Robust output predictive sequential controller design

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The paper addresses design problem of a robust parameter dependent quadratically stabilizing output/state feedback model predictive control for linear polytopic systems without constraints using original sequential approach. The design procedure ensures stability, robustness properties and guaranteed cost for the closed-loop uncertain system.

Key words: model predictive control, robust control, parameter dependent quadratic stability, Lyapunov function, polytopic system, sequential approach

1. Introduction

Model predictive control (MPC) has attracted notable attention in control of dynamic systems in the recent two decades. The idea of MPC can be summarized as follows, ([3], [11], [20]):

- Predict the future behavior of the process state/output over the finite time horizon.

- Compute the future input signals on line at each step by minimizing a cost function under inequality constraints on the manipulated (control) and/or controlled variables horizon.

- Apply on the controlled plant only the first element of vector control variable and repeat the previous steps with new measured input/state/output variables.

The presence of the plant model is therefore a necessary condition for the development of the predictive control. The success of MPC depends on the degree of precision of the plant model. Two typical description of model uncertainty, state space polytope and bounded unstructured uncertainty, are extensively considered in the field of robust model predictive control. Most of existing techniques for the robust MPC assume measurable state, and apply plant state feedback or, if the state estimator is utilized then

the output feedback is applied. Some results in the field of the robust MPC design can be summarized as follows:

Analysis of robustness properties of MPC.

For single-input and single-output (SISO) systems and impulse response model Zafiriou and Marchal [23] have used the contraction properties of MPC to develop necessary-sufficient conditions for robust stability of MPC with input and output constraints. Polak and Yang [16] have analyzed robust stability of MPC using a contraction constraint on the state.

MPC with explicit uncertainty description.

For SISO finite impulse response (FIR) plants Zheng and Morari [25] have presented robust MPC schemes with uncertainty bounds on the impulse response coefficients. Some MPC approaches consider additive type of uncertainty, as de la Pena *et al.* in [15] or parametric (structured) type uncertainty using CARIMA model and linear matrix inequality, as Bouzouita *et al.* in [2]. In Wang *et al.* [24] generalized predictive control (GPC) design technique is used for multi-input and multi-output (MIMO) uncertain system. In Lovas *et al.* [10] the unstructured uncertainty is used for open-loop stable systems with input constraints. The robust stability can be established by choosing the large value for the control input weighting matrix R in the cost function. The authors have proposed a new less conservative stability test for determining a sufficiently large control penalty R using bilinear matrix inequality (BMI). The other technique constrained tightening to design of the robust MPC have been proposed by Kuwata *et al.* [9]. The above approaches are based on idea of increasing the robustness of the control system by tightening the constraints on the predicted states.

The mixed H_2/H_∞ control approach.

This design method has been proposed by Orukpe *et al.* [13].

Robust constrained MPC using linear matrix inequality (LMI).

LMI have been proposed by Kothare *et al.* [8]. Here, the polytopic model or structured feedback uncertainty model have been used. The main idea is using infinite control horizon laws which for state feedback guarantees robust stability. In Ding *et al.* [7] output feedback robust MPC for systems with both polytopic and bounded uncertainty with input/state constraints is presented. In an off-line stage a sequence of output feedback laws based on the state estimators is calculated, by solving LMI optimization problem. in an on-line stage, at each sampling time, an appropriate output feedback law from this sequence is chosen. Robust MPC design with one step ahead prediction is proposed in Veselý and Rosinová [21]. An extension of two degree of generalized predictive control for multivariable systems is proposed in Yanou *et al.* [22].

