\ln|\det\mathcal{A}(n)| &= -\limsup_{n\to\infty}\frac{1}{n}(-\ln|\det\mathcal{A}(n)|) = \\
-\limsup_{n\to\infty}\frac{1}{n}(\ln|\det\mathcal{A}^{-1}(n)|) &= -\limsup_{n\to\infty}\frac{1}{n}(\ln|\det\mathcal{G}(0,n)|) \geqslant \\
&\geqslant \gamma s+\sum_{i=1}^{s}\mu_i
\end{aligned} \tag{13}$$

Using the Lyapunov inequality

$$\sum_{i=1}^{s}\mu_i \geqslant \limsup_{n\to\infty}\frac{1}{n}(\ln|\det\mathcal{A}(n)|),$$

combining the above result with (13) and taking into account, that γ is arbitrarily small, we obtain

$$\limsup_{n\to\infty}\frac{1}{n}\ln|\det\mathcal{A}(n)| = \sum_{i=1}^{s}\mu_i = \sum_{i=1}^{s}\lambda_i'(A). \tag{14}$$

3. Applying the Lyapunov inequality (2) to the disturbed system (4) we have

$$\begin{aligned}
&\sum_{i=1}^{s}\lambda_i'(A+\Delta) \geqslant \limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln|\det(A(n)+\Delta(n))| \geqslant \\
&\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln|\det A(n)(I+A^{-1}(n)\Delta(n))| = \\
&\sum_{i=1}^{s}\lambda_i'(A)+\limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln|\det(I+A^{-1}(n)\Delta(n))|,
\end{aligned} \tag{15}$$

where I is the identity matrix of size s-by-s. Fix $\varepsilon>0$. Since the sequence $(A^{-1}(n))_{n\in\mathbb{N}}$ is bounded, then there exists $\delta_1>0$ such that for all matrices X such that $\|X\|<\delta_1$ the following inequality is true

$$|\ln|\det(I+A^{-1}(n)X)|\,| \leqslant \frac{\varepsilon}{s}$$

for all $n \in \mathbb{N}$. For the perturbation satisfying (5) with δ_1, we have from (15) the following inequality

$$\sum_{i=1}^{s} \lambda_i'(A+\Delta) \geqslant \sum_{i=1}^{s} \lambda_i'(A) - \frac{\varepsilon}{s}. \tag{16}$$

Moreover, according to (12), we can find $\delta_2 > 0$ such that $\lambda_i'(A+\Delta) \leqslant \lambda_i'(A) + \frac{\varepsilon}{s}$ for $i = 1,\dots,s$ and

$$\sup_{n\in\mathbb{N}} \|\Delta(n)\| < \delta_2.$$

Consider now perturbations $(\Delta(n))_{n\in\mathbb{N}}$ with $\delta = \min(\delta_1, \delta_2)$. Introduce $\gamma_i > 0$ such that

$$\lambda_i'(A+\Delta) = \lambda_i'(A) + \frac{\varepsilon}{s} - \gamma_i, \; i = 1,\dots,s. \tag{17}$$

By substituting this expression to (16) we obtain

$$\left(1 + \frac{1}{s}\right)\varepsilon \geqslant \sum_{i=1}^{s} \gamma_i \geqslant \gamma_i.$$

From this bound and (17) we have

$$\lambda_i'(A+\Delta) \geqslant \lambda_i'(A) - \varepsilon,$$

which ends the proof. □

Using this result we will show that the Lyapunov exponents of time-invariant system are stable.

Theorem 4 *Lyapunov exponents of time-invariant system*

$$x(n+1) = Ax(n) \tag{18}$$

with invertible matrix A are stable.

