Central European Journal of Economic Modelling and Econometrics

A Note on Some Extensions of the Matrix Angular Central Gaussian Distribution

Justyna Wróblewska*

Submitted: 2.04.2025, Accepted: 19.09.2025

Abstract

In this paper, we extend the concept of a matrix angular central Gaussian (MACG) distribution to the complex domain. First, we consider a normally distributed random complex matrix (Z) and demonstrate that its orientation ($H_Z = Z(\bar{Z}'Z)^{-1/2}$) exhibits a complex MACG (CMACG) distribution. We then discuss the distribution of the orientation of the linear transformation of the random matrix, the orientation of which has a CMACG distribution. Finally, we examine the family of distributions that leads to the CMACG distribution.

Keywords: complex Stiefel manifold, complex Grassmann manifold, matrix angular central Gaussian distribution

JEL Classification: C11, C15, C46

*Krakow University of Economics, Department of Econometrics and Operational Research, Poland; e-mail: eowroble@cyf-kr.edu.pl, wroblewj@uek.krakow.pl; ORCID: 0000-0002-9789-2601

1 Introduction

The matrix angular central Gaussian (MACG) distribution has been introduced by Chikuse (1990), with its density and properties described therein. The distribution is defined for the elements of the Stiefel manifold. The MACG distribution proved to be very useful in the Bayesian analysis of cointegration (see, e.g., Koop, León-González, Strachan, 2009) and in the Bayesian models that combine cointegration with the idea of common cyclical features (see Wróblewska, 2011, 2012, 2015). One of the main advantages of using the MACG distribution in Bayesian analyses is that it is easy to obtain a pseudo-random sample from it. Moreover, as the MACG distribution is invariant to the right orthonormal transformations, it can be treated as the distributions defined on the Grassmann manifolds. This feature is advantageous for the aforementioned analysis, since the data contains only information about cointegration and common feature spaces, rather than the vectors that span them. Finally, through the parameter of the MACG distribution, the researcher can easily and transparently incorporate prior information about the analyzed spaces. However, if the researcher is interested in the analysis of the seasonally cointegrated process (see, e.g., Hylleberg et al., 1990, Johansen, Schaumburg, 1999, Cubadda, Omtzigt, 2005) within the Bayesian paradigm, the generalization of MACG to the complex case proves useful (see Wróblewska, 2025). Another potential area of application for the discussed distribution is represented by cyclically cointegrated processes, see, e.g., Arteche, Robinson (1999), Gregoir (1999a, 1999b).
The proposed distribution extends a complex angular central Gaussian distribution (ACG, see, e.g., Mardia, Jupp, 2009, pp. 343-344) to the matrix case. ACG has been employed in various engineering applications, such as shape analysis and signal processing (see Kent, 1997, Micheas et al., 2006, Ollia et al., 2012, Abramovich, Besson, 2013a, 2013b, Dryden, Mardia, 2016).
Before defining the complex central Gaussian matrix (CMACG) distribution and its properties, we present the definition of the complex Stiefel manifold and the measure defined on it.
The set of $m \times r$ ($m \geq r$) semi-unitary matrices, i.e. matrices fulfilling the condition $\bar{X}'X = I_r$, where $\bar{X}'$ denotes the conjugate transpose of X and I_r is the $r \times r$ identity matrix, is called the complex Stiefel manifold ($V_{r,m}^{\mathbb{C}}$):

$$V_{r,m}^{\mathbb{C}} = \left\{ X_{m\times r} : \bar{X}'X = I_r,\ m \geq r \right\}.$$

An invariant measure on $V_{r,m}^{\mathbb{C}}$ is given by the differential form (Díaz-García, Gutiérrez-Jáimez, 2011):

$$(\bar{X}'\ \mathrm{d}X) = \bigwedge_{i=1}^{r} \bigwedge_{j=i+1}^{m} \bar{x}_j'\ \mathrm{d}x_i,$$

where $\bigwedge$ denotes the exterior product and the matrix X_1 is chosen such that $\mathbf{X} = (X \mid X_1) = (x_1, \ldots, x_r \mid x_{r+1}, \ldots, x_m)$ is an element of the unitary group, $U(m)$,

i.e. the group of all $m \times m$ complex unitary matrices: $\bar{\mathbf{X}}'\mathbf{X} = I_m$. It can be shown that this differential form does not depend on the choice of the matrix X_1.
The volume of the complex Stiefel manifold is