In this paper a new MPC algorithm is proposed pursuing the idea of Veselý and Rosinová [21]. Proposed robust MPC control algorithm is designed sequentially. Note that

within sequentially design procedure the degree of plant model does not change when the output prediction horizon changes. The proposed sequential robust MPC design procedure consists of two steps: In the first step and one step ahead prediction horizon the necessary and sufficient robust stability conditions have been developed for MPC and polytopic system with output feedback using generalized parameter dependent Lyapunov matrix $P(\alpha)$. The proposed robust MPC algorithm ensures parameter dependent quadratic stability (PDQS) and guaranteed cost. In the second step of design procedure the nominal plant model is used to design the predicted input variables $u(t+1),\ldots,u(t+N-1)$ so that the robust closed-loop stability of MPC and guaranteed cost are ensured. Thus, input variable $u(t)$ guarantees the performance and robustness properties of the closed-loop system and predicted input variables $u(t+1),\ldots,u(t+N-1)$ guarantee the performance and closed-loop stability of uncertain plant model and nominal model prediction.

The paper is organized as follows: Section 2 presents preliminaries and problem formulation. In Section 3 the main results are given and finally, in Section 4 two examples using Yalmip BMI solvers show the effectiveness of the proposed method.

Hereafter, the following notational conventions will be adopted: given a symmetric matrix $P = P^T \in R^{n \times n}$, the inequality $P > 0, (P \geqslant 0)$ denotes matrix positive definiteness (semi-definiteness). Given two symmetric matrices P, Q, the inequality $P > Q$ indicates that $P - Q > 0$. The notation $x(t+k)$ will be used to define at time t k-steps ahead prediction of a system variable x from time t onwards under specified initial state and input scenario. That is estimated predicted output at time $k = 1, 2, \ldots$ $y(t+k|t)$ will be denoted as $y(t+k)$. I denotes the identity matrix of the corresponding dimensions.

2. Problem formulation and preliminaries

Consider a time invariant linear discrete-time uncertain polytopic system

$$x(t+1) = A(\alpha)x(t) + B(\alpha)u(t)$$
$$y(t) = Cx(t) \tag{1}$$

where $x(t) \in R^n, u(t) \in R^m, y(t) \in R^l$ are state, control and output variables of the system, respectively; $A(\alpha), B(\alpha)$ belong to the convex set

$$S = \{A(\alpha) \in R^{n \times n}, B(\alpha) \in R^{n \times m}\}$$
$$\{A(\alpha) = \sum_{j=1}^{M} A_j \alpha_j \quad B(\alpha) = \sum_{j=1}^{M} B_j \alpha_j, \alpha_j \geqslant 0\} \tag{2}$$
$$j = 1, 2, \ldots, M, \sum_{j=1}^{M} \alpha_j = 1.$$

Matrix C is known matrix of corresponding dimension. Jointly with the system (1), the following nominal plant model will be used

$$x(t+1) = A_n x(t) + B_n u(t)$$
$$y(t) = Cx(t)$$
(3)

where $(A_n, B_n) \in S$ are any matrices with constant entries.

The problem studied in this paper is summarized as follows: in the first step, parameter dependent quadratic stability conditions for output feedback and one step ahead robust model predictive control are derived for the polytopic system. The control algorithm is assumed to be given in the following form:

$$u(t) = F_{11}y(t) + F_{12}y(t+1).$$
(4)

In the second step of the design procedure, the nominal model and a given prediction horizon N is considered. The model predictive control is designed in the form:

$$u(t+k-1) = F_{kk}y(t+k-1) + F_{kk+1}y(t+k)$$
(5)

where $F_{ki} \in R^{m \times l}$, $k = 2, 3, \ldots, N$, $i = k+1$ is the output (state) feedback gain matrices to be determined so that the cost function given below is optimal with respect to the system variables. We would like to stress that $y(t+k-1), y(t+1)$ are predicted outputs obtained from predictive model (12).

