

# Lyapunov functional for a linear system with two delays both retarded and neutral type

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The paper presents a method of determining of the Lyapunov quadratic functional for linear time-invariant system with two delays both retarded and neutral type. The Lyapunov functional is constructed for a given its time derivative which is calculated on the trajectory of the system with two delays both retarded and neutral type. The presented method gives analytical formulas for the coefficients of the Lyapunov functional.

**Key words:** Lyapunov functional, time delay system, neutral system, LTI system

## 1. Introduction

Lyapunov quadratic functionals are used to test the stability of the systems, to computation of the critical delay values for time delay systems, to computation of the exponential estimates for the solutions of time delay systems, to calculation of the robustness bounds for uncertain time delay systems, to calculation of a quadratic performance index of quality for the process of parametric optimization for time delay systems. We construct the Lyapunov functionals for the systems with time delay with a given time derivative. For the first time such Lyapunov functional was introduced by Repin [19] for the case of retarded time delay linear systems with one delay. Repin [19] delivered also the procedure for the designation of the coefficients of functional. The Lyapunov functional, which was proposed by Repin, was used in [1] for the calculation of the value of a quadratic performance index of quality in the process of parametric optimization for systems with time delay of retarded type. In [2] the Repin's method was extended to the case of neutral type systems and in [3] to the case of linear time invariant system with two lumped retarded type time delays. Infante and Castelan's construction of the Lyapunov functional, presented in [12], was based on a solution of a matrix differential-difference equation on a finite time interval. This solution satisfies symmetry and boundary conditions. In [18] Kharitonov and Zhabko extended results of Infante and Castelan [12] and proposed a procedure of construction of quadratic functionals for linear retarded type delay systems which could be used for the robust stability analysis of time delay systems.

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This functional was expressed by means of Lyapunov matrix which depended on the fundamental matrix of time delay system. In [14] Kharitonov extended some basic results obtained for the case of retarded type time delay systems to the case of neutral type time delay systems, and in [15] to the neutral type time delay systems with discrete and distributed delay. In [16] Kharitonov and Hinrichsen used the Lyapunov matrix to derive exponential estimates for the solutions of exponentially stable time delay systems. In [17] Kharitonov and Plischke received necessary and sufficient conditions for the existence and uniqueness of the delay Lyapunov matrix for the case of retarded system with one delay.

The numerical scheme for construction of the Lyapunov functionals has been proposed in [6]. This method starts with the discretisation of the Lyapunov functional. The scheme is based on linear matrix inequality (LMI) techniques. Fridman in [4] introduced the Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems with discrete and distributed delays which were based on equivalent descriptor form of the original system and obtained delay-dependent and delay-independent conditions in terms of LMI. Ivanescu et al. in [13] proceeded with the delay-dependent stability analysis for linear neutral systems, constructed the Lyapunov functional and derived sufficient delay-dependent conditions in terms of linear matrix inequalities (LMIs). Han in [8] obtained a delay-dependent stability criterion for neutral systems with time varying discrete delay. This criterion was expressed in the form of LMI and was received using the Lyapunov direct method. Han in [9] investigated the robust stability of uncertain neutral systems with discrete and distributed delays, which has been based on the descriptor model transformation and the decomposition technique, and formulated the stability criteria in the form of LMIs. Han in [10] developed the discretized Lyapunov functional approach to investigation of the stability of linear neutral systems with mixed neutral and discrete delays. Stability criteria which are applicable to linear neutral systems with both small and non-small discrete delays are formulated in the form of LMIs. Han in [11] studied the problem of the stability of linear time delay systems both retarded and neutral types using the discrete delay N-decomposition approach to derive some new more general discrete delay dependent stability criteria. Gu and Liu in [7] investigated the stability of coupled differential-functional equations using the discretized Lyapunov functional method and delivered the stability condition in the form of LMI, suitable for numerical computation.

This paper presents a method of determining of the Lyapunov functional for linear dynamic system with two delays both retarded and neutral type time delay. The novelty of the result lies in the extension of the Repin method to the system with two delays both retarded and neutral type time delay. To the best of author's knowledge, such extension has not been reported in the literature. There is also presented an example illustrating that method.