$$Vol(V_{r,m}^{\mathbb{C}}) = \int_{X \in V_{r,m}^{\mathbb{C}}} (\bar{X}' \, \mathrm{d}X) = \frac{2^r \pi^{mr}}{\Gamma_r^{\mathbb{C}}[m]},$$

where $\Gamma_r^{\mathbb{C}}[a]$ denotes the complex multivariate Gamma function, and is defined by:

$$\Gamma_r^{\mathbb{C}}[a] = \int_{A_{r\times r}>0, \bar{A}'=A} \exp\{-tr(A)\}|A|^{a-r}(\mathrm{d}A) = \pi^{r(r-1)/2} \prod_{i=1}^{r} \Gamma[a-i+1],$$

where $tr(\cdot)$ denotes the trace, $|\cdot|$ – the determinant and $Re(a) > m-1$ (see Gross, Richards, 1987, Díaz-García, Gutiérrez-Jáimez, 2011).
The normalized invariant measure ($[dX]$) of unit mass on the considered manifold is defined as:

$$[\mathrm{d}X] = \frac{(\bar{X}' \, \mathrm{d}X)}{Vol(V_{r,m}^{\mathbb{C}})} = \frac{\Gamma_r^{\mathbb{C}}[m]}{2^r \pi^{mr}} (\bar{X}' \, \mathrm{d}X). \tag{1}$$

The next two theorems provide Jacobians with a transformation which will be used in the paper.

Theorem 1 (Díaz-García, Gutiérrez-Jáimez, 2011)**.** *If* $Y = AXB + C$, *where* $X \in \mathbb{C}^{m\times r}$ *and* $Y \in \mathbb{C}^{m\times r}$ *are random matrices and* $A \in \mathbb{C}^{m\times m}$, $|A| \neq 0$, $B \in \mathbb{C}^{r\times r}$, $|B| \neq 0$, $C \in \mathbb{C}^{m\times r}$ *are matrices of constants, then*

$$(\mathrm{d}Y) = |\bar{A}'A|^r |\bar{B}'B|^m (\mathrm{d}X), \tag{2}$$

so that $J(Y \to X) = |\bar{A}'A|^r |\bar{B}'B|^m$.

Theorem 2 (Polcari, 2017)**.** *If* $Y = BX\bar{B}'$, *where* $X \in \mathbb{C}^{m\times m}$ *and* $Y \in \mathbb{C}^{m\times m}$ *are random Hermitian matrices* ($\bar{X}' = X$, $\bar{Y}' = Y$) *and* $B \in \mathbb{C}^{m\times m}$ *is a non-singular* ($|B| \neq 0$) *matrix of constants, then*

$$(\mathrm{d}Y) = |B|^{2m} (\mathrm{d}X), \tag{3}$$

so that $J(Y \to X) = |B|^{2m}$.

2 Complex matrix angular central gaussian distribution

Following the idea of the MACG distribution of Chikuse (1990, 2003) we analyze the distribution of the "orientation" part (H_Z) of polar decomposition of the full column rank random matrix $Z_{m\times r}$, $m \geq r$, $r(Z) = r$.
The unique polar decomposition of Z is defined as:

$$Z = H_Z T_Z^{\frac{1}{2}}, \quad H_Z = Z(\bar{Z}'Z)^{-\frac{1}{2}}, \quad T_Z = \bar{Z}'Z.$$

Lemma 3. *The measure* $(\mathrm{d}Z)$ *is decomposed as*

$$(\mathrm{d}Z) = \frac{\pi^{mr}}{\Gamma_r^{\mathbb{C}}[m]} |T_Z|^{m-r} (\mathrm{d}T_Z)[\mathrm{d}H_Z]. \tag{4}$$

Proof. This is a direct consequence of the definition of the normalised invariant measure on the complex Stiefel manifold (see Equation 1) and decomposition of the measure $(\mathrm{d}Z)$, obtained by applying Proposition 4 in Díaz-García, Gutiérrez-Jáimez (2011) to the complex case ($\beta = 2$ in their notation):
$(\mathrm{d}Z) = 2^{-r}|T_Z|^{2(m-r+1)/2-1}(\mathrm{d}T_Z)(\bar{H}_Z' \,\mathrm{d}H_Z) = 2^{-r}|T_Z|^{m-r}(\mathrm{d}T_Z)(\bar{H}_Z' \,\mathrm{d}H_Z)$.