Substituting control algorithm (4) to (3) we obtain

$$x(t+1) = D_1(j)x(t)$$
(6)

where

$$
\begin{aligned}
D_1(j) &= A_j + B_j K_1(j) \\
K_1(j) &= (I - F_{12}CB_j)^{-1}(F_{11}C + F_{12}CA_j), \quad j = 1, 2, \ldots, M.
\end{aligned}
$$

For the first step of design procedure, the cost function to be minimized is given as follows:

$$J_1 = \sum_{t=0}^{\infty} J_1(t)$$
(7)

where

$$J_1(t) = x(t)^T Q_1 x(t) + u(t)^T R_1 u(t)$$

and Q_1, R_1 are positive definite matrices of the corresponding dimensions. For the case of $k = 2$ we obtain

$$u(t+1) = F_{22}CD_1(j)x(t) + F_{23}C(A_n D_1(j)x(t) + B_n u(t+1))$$

or

$$u(t+1) = K_2(j)x(t)$$

and closed-loop system

$$x(t+2) = (A_n D_1(j) + B_n K_2(j))x(t) = D_2(j)x(t), \quad j = 1, 2, \ldots, M.$$

Sequentially, for the case k-step ahead prediction ($k = N \geqslant 2$), we obtain the following closed-loop system

$$x(t+k) = (A_n D_{k-1}(j) + B_n K_k(j))x(t) = D_k(j)x(t) \tag{8}$$

where

$$
\begin{aligned}
D_0 &= I \\
D_k(j) &= A_n D_{k-1}(j) + B_n K_k(j) \\
K_k(j) &= (I - F_{kk+1}CB_n)^{-1}(F_{kk}C + F_{kk+1}CA_n)D_{k-1}(j) \\
&\qquad k = 2, 3, \ldots, N, \quad j = 1, 2, \ldots, M.
\end{aligned}
$$

For the second step of the robust MPC design procedure and k prediction horizon the cost function to be minimized is given as follows:

$$J_k = \sum_{t=0}^{\infty} J_k(t) \tag{9}$$

where

$$J_k(t) = x(t)^T Q_k x(t) + u(t+k-1)^T R_k u(t+k-1)$$

and $Q_k, R_k, k = 2, 3, \ldots, N$ are positive definite matrices of the corresponding dimensions. We proceed with following two lemmas and definition.

Lemma 1 *The closed-loop system matrix of discrete-time system* (1) *is robustly stable if and only if there exists a symmetric positive definite parameter dependent Lyapunov matrix* $0 < P(\alpha) = P(\alpha)^T < I\rho$ *such that*

$$-P(\alpha) + D_1(\alpha)^T P(\alpha) D_1(\alpha) \leqslant 0 \tag{10}$$

where $D_1(\alpha)$ *is the closed-loop polytopic system matrix for system* (1).

Definition 1 *Consider the system* (1). *If there exists a control algorithm* $u(t)^*$ *and a positive scalar* J_1^* *such that the closed-loop system* (1) *with* (4) *is stable and closed-loop value of cost function* (7) *satisfies* $J_1 \leqslant J_1^*$, *then* J_1^* *is said to be guaranteed cost and* $u(t)^*$ *is said to be guaranteed cost control law for the system* (1).

The necessary and sufficient robust stability condition for closed-loop polytopic system with guaranteed cost is given by recent result (Rosinová *et al.* [19]).

Lemma 2 *Consider the system* (1) *with control algorithm* (4). *The control algorithm* (4) *is the guaranteed cost control law for the closed-loop system if and only if the following condition holds*

$$
\begin{aligned}
B_e \;=\;& D_1(\alpha)^T P(\alpha) D_1(\alpha) - P(\alpha) + Q_1 + \\
& (F_{11}C + F_{12}CD_1(\alpha))^T R_1 (F_{11}C + F_{12}CD_1(\alpha)) \leqslant 0.
\end{aligned}
\tag{11}
$$