Proof Using the theorem, shown with proof in [12, page 239], we can represent the matrix A as exponent of a matrix B

$$A = e^B.$$

In [12] the matrix B is called a logarithm of A. The transition matrix of the system (18) can be presented in the form:

$$\mathcal{A}(m,n) = A^{m-n} = e^{B(m-n)}.$$

Let us introduce a matrix S that transforms B to the Jordan canonical form,

$$C = S^{-1}BS = \operatorname{diag}[J_{\rho_1}(\lambda_1),\dots,J_{\rho_k}(\lambda_k)],$$

where $\lambda_1,\dots,\lambda_k$ are the eigenvalues of the matrix B (not necessarily distinct), $J_v(\lambda)$ is the Jordan block of order v, and $\sum_{i=0}^{k}\rho_i = s$. Hence,

$$e^{B(m-n)} = Se^{C(m-n)}S^{-1}. \tag{19}$$

Moreover,

$$e^{C(m-n)} = \operatorname{diag}\left[e^{J_{\rho_1}(\lambda_1)(m-n)},\dots,e^{J_{\rho_k}(\lambda_k)(m-n)}\right], \tag{20}$$

where

$$e^{J_v(\lambda)(m-n)} = e^{\lambda(m-n)}\begin{pmatrix} 1 & \cdots & & 0 \\ m-n & & & \vdots \\ \vdots & \ddots & & \vdots \\ \frac{(m-n)^{v-1}}{(v-1)!} & \cdots & m-n & 1 \end{pmatrix}.$$

Each element of the matrix $\exp^{C(m-n)}$ satisfies the estimates (6) and (7). Indeed, it has the following general form:

$$\frac{1}{l!}(m-n)^l e^{\lambda(m-n)}, \qquad l = 0,1,\dots,s-1\,. \tag{21}$$

The inequalities (6) and (7) for this function have the form

$$\begin{aligned} |\tfrac{1}{k!}(m-n)^k e^{\lambda(m-n)}| &\leqslant ae^{(Re\lambda+\gamma)(m-n)}, \quad m \geqslant n \\ |\tfrac{1}{k!}(m-n)^k e^{\lambda(m-n)}| &\leqslant ae^{(Re\lambda-\gamma)(m-n)}, \quad n \geqslant m. \end{aligned} \tag{22}$$

Dividing by the exponent, we obtain

$$\begin{aligned} |\tfrac{1}{k!}(m-n)^k| &\leqslant ae^{\gamma(m-n)}, \quad m \geqslant n \\ |\tfrac{1}{k!}(m-n)^k| &\leqslant ae^{\gamma(n-m)}, \quad n \geqslant m. \end{aligned}$$

Both inequalities are reduced to a single one, and there exists a constant a depending on γ and independenting of n that realizes the inequality $\frac{1}{k!} \leqslant a\exp(\gamma\theta), \theta > 0$, namely

$$a = \max_{\theta\in\mathbb{R}_+} \frac{1}{k!}\theta^k \exp(-\gamma\theta).$$

Let us return now to the matrix (19):

$$e^{B(m-n)} = V(m)S^{-1} =$$

$$= \{v_1^{(1)}(m),\dots,v_{\rho_1}^{(1)}(m),v_1^{(2)}(m),\dots,v_{\rho_2}^{(2)}(m),\ \dots\ ,v_1^{(k)}(m),\dots,v_{\rho_k}^{(k)}(m)\}S^{-1}.$$

The columns of the matrix $V(t)$ are solutions of system (1). They are divided into k groups. The first one is the result of multiplication of the matrix S by the first ρ_1 columns of the matrix $e^{C(m-n)}$, etc., and the last one is the result of multiplication of S by the last ρ_k columns of the matrix $e^{C(m-n)}$. The solutions of the m-th group have the characteristic exponent