□

Using Lemma 3 we obtain the density of the orientation H_Z

$$f_{H_Z}(H_Z) = \frac{\pi^{mr}}{\Gamma_r^{\mathbb{C}}[m]} \int_{T>0, \bar{T}'=T} f_Z(H_Z T^{\frac{1}{2}}) |T|^{m-r} (\mathrm{d}T) \tag{5}$$

and of the product matrix T_Z

$$f_{T_Z}(T_Z) = \frac{\pi^{mr}}{\Gamma_r^{\mathbb{C}}[m]} |T_Z|^{m-r} \int_{V_{m,r}^{\mathbb{C}}} f_Z(H T_Z^{\frac{1}{2}})[\mathrm{d}H]. \tag{6}$$

Theorem 4. *Assume that* $Z_{m\times r}$ *has the* $m\times r$ *matrix-variate complex central normal distribution with the parameter* P, $Z \sim mN^{\mathbb{C}}(0, I_r, P)$, *where* P *is an* $m \times m$ *positive definite matrix and define* $H_Z = Z(\bar{Z}'Z)^{-\frac{1}{2}} \in V_{r,m}^{\mathbb{C}}$.
Then it is said that H_Z *has a complex matrix angular central Gaussian distribution with parameter* P, *denoted as* $H_Z \sim CMACG(P)$, *and its density is*

$$f_{H_Z}(H_Z) = |P|^{-r} |\bar{H}_Z' P^{-1} H_Z|^{-m}. \tag{7}$$

The density of the product matrix T_Z *is*

$$f_{T_Z}(T_Z) = \frac{|P|^{-r}|T_Z|^{m-r}}{\Gamma_r^{\mathbb{C}}[m]} \, {}_0F_0^{(r)}(P^{-1}, -T_Z), \tag{8}$$

where ${}_0F_0^{(k)}(M_1, M_2)$ *is the hypergeometric function with complex Hermitian matrix arguments.*

For the definition and properties of hypergeometric functions with (complex) matrix arguments, see, e.g. Constantine (1963) and James (1964).

Proof. The density of Z is

$$f_Z(Z) = \pi^{-mr}|P|^{-r} \exp[-tr(\bar{Z}' P^{-1} Z)],$$

so according to (4) the density of H_Z is obtained as

$$\begin{aligned}
f_{H_Z}(H_Z) \underbrace{=}_{(5)} & \frac{\pi^{mr}}{\Gamma_r^{\mathbb{C}}[m]} \int_{T>0,\bar{T}'=T} f_Z(H_Z T^{\frac{1}{2}})|T|^{m-r}(\mathrm{d}T) = \\
= & \frac{\pi^{mr}}{\Gamma_r^{\mathbb{C}}[m]} \int_{T>0,\bar{T}'=T} \pi^{-mr}|P|^{-r}\exp[-tr(T^{\frac{1}{2}}\bar{H}_Z' P^{-1} H_Z T^{\frac{1}{2}})]|T|^{m-r}(\mathrm{d}T) = \\
= & \frac{|P|^{-r}}{\Gamma_r^{\mathbb{C}}[m]} \int_{T>0,\bar{T}'=T} \exp[-tr(T^{\frac{1}{2}}\bar{H}_Z' P^{-1} H_Z T^{\frac{1}{2}})]|T|^{m-r}(\mathrm{d}T).
\end{aligned}$$

In the integral, make the change of variables $V = M^{\frac{1}{2}} T M^{\frac{1}{2}}$, where M stands for $\bar{H}_Z' P^{-1} H_Z$. By (3) $(\mathrm{d}T) = |M|^{-r}(\mathrm{d}V)$ so the integral becomes

$$\begin{aligned}
f_{H_Z}(H_Z) &= \frac{|P|^{-r}}{\Gamma_r^{\mathbb{C}}[m]} \int_{V>0,\bar{V}'=V} \exp[-tr(V)]|VM^{-1}|^{m-r}|M|^{-r}(\mathrm{d}V) = \\
&= \frac{|P|^{-r}}{\Gamma_r^{\mathbb{C}}[m]}|M|^{-m} \int_{V>0,\bar{V}'=V} \exp[-tr(V)]|V|^{m-r}(\mathrm{d}V) = \\
&= \frac{|P|^{-r}}{\Gamma_r^{\mathbb{C}}[m]}|M|^{-m}\Gamma_r^{\mathbb{C}}[m] = \\
&= |P|^{-r}|\bar{H}_Z' P^{-1} H_Z|^{-m}.
\end{aligned}$$