For the nominal model and $k = 1, 2, \ldots, N$ the model prediction can be obtained in the form

$$
\begin{aligned}
z(t+1) &= A_f z(t) + B_f v(t) \\
y_f(t) &= C_f z(t)
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
z(t)^T &= [x(t)^T, \ldots, x(t+N-1)^T] \\
v(t)^T &= [u(t)^T, \ldots, u(t+N-1)^T] \\
y_f(t)^T &= [y(t)^T, \ldots, y(t+N-1)^T]
\end{aligned}
$$

$$
A_f = \begin{bmatrix}
A_n & 0 & 0 & \ldots & 0 \\
A_n D_1 & 0 & 0 & \ldots & 0 \\
A_n D_2 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_n D_{N-1} & 0 & 0 & \ldots & 0
\end{bmatrix} \in R^{nN \times nN}
$$

$$
\begin{aligned}
B_f &= \text{blockdiag}\{B\}_{nN \times mN} \\
C_f &= \text{blockdiag}\{C\}_{lN \times nN}.
\end{aligned}
$$

Remarks

- Control algorithm for $k = N$ is $u(t+N-1) = F_{NN} y(t+N-1)$.

- If one wants to use the control horizon $N_u < N$ [3], the control algorithm $u(t+k-1) = 0$, $K_k = 0$, $F_{N_{u+1}N_{u+1}} = 0$, $F_{N_{u+1}N_{u+2}} = 0$ for $k > N_u$.

- Note that model prediction (12) is calculated using nominal model (3), that is $D_0 = I$, $D_k = A_n D_{k-1} + B_n K_k$, $D_k(j)$ is used for robust controller design.

3. Main results

3.1. Robust MPC controller design – first step

The main results of this paper for the first step of design procedure can be summarized in the following theorem.

Theorem 1 *The system* (1) *with control algorithm* (4) *is parameter dependent quadratically stable with parameter dependent Lyapunov function* $V(t) = x(t)^T P(\alpha) x(t)$ *if and only if there exit matrices* N_{11}, N_{12} *and* F_{11}, F_{12} *such that the following bilinear matrix inequality holds:*

$$B_e = \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^T & G_{22} \end{bmatrix} \leqslant 0 \tag{13}$$

where

$$
\begin{aligned}
G_{22} &= N_{12}^T A_c(\alpha) + A_c(\alpha)^T N_{12} - P(\alpha) + Q_1 + C^T F_{11}^T R_1 F_{11} C \\
G_{12}^T &= A_c(\alpha)^T N_{11} + N_{12}^T M_c(\alpha) + C^T F_{11}^T R_1 F_{12} C \\
G_{11} &= N_{22}^T M_c(\alpha) + M_c(\alpha)^T N_{22} + C^T F_{12}^T R_1 F_{12} C + P(\alpha) \\
&\quad M_c(\alpha) = B(\alpha) F_{12} C - I \\
&\quad A_c(\alpha) = A(\alpha) + B(\alpha) F_{11} C.
\end{aligned}
$$

Note that (13) is affine with respect to α. Substituting (2) and $P(\alpha) = \sum_{i=1}^{M} \alpha_i P_i$ to (13) for the polytopic system the following BMI is obtained

$$B_{ie} = \begin{bmatrix} G_{11i} & G_{12i} \\ G_{12i}^T & G_{22i} \end{bmatrix} \leqslant 0 \quad i = 1, 2, \dots, M \tag{14}$$

where

$$
\begin{aligned}
G_{11i} &= N_{22}^T M_{ci} + M_{ci}^T N_{22} + C^T F_{12}^T R_1 F_{12} C + P_i \\
G_{12i}^T &= A_{ci}^T N_{22} + N_{12}^T M_{ci} + C^T F_{11}^T R_1 F_{12} C \\
G_{22i} &= N_{12}^T A_{ci} + A_{ci}^T N_{12} - P_i + Q_1 + C^T F_{11}^T R_1 F_{11} C \\
&\quad M_{ci} = B_i F_2 C - I \\
&\quad A_{ci} = A_i + B_i F_1 C.
\end{aligned}
$$

Proof. For the proof of this theorem see the proof of theorem 2.

If the solution of (14) is feasible with respect to symmetric matrices $P_i = P_i^T > 0$, $i = 1, 2, \dots, M$, and matrices N_{11}, N_{12}, within the convex set defined by (2) then the gain matrices F_{11}, F_{12} ensure the guaranteed cost and parameter dependent quadratic stability (PDQS) of the closed-loop polytopic system for one step ahead predictive control.