$$\mathrm{Re}\lambda_j, \qquad j = 1,2,\ldots,k.$$

Multiplying the matrix $V(t)$ by S^{-1} from the right, we have

$$e^{B(m-n)} = \{x_1(m,n), x_2(m,n), \ldots, x_s(m,n)\},$$

where each vector $x_i(m,n)$ is a linear combination of the solutions $v_i(m),\ldots,v_s(m)$. Therefore, the components of any solution $x_i(m,n)$ represent linear combinations of sequences of the form (21) with coefficients depending on the constant matrices S and S^{-1}. For each of these functions, the estimates (22) are satisfied and the inequalities can only be strengthened if in the right-hand side $\mathrm{Re}\lambda$ is replaced with the maximal exponent of the linear combination, i.e., with the exponent of the solution $x_i(m,n)$. The constant a changes its value because of the multiplication by the constant matrices S and S^{-1}. Estimating the vectors $x_i(m,n)$, $i = 1,\ldots,s$, component-wise, we verify that inequalities (6) and (7) hold. □

Corollary 5 *Lyapunov exponents of time-invariant system (18) are stable.*

Proof According to the Theorem VI.1.2 in [4, page 154] eigenvalues of the matrix are continuous as functions of coefficients. It is also known, that eigenvalues of matrix $B = cA + dI$, where $c,d \in \mathbb{R}, c \neq 0$, are equal to $\mu = c\lambda + d$, where λ are eigenvalues of matrix A.

Consider now matrix $X = A + \eta I$, where $|\eta| \notin \sigma(A)$ (where $\sigma(A)$ is a spectrum of matrix A). In this case matrix X does not have eigenvalue equals 0, so it is nonsingular.

Using Theorem (4) above dependences we can write that for all $\varepsilon > 0$ there exists $\delta_1 > 0$, that if only $|\delta| < \delta_1$ and $|\delta| \notin \sigma(A)$ then

$$|\lambda_i'(A + \delta I) - \lambda_i'(A)| < \frac{\varepsilon}{2} \tag{23}$$

for all $i \in \{1,\ldots,s\}$.

Consider now δ_1 lesser than the minimal, nonzero module of matrix A eigenvalue.

We will show now, that for all $\varepsilon > 0$ there exists δ_2, $0 < \delta_2 < \delta_1$, that for $\eta < \delta_2$

$$\sup_{n\in\mathbb{N}} \|\Delta(n) - \eta I\| < \delta_2 \Rightarrow |\lambda_i'(A+\Delta) - \lambda_i'(A+\eta I)| < \frac{\varepsilon}{2} \tag{24}$$

for all $i \in \{1,\ldots,s\}$.

Observe, that assumption shown above can be replaced by

$$\sup_{n\in\mathbb{N}} \|\Delta(n)\| < \frac{\delta_2}{2}, \eta = \frac{\delta_2}{2}, \delta_2 < \delta_1 .$$

With this assumption the matrix A is nonsingular. Moreover $\sup_{n\in\mathbb{N}}\|\Delta(n)-\eta I\| \leqslant \frac{\delta_2}{2}+\eta=\delta_2$, so, according to Theorem 4 dependence (24) occurs.

With defined η,δ_1,δ_2 we have:

$$|\lambda_i'(A+\Delta)-\lambda_i'(A)| = |\lambda_i'(A+\Delta)-\lambda_i'(A+\eta I)+\lambda_i'(A+\eta I)-\lambda_i'(A)| \leqslant$$
$$\leqslant |\lambda_i'(A+\Delta)-\lambda_i'(A+\eta I)|+\frac{\varepsilon}{2} = |\lambda_i'(A+\eta I+\Delta-\eta I)-\lambda_i'(A+\eta I)|+\frac{\varepsilon}{2} \leqslant$$
$$\leqslant \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

for all $i\in\{1,\ldots,s\}$. □

3. Conclusion

In this paper we have presented the sufficient condition for continuity (called here as a stability) of Lyapunov exponents of discrete time-varying linear system. Basing on this condition it has been also prooved that Lyapunov exponents of time-invariant systems depend continuously on the time-varying convergent to zero perturbations. The results maybe used to analyse stability problem for the systems with uncertainties ([7], [15]).

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