To obtain the density of T_Z we also use the decomposition (4), which leads to:

$$\begin{aligned}
f_{T_Z}(T_Z) \underbrace{=}_{(6)} & \frac{\pi^{mr}}{\Gamma_r^{\mathbb{C}}[m]}|T_Z|^{m-r} \int_{V_{m,r}^{\mathbb{C}}} f_Z(HT_Z^{\frac{1}{2}})[\mathrm{d}H] = \\
= & \frac{\pi^{mr}}{\Gamma_r^{\mathbb{C}}[m]}|T_Z|^{m-r} \int_{V_{m,r}^{\mathbb{C}}} \pi^{-mr}|P|^{-r}\exp[-tr(T_Z^{\frac{1}{2}}\bar{H}' P^{-1} H T_Z^{\frac{1}{2}})][\mathrm{d}H] = \\
= & \frac{|T_Z|^{m-r}|P|^{-r}}{\Gamma_r^{\mathbb{C}}[m]} \int_{V_{m,r}^{\mathbb{C}}} \exp[-tr(P^{-1}HT_Z\bar{H}')][\mathrm{d}H]
\end{aligned}$$

From Theorem 5 in Shimizu, Hashiguchi (2021) we have:

$$f_{T_Z}(T_Z) = \frac{|T_Z|^{m-r}|P|^{-r}}{\Gamma_r^{\mathbb{C}}[m]}\ {}_0F_0^{(r)}(P^{-1}, -T_Z).$$

□

Note that the distribution in question inherits the properties of its real counterpart. There is an indeterminacy in the matrix parameter P by multiplication by a positive scalar (i.e. CMACG(P)=CMACG(cP), where $c > 0$). For $P = I_m$ the orientation H_Z is uniformly distributed over the complex Stiefel manifold and T_Z has a complex

Wishart distribution, $CW(m, I_r)$.
It should also be emphasized that the CMACG distribution is invariant under the right unitary transformations ($H_Z \to H_Z Q,\ Q_{r\times r} \in O^{\mathbb{C}}(r)$, that is, $\bar{Q}'Q = I_r$), so it can be treated as the distribution defined on the complex Grassmann manifold.
The decomposition (4) leads to the feature stated below (see Chikuse, 1990, Theorem 2.3 for a more extended discussion of the characterization of such distribution in the real case).

Theorem 5. *If the $m \times r$ complex random matrix Z has the density of the form $g(\bar{Z}'Z)$ then its orientation H_Z is uniformly distributed on $V^{\mathbb{C}}_{r,m}$.*

Proof. With the help of (5) we obtain:

$$\begin{aligned} f_{H_Z}(H_Z) &= \frac{\pi^{mr}}{\Gamma^{\mathbb{C}}_r[m]} \int_{T>0, \bar{T}'=T} f_Z(H_Z T^{\frac{1}{2}}) |T|^{m-r} (\mathrm{d}T) = \\ &= \frac{\pi^{mr}}{\Gamma^{\mathbb{C}}_r[m]} \int_{T>0, \bar{T}'=T} g(T^{\frac{1}{2}} \bar{H}'_Z H_Z T^{\frac{1}{2}}) |T|^{m-r} (\mathrm{d}T) = \\ &\underbrace{=}_{\bar{H}'_Z H_Z = I_r} \frac{\pi^{mr}}{\Gamma^{\mathbb{C}}_r[m]} \int_{T>0, \bar{T}'=T} g(T) |T|^{m-r} (\mathrm{d}T) = \\ &= const. \end{aligned}$$

□

Theorem 6. *Let Z be an $m \times r$ complex random matrix with the density $f_Z(Z)$ invariant under right unitary transformation ($Z \to ZQ,\ \bar{Q}'Q = I_r$). Define a new $m \times r$ random matrix $Y = BZ$ with an $m \times m$ non-singular matrix B, ($|B| \neq 0$). Consider polar decomposition of these matrices:*

- *$Z = H_Z T_Z^{1/2}$ with $H_Z = Z(\bar{Z}'Z)^{-1/2}$ and $T_Z = \bar{Z}'Z$,*
- *$Y = H_Y T_Y^{1/2}$ with $H_Y = Y(\bar{Y}'Y)^{-1/2}$ and $T_Y = \bar{Y}'Y$.*

and let $f_{H_Z}(H_Z)$ be the density of H_Z (see Theorem 4). Then the density of H_Y, the orientation of the random matrix Y, is of the form:

$$f_{H_Y}(H_Y) = |\bar{B}'B|^{-r} |\bar{W}'W|^{-m} f_{H_Z}(H_W), \tag{9}$$

where $W = B^{-1}H_Y$ and H_W is the orientation of W, i.e. $H_W = W(\bar{W}'W)^{-1/2}$.