Note that:

- For concrete matrix $P(\alpha) = \sum_{i=1}^{M} \alpha_i P_i$ BMI robust stability conditions 'if and only if' in (13) reduces in (14) to BMI conditions 'if'.

- If in (14) $P_i = P_j = P$, $i = j = 1, 2, \ldots, M$ then the feasible solution of (14) with respect to matrices N_{11}, N_{12}, symmetric positive definite matrix P and the gain matrices F_{11}, F_{12} guarantee quadratic stability and guaranteed cost for one step ahead predictive control closed-loop polytopic system within the convex set defined by (2). Quadratic stability gives more conservative results than PDQS. Conservatism of real results depends on the concrete examples.

Assume that the BMI solution of (14) is feasible, then for nominal plant one can calculate the matrices D_1 and K_1 using (6). For second step of MPC design procedure, the obtained nominal model will be used.

3.2.   Model predictive controller design – second step

The aim of the second step of predictive control design procedure is to design gain matrices F_{kk}, F_{kk+1}, $k = 2, 3, \ldots, N$ such that the closed-loop system with nominal model is stable with guaranteed cost. In order to design model predictive controller with output feedback in the second step of design procedure we proceed with the following corollary and theorem.

Corollary (Lemma 2) *The closed-loop system* (8) *or rewritten as* (17) *is stable with guaranteed cost iff the following inequality holds*

$$B_{ek}(t) = \Delta V_k(t) + x(t)^T Q_k x(t) + u(t+k-1)^T R_k u(t+k-1) \leqslant 0 \qquad (15)$$

where $\Delta V_k(t) = V_k(t+k) - V_k(t)$ *and* $V_k(t) = x(t)^T P_k x(t)$, $P_k = P_k^T > 0$, $k = 2, 3, \ldots, N$.

Theorem 2 *The closed-loop system* (8) *is stable with guaranteed cost iff for* $k = 2, 3, \ldots, N$ *there exist matrices*

$$F_{kk}, \ F_{kk+1}, \ N_{k1} \in R^{n \times n}, \ N_{k2} \in R^{n \times n}$$

and positive definite matrix $P_k = P_k^T \in R^{n \times n}$ *such that the following bilinear matrix inequality holds*

$$B_{e2} = \left[\begin{array}{cc} G_{k11} & G_{k12} \\ G_{k12}^T & G_{k22} \end{array} \right] \leqslant 0 \qquad (16)$$

where

$$
\begin{aligned}
G_{k11} &= N_{k1}^T M_{ck} + M_{ck}^T N_{k1} + C^T F_{kk+1}^T R_k F_{kk+1} C + P_k \\
G_{k12}^T &= D_{k-1}(j)^T C^T F_{kk}^T R_k F_{kk+1} C + D_{k-1}(j)^T A_{ck}^T N_{k1} + N_{k2}^T M_{ck}
\end{aligned}
$$

$$G_{k22} = Q_k - P_k + D_{k-1}(j)^T C^T F_{kk}^T R_k F_{kk} C D_{k-1}(j) + N_{k2}^T A_{ck} D_{k-1}(j) + D_{k-1}(j)^T A_{ck}^T N_{k2}$$

and

$$M_{ck} = B_n F_{kk+1} C - I$$
$$A_{ck} = A_n + B_n F_{kk} C$$
$$D_k(j) = A_n D_{k-1}(j) + B_n K_k(j)$$
$$K_k(j) = (I - F_{kk+1} C B_n)^{-1}(F_{kk} C + F_{kk+1} C A_n) D_{k-1}(j), j = 1, 2, \ldots, M.$$

Proof. *Sufficiency.*

The closed-loop system (8) can be rewritten as follows

$$x(t+k) = -(M_{ck})^{-1} A_{ck} D_{k-1}(j) x(t) = A_{clk} x(t). \tag{17}$$

Because the matrix (j is omitted)

$$U_k^T = [-D_{k-1}^T A_{ck}^T (M_{ck})^{-1} \quad I]$$

has full row rank, multiplying from the left and right hand side of (16) the inequality equivalent to (16) is obtained. Multiplying the results from left by $x(t)^T$ and right by $x(t)$, taking into account the closed-loop matrix (17) the inequality (15) is obtained, which proves the sufficiency.