## 2. Formulation of the problem

Let us consider the linear system with two delays both retarded and neutral type time delay, which dynamics is described by equations

$$\begin{cases} \frac{dx(t)}{dt} - D \frac{dx_t(-\tau)}{dt} = A \cdot x(t) + B \cdot x_t(-r) + C \cdot x_t(-\tau) \\ x(t_0) = x_0 \in R^n \\ x_{t_0} = \Phi \in W^{1,2}([-r, 0], R^n) \end{cases} \quad (1)$$

$r \geq \tau \geq 0$ ;  $x(t) \in R^n$ ;  $A, B, C, D \in R^{n \times n}$ ;  $\text{rank } D = n$ ,  $x_t \in W^{1,2}([-r, 0], R^n)$   
 $x_t(\theta) = x(t + \theta)$ ,  $t \geq t_0$ ,  $\theta \in [-r, 0]$

where  $W^{1,2}([-r, 0], R^n)$  is a space of all absolutely continuous functions  $[-r, 0] \rightarrow R^n$  with derivatives in  $L^2([-r, 0], R^n)$  a space of a Lebesgue square integrable functions on interval  $[-r, 0]$  with values in  $R^n$ .

We introduce a new variable  $y$ , defined by the formula

$$y(t) = x(t) - Dx_t(-\tau) \quad \text{for } t \geq t_0. \quad (2)$$

Thus the equations (1) take the form of (3)

$$\begin{cases} \frac{dy(t)}{dt} = Ay(t) + Bx_t(-r) + (C + D)x_t(-\tau) \\ y(t_0) = y_0 \\ x_{t_0} = \phi \in W^{1,2}([-r, 0], R^n) \\ y(t) = x(t) - Dx_t(-\tau) \end{cases} \quad (3)$$

for  $t \geq t_0$  where  $y_0 = x_0 - D\phi(-\tau)$ .

State of the system (3) is a vector

$$S(t) = \begin{bmatrix} y(t) \\ x_t \end{bmatrix} \quad \text{for } t \geq t_0. \quad (4)$$

The state space is defined by the formula

$$X = R^n \times W^{1,2}([-r, 0], R^n). \quad (5)$$

On the state space  $X$  we define a Lyapunov functional, positively defined, differentiable, which derivative computed on the trajectory of the system (3) is negatively defined.

$$\begin{aligned}
 V(S(t)) &= y^T(t) \cdot \alpha \cdot y(t) + \int_{-r}^0 y^T(t) \cdot \beta(\theta) \cdot x_t(\theta) d\theta + \int_{-r}^0 x_t^T(\theta) \cdot \gamma(\theta) \cdot x_t(\theta) d\theta \\
 &+ \int_{-r}^0 \int_{\theta}^0 x_t^T(\theta) \cdot \delta(\theta, \xi) \cdot x_t(\xi) d\xi d\theta,
 \end{aligned} \tag{6}$$

$$\alpha = \alpha^T \in R^{n \times n}; \beta, \gamma \in C^1([-r, 0], R^{n \times n}); \delta \in C^1(\Omega, R^{n \times n})$$

$$\Omega = \{(\theta, \xi) : \theta \in [-r, 0], \xi \in [\theta, 0]\}; \gamma(\theta) = \gamma^T(\theta)$$

$C^1$  is a space of continuous functions with continuous derivative.

### 3. Designation of the coefficients of the functional (6)

We compute the derivative of the functional (6) on the trajectory of the system (3) according to the formula

$$\frac{dV(S(t))}{dt} = \text{grad}(V(S(t))) \cdot \frac{dS(t)}{dt} \quad \text{for } t \geq t_0. \tag{7}$$

Derivative of the functional (6) calculated on the basis of the formula (7) is given by formula