Proof. Knowing the density of Z and the Jacobian of transformation $Z \to BZ = Y$, $(\mathrm{d}Y) = |\bar{B}'B|^r (\mathrm{d}Z)$, we may obtain the density of Y:

$$f_Y(Y) = |\bar{B}'B|^{-r} f_Z(B^{-1}Y), \tag{10}$$

which together with (5) leads to the density of H_Y:

$$f_{H_Y}(H_Y) = \frac{\pi^{mr}}{\Gamma_r^{\mathbb{C}}[m]} |\bar{B}'B|^{-r} \int_{T>0,\bar{T}'=T} f_Z(B^{-1}H_Y T^{\frac{1}{2}})|T|^{m-r}(\mathrm{d}T). \tag{11}$$

We follow Chikuse (1990) and apply the idea of her transformation (3.4) to the complex case:

$$T = (\bar{W}'W)^{-1/2} S (\bar{W}'W)^{-1/2}, \text{ with } W = B^{-1}H_Y, \tag{12}$$

the Jacobian od this transformation leads to the relationship between measures $(\mathrm{d}T) = |(\bar{W}'W)^{-1/2}|^{2r}(\mathrm{d}S) = |\bar{W}'W|^{-r}(\mathrm{d}S)$.
From the invariance property of the density of Z we have

$$f_Z(WT^{1/2}) = f_Z(H_W S^{1/2}). \tag{13}$$

Now we can combine the above stated transformation and present the density of H_Y as:

$$\begin{aligned} f_{H_Y}(H_Y) \quad &= \quad \frac{\pi^{mr}}{\Gamma_r^{\mathbb{C}}[m]} |\bar{B}'B|^{-r} \times \\ &\times \quad \int_{S>0,\bar{S}'=S} f_Z(H_W S^{\frac{1}{2}}) |(\bar{W}'W)^{-1/2} S (\bar{W}'W)^{-1/2}|^{m-r} |\bar{W}'W|^{-r}(\mathrm{d}S) = \\ &= \quad |\bar{B}'B|^{-r} |\bar{W}'W|^{-m} \frac{\pi^{mr}}{\Gamma_r^{\mathbb{C}}[m]} \int_{S>0,\bar{S}'=S} f_Z(H_W S^{\frac{1}{2}}) |S|^{m-r}(\mathrm{d}S) = \\ &\underbrace{=}_{(5)} \quad |\bar{B}'B|^{-r} |\bar{W}'W|^{-m} f_{H_Z}(H_W). \end{aligned}$$

□

Theorem 6 leads to the following feature of the CMACG distribution for the linear transformations of complex random matrices.

Corollary 7. *If H_Z, the orientation of Z, has the CMACG(P) distribution, then H_Y, the orientation of $Y = BZ$, has the CMACG($BP\bar{B}'$) distribution.*

Proof. As the orientation H_Z has CMACG(P) distribution its density is $f_{H_Z}(H_Z) = |P|^{-r} |\bar{H}_Z' P^{-1} H_Z|^{-m}$, see (7). Using (9) from Theorem 6 we obtain

$$\begin{aligned} f_{H_Y}(H_Y) &= |\bar{B}'B|^{-r} |\bar{W}'W|^{-m} f_{H_Z}(H_W) = \\ &= |\bar{B}'B|^{-r} |\bar{W}'W|^{-m} |P|^{-r} |\bar{H}_W' P^{-1} H_W|^{-m} = \\ &= |\bar{B}'B|^{-r} |\bar{W}'W|^{-m} |P|^{-r} |(\bar{W}'W)^{-1/2} \bar{W}' P^{-1} W (\bar{W}'W)^{-1/2}|^{-m} = \\ &= |\bar{B}'B|^{-r} |\bar{W}'W|^{-m} |P|^{-r} |\bar{W}'W|^{m} |\bar{W}' P^{-1} W|^{-m} = \\ &= |\bar{B}'B|^{-r} |P|^{-r} |\bar{H}_Y' (\bar{B}^{-1})' P^{-1} B^{-1} H_Y|^{-m} = \\ &= |BP\bar{B}'|^{-r} |\bar{H}_Y' (