Necessity.

Suppose that for *k*-step ahead model predictive control there exists such matrix $0 < P_k = P_k^T < I\rho$ that (15) holds. Necessary, there exists a scalar $\beta > 0$ such that for the first difference of Lyapunov function (15) holds the following

$$A_{clk}^T P_k A_{clk} - P_k \leqslant -\beta(A_{clk}^T A_{clk}). \tag{18}$$

Inequality (18) can be rewritten as follows

$$A_{clk}^T (P_k + \beta I) A_{clk} - P_k \leqslant 0.$$

Using Schur complement formula one obtains

$$\begin{bmatrix} -P_k & -A_{clk}^T(P_k + \beta I) \\ (P_k + \beta I) A_{clk} & -(P_k + \beta I) \end{bmatrix} \leqslant 0 \tag{19}$$

and taking

$$N_{k1} = -(M_{ck})^{-1}(P_k + \beta I/2)$$
$$N_{k2}^T = -D_{k-1}^T A_{ck}^T (M_{ck}^{-1})^T M_{ck}^{-1} \beta/2$$

results in

$$
\begin{aligned}
&-A_{clk}^T(P_k + \beta I) = D_{k-1}^T A_{ck}^T N_{k1} + N_{k2}^T M_{ck} \\
&-P_k = -P_k + N_{k2}^T A_{ck} D_{k-1} + D_{k-1}^T \\
&A_{ck}^T N_{k2} + \beta(D_{k-1}^T A_{ck}^T (M_{ck}^{-1})^T M_{ck}^{-1} A_{ck} D_{k-1}) - (P_k + \beta I) = 2M_{ck}N_{k1} + P_k.
\end{aligned}
\tag{20}
$$

Substituting (20) to (19) for $\beta \to 0$ one has got the inequality (16) for the case of $Q_k = 0$, $R_k = 0$. If one substitute to second part of (15) instead of $u(t+k-1)$ (5), rewrite the obtained result into the matrix form and add previously obtained matrix inequality (16) then the necessity condition of the theorem is proven. This completes the proof. $\qquad\square$

If there exist feasible solution of (16) with respect to matrices $F_{kk}, F_{kk+1}, N_{k1} \in R^{n\times n}$, $N_{k2} \in R^{n\times n}$, $k = 2,3,\ldots,N$ and positive definite matrix $P_k = P_k^T \in R^{n\times n}$, then the designed MPC ensure the quadratic stability of the closed-loop system and guaranteed cost.

Remarks

- Due to the proposed design philosophy predictive control algorithm $u(t+k), k \geqslant 1$ is the function of corresponding performance index (9) and previous closed-loop system matrix.

- In the proposed design approach constraints on system variables are easy to be included by LMI (BMI) using a notion of invariant set [1], [18].

- The proposed MPC with sequential design is a special case of classical MPC. Sequential MPC may not provide 'better' dynamic behavior than classical one but it is another approach to design of MPC.

- Note that in the proposed model predictive control sequential design procedure, the size of system does not change with increasing N.

- If in the convex set (2) there exists feasible solution for both step, the proposed algorithm (4) and model predictive control (5) guarantee the PDQS and robustness properties of the closed-loop MPC system with guaranteed cost.

The sequential robust MPC design procedure can be summarized in the following steps:

- Design of robust MPC controller with control algorithm (4) using (14).

- For nominal and uncertain model of system calculate matrices K_1, D_1 and $K_1(j)$, $D_1(j)$, $j = 1,2,\ldots,M$ given in (6).