$$\begin{aligned}
 \frac{dV(S(t))}{dt} &= y^T(t) [A^T \alpha + \alpha A + \frac{\beta(0) + \beta^T(0)}{2} + \gamma(0)] y(t) \\
 &+ y^T(t) [2\alpha(C + D) + \beta(0)D + 2\gamma(0)D] x_t(-\tau) + y^T(t) [2\alpha B - \beta(-r)] x_t(-r) \\
 &- x_t^T(-r) \gamma(-r) x_t(-r) + \int_{-r}^0 y^T(t) [A^T \beta(\theta) - \frac{d\beta(\theta)}{d\theta} + \delta^T(\theta, 0)] x_t(\theta) d\theta \\
 &+ \int_{-r}^0 x_t^T(-\tau) [(C + D)^T \beta(\theta) + D^T \delta^T(\theta, 0)] x_t(\theta) d\theta \\
 &+ \int_{-r}^0 x_t^T(-r) [B^T \beta(\theta) - \delta(-r, \theta)] x_t(\theta) d\theta \\
 &- \int_{-r}^0 x_t^T(\theta) \frac{d\gamma(\theta)}{d\theta} x_t(\theta) d\theta - \int_{-r}^0 \int_{\theta}^0 x_t^T(\theta) \left[ \frac{\partial \delta(\theta, \xi)}{\partial \theta} + \frac{\partial \delta(\theta, \xi)}{\partial \xi} \right] x_t(\xi) d\xi d\theta \quad t \geq t_0.
 \end{aligned} \tag{8}$$

We identify the coefficients of the functional (6) assuming that the derivative (8) satisfies the relationship

$$\frac{dV(S(t))}{dt} = -y^T(t)W y(t) \quad \text{for } t \geq t_0 \quad (9)$$

where  $W \in R^{n \times n}$  is symmetric positively defined matrix.

When the system (3) is asymptotically stable and the relationship (9) holds, one can easily determine the value of a square indicator of the quality of the parametric optimization, knowing the Lyapunov functional (6), because

$$J = \int_{t_0}^{\infty} y^T(t)W y(t)dt = V(S(t_0)). \quad (10)$$

From equation (8) and (9) we receive the system of equations

$$A^T \alpha + \alpha A + \frac{\beta(0) + \beta^T(0)}{2} + \gamma(0) = -W \quad (11)$$

$$2\alpha(C + D) + \beta(0)D + 2\gamma(0)D = 0 \quad (12)$$

$$2\alpha B - \beta(-r) = 0 \quad (13)$$

$$\gamma(-r) = 0 \quad (14)$$

$$A^T \beta(\theta) - \frac{d\beta(\theta)}{d\theta} + \delta^T(\theta, 0) = 0 \quad (15)$$

$$(C + D)^T \beta(\theta) + D^T \delta^T(\theta, 0) = 0 \quad (16)$$

$$B^T \beta(\theta) - \delta(-r, \theta) = 0 \quad (17)$$

$$\frac{d\gamma(\theta)}{d\theta} = 0 \quad (18)$$

$$\frac{\partial \delta(\theta, \xi)}{\partial \theta} + \frac{\partial \delta(\theta, \xi)}{\partial \xi} = 0 \quad (19)$$

for  $\theta \in [-r, 0]$ ,  $\xi \in [\theta, 0]$ .

From equation (14) and (18) results, that

$$\gamma(\theta) = 0 \quad \text{for } \theta \in [-r, 0]. \quad (20)$$

Then equations (11) and (12) take the form

$$A^T \alpha + \alpha A + \frac{\beta(0) + \beta^T(0)}{2} = -W, \quad (21)$$

$$2\alpha(C + D) + \beta(0)D = 0. \quad (22)$$

From the equation (22) we receive  $\beta(0)$  and we put it into (21). After some calculations we obtain the relationship

$$\alpha G + G^T \alpha = -W \quad (23)$$

where

$$G = A - CD^{-1} - I. \quad (24)$$

Matrix  $G$  should be negatively defined because matrix  $W$  is positively defined. From the formula (23) we can obtain the matrix  $\alpha$ .

Now we take into account the equations (15) and (16). From equation (16) we receive

$$\delta^T(\theta, 0) = [-(CD^{-1})^T - I] \beta(\theta) \quad \text{for } \theta \in [-r, 0] \quad (25)$$

and put it into (15). After some calculations we have

$$\frac{d\beta(\theta)}{d\theta} = G^T \beta(\theta) \quad \text{for } \theta \in [-r, 0] \quad (26)$$

where matrix  $G$  is given by formula (24).