- For a given $k = 2,3,\ldots,N$ and control algorithm (5) sequentially calculate F_{kk}, F_{kk+1} using (16) and K_k, D_k given in (8).

- Calculate the matrices A_f, B_f and C_f (12) for model prediction.

4. Examples

Example 1. First example serves as a benchmark. The model of a double integrator turns to (3) where

$$A_n = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$B_n = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and uncertainty matrices are

$$A_{1u} = \begin{bmatrix} 0.01 & 0.01 \\ 0.02 & 0.03 \end{bmatrix}$$

$$B_{1u} = \begin{bmatrix} 0.001 \\ 0 \end{bmatrix}.$$

For the case when number of uncertainties $p = 1$ the number of vertices is $M = 2^p = 2$, the matrices (2) are calculated as follows

$$A_1 = A_n - A_{1u}, A_2 = A_n + A_{1u}$$

$$B_1 = B_n - B_{1u}, B_2 = B_n + B_{1u}.$$

For the parameters of $\rho = 20000$, prediction $N = 4$, $N_u = 4$, performance $R1 = \cdots = R4 = 1$, $Q1 = 0.1I$, $Q2 = 0.5I$, $Q3 = I$, $Q4 = 5I$ the following results are obtained using the sequential design approach proposed in the paper:

- For prediction $k = 1$, the robust control algorithm is given as follows

$$u(t) = F_{11}y(t) + F_{12}y(t+1).$$

  Using (14), one obtains the following gain matrices $F_{11} = 0.9189$, $F_{12} = -1.4149$. The eigenvalue of closed-loop first vertex model system are as follows

$$\text{Eig(Closed-loop)} = \{0.2977 \pm 0.0644i\}.$$

- For $k = 2$, control algorithm is

$$u(t+1) = F_{22}y(t+1) + F_{23}y(t+2).$$

  In the second step of design procedure gain matrices obtained using (16) are $F_{22} = 0.4145$, $F_{23} = -0.323$. The eigenvalues of closed-loop first vertex model system are as follows

$$\text{Eig(Closed-loop)} = \{0.1822 \pm 0.1263i\}.$$

- For $k = 3$, control algorithm is given

$$u(t+2) = F_{33}y(t+2) + F_{34}y(t+3).$$

In the second step of design procedure the obtained gain matrices are $F_{33} = 0.2563; F_{34} = -0.13023$. The eigenvalues of closed-loop first vertex model system are as follows

$$\text{Eig(Closed-loop)} = \{0.1482 \pm 0.051i\}.$$

- For prediction $k = N = 4$, control algorithm is given

$$u(t+3) = F_{44}y(t+3) + F_{45}y(t+4).$$

In the second step the obtained gain matrices are $F_{44} = 0.5797$, $F_{45} = 0.0$. The eigenvalues of closed-loop first vertex model system are as follows

$$\text{Eig(Closed-loop)} = \{0.1002 \pm 0.145i\}.$$

Example 2. Nominal model for the second example is given as follows

$$A_n = \begin{bmatrix} 0.6 & 0.0097 & 0.0143 & 0 & 0 \\ 0.012 & 0.9754 & 0.0049 & 0 & 0 \\ -0.0047 & 0.01 & 0.46 & 0 & 0 \\ 0.0488 & 0.0002 & 0.0004 & 1 & 0 \\ -0.0001 & 0.0003 & 0.0488 & 0 & 1 \end{bmatrix}$$

$$B_n = \begin{bmatrix} 0.0425 & 0.0053 \\ 0.0052 & 0.01 \\ 0.0024 & 0.0001 \\ 0 & 0.0012 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The linear affine type model of uncertain system (2) can be obtained in the form

$$A_i = A_n + \theta_1 A_{1u}, \quad B_i = B_n + \theta_1 B_{1u}$$

$$C_i = C, \quad i = 1, 2$$

where A_{1u}, B_{1u} are uncertainty model of the system with constant entries, θ_1 is an uncertain real parameter with $\theta_1 \in <\underline{\theta_1}, \overline{\theta_1}>$. When lower and upper bounds of the uncertain