The solution of the differential equation (26) with the initial condition given by relation (13) is given by formula

$$\beta(\theta) = \exp(G^T(\theta + r))\beta(-r) = 2\exp(G^T(\theta + r))\alpha B \quad \theta \in [-r, 0]. \quad (27)$$

The solution of equation (19) is as below

$$\delta(\theta, \xi) = \varphi(\theta - \xi) \quad \text{for } \theta \in [-r, 0] \quad \xi \in [\theta, 0] \quad (28)$$

where  $\varphi \in C^1([-r, r], \mathbb{R}^{n \times n})$ ,  $C^1$  is a space of continuous functions with continuous derivative.

The relation (17) takes the form of

$$B^T \beta(\theta) - \varphi(-r - \theta) = 0 \quad \text{for } \theta \in [-r, 0]. \quad (29)$$

Hence

$$\varphi(\theta) = B^T \beta(-r - \theta) \quad \text{for } \theta \in [-r, 0]. \quad (30)$$

Taking into account the formula (30) we receive

$$\delta(\theta, \xi) = B^T \beta(\xi - \theta - r) \quad \text{for } \theta \in [-r, 0], \quad \xi \in [\theta, 0]. \quad (31)$$

In this way we obtained all parameters of the Lyapunov functional (6).

#### 4. The example

Let us consider the system described by equation

$$\begin{cases} \frac{dx(t)}{dt} - d \frac{dx(t-\tau)}{dt} = ax(t) + bx(t-r) + cx(t-\tau) \\ x(t_0) = x_0 \in R \\ x_{t_0} = \Phi \in W^{1,2}([-r, 0], R) \end{cases} \quad (32)$$

$$t \geq t_0; x(t) \in R; x_t \in W^{1,2}([-r, 0], R); x_t(\theta) = x(t + \theta); a, b, c, d \in R; d \neq 0; r \geq \tau \geq 0.$$

We introduce the new variable  $y$

$$y(t) = x(t) - dx(t-\tau). \quad (33)$$

Now the equation (32) takes the form of

$$\begin{cases} \frac{dy(t)}{dt} = ay(t) + bx(t-r) + (c+d)x(t-\tau) \\ y(t_0) = x_0 - d\phi(-\tau) \\ x_{t_0} = \phi \in W^{1,2}([-r, 0], R) \end{cases} \quad (34)$$

$$t \geq t_0; y(t) \in R; x_t \in W^{1,2}([-r, 0], R); a, b, c, d \in R; d \neq 0; r \geq \tau \geq 0$$

The Lyapunov functional is defined by the formula

$$\begin{aligned} V(S(t)) = & \alpha \cdot y^2(t) + \int_{-r}^0 y(t) \cdot \beta(\theta) \cdot x_t(\theta) d\theta + \int_{-r}^0 \gamma(\theta) \cdot x_t^2(\theta) d\theta \\ & + \int_{-r}^0 \int_{\theta}^0 x_t(\theta) \cdot \delta(\theta, \xi) \cdot x_t(\xi) d\xi d\theta. \end{aligned} \quad (35)$$

We obtain coefficients of the functional (35) as below.

According to the equation (20)

$$\gamma(\theta) = 0 \quad \text{for } \theta \in [-r, 0]. \quad (36)$$

The equations (23) and (24) take the form

$$2g\alpha = -w, \quad (37)$$

$$g = a - c/d - 1. \quad (38)$$

where  $w > 0$  and  $g < 0$ .

It is because the Lyapunov functional is positively defined and its derivative on the trajectory of the system (32) is negatively defined. From equation (35) we obtain

$$\alpha = -\frac{w}{2g}. \quad (39)$$

According to the equation (27)

$$\beta(\theta) = -\frac{wb}{g} \exp(g(\theta + r)) \quad \text{for } \theta \in [-r, 0]. \quad (40)$$

From the formula (31) we obtain

$$\delta(\theta, \xi) = b\beta(\xi - \theta - r) = -\frac{wb^2}{g} \exp(g(\xi - \theta)). \quad (41)$$

## 5. Conclusions

The paper presents the procedure of determining of the coefficients of the Lyapunov functional given by the formula (6) for the linear system with two delays both retarded and neutral type time delay, described by equation (1). This paper extends the method presented by Repin to the systems with two delays both retarded and neutral type time delay. Presented method allows achieving the analytical formula on the factors occurring in the Lyapunov functional, which can be used to examine the stability and in the process of the parametric optimization to designate the square index of the quality given by the formula (10).

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