parameter θ_1 are substituted to affine type model, the polytopic system (1) is obtained. Let $\theta_1 \in < -1, 1 >$ and

$$A_{1u} = \begin{bmatrix} 0.025 & 0 & 0 & 0 & 0 \\ 0 & 0.021 & 0 & 0 & 0 \\ 0 & 0 & 0.0002 & 0 & 0 \\ 0.001 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0001 & 0 & 0 \end{bmatrix}$$

$$B_{1u} = \begin{bmatrix} 0.0001 & 0 \\ 0 & 0.001 \\ 0 & 0.0021 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

In this example two vertices ($M = 2$) are calculated (see *Example 1*). The problem is to design two PS(PI) model predictive robust decentralized controllers for plant input $u(t)$ and prediction horizon $N = 5$ using sequential design approach. The cost function is given by following matrices

$$Q_1 = Q_2 = Q_3 = I, \quad R_1 = R_2 = R_3 = I,$$

$$Q_4 = Q_5 = 0.5I, \quad R_4 = R_5 = I.$$

In the first step calculation for the uncertain system (1) yields the robust control algorithm

$$u(t) = F_{11} y(t) + F_{12} y(t+1)$$

where the matrix F_{11} with decentralized output feedback structure containing two PS controllers, is designed.

From (14), the obtained gain matrices F_{11} and F_{12} are

$$F_{11} = \begin{bmatrix} -18.7306 & 0 & -42.4369 & 0 \\ 0 & 8.8456 & 0 & 48.287 \end{bmatrix}$$

where decentralized proportional and integral gains for the first controller are

$$K_{1p} = 18.7306, \quad K_{1i} = 42.4369$$

and for the second one

$$K_{2p} = -8.8456, K_{2i} = -48.287.$$

Note that in F_{11} sign - shows the negative feedback. Because predicted output $y(t+1)$ is obtained from model prediction (12), for output feedback gain matrix F_{12} there is no need to use decentralized control structure

$$F_{12} = \begin{bmatrix} -22.0944 & 20.2891 & -10.1899 & 18.2789 \\ -29.3567 & 8.5697 & -28.7374 & -40.0299 \end{bmatrix}.$$

In the second step of the design procedure, using (16) for nominal model, the matrices (5) $F_{kk}, F_{kk+1}, k = 2, 3, 4, 5$ are calculated. The eigenvalues of the closed-loop first vertex model system for $N = N_u = 5$ are as follows

$$\text{Eig(Closed-loop)} = \{-0.0009; -0.0087; 0.9789; 0.8815; 0.8925\}.$$

The feasible solutions of bilinear matrix inequality have been obtained by YALMIP with PENBMI solver.

5. Conclusion

The paper addresses the design problem of a new MPC with special control algorithm. Because proposed robust MPC control algorithm is designed sequentially, the degree of plant model does not change when the output prediction horizon changes. The proposed sequential robust MPC design procedure consists of two steps. In the first step for one step ahead prediction horizon the necessary and sufficient robust stability conditions have been developed for MPC and the polytopic system with output feedback using generalized parameter dependent Lyapunov matrix $P(\alpha)$. The proposed robust MPC ensures parameter dependent quadratic stability (PDQS) and guaranteed cost. In the second step of the design procedure the plant uncertain, nominal model and sequential design approach is used to design the predicted input variables $u(t+1), \ldots, u(t+N-1)$ so the robust closed-loop stability of MPC and guaranteed cost is ensure. Main advantages of the proposed sequential method are: the design plant model degree is independent on prediction horizon N; robust controller design procedure ensures PDQS and guaranteed cost and, the obtained results are easy to be implemented in real plant. In the proposed design approach constraints on system variables are easy to be implemented by LMI (BMI) using a notion of invariant set. The feasible solution of BMI has been obtained by YALMIP with PENBMI solver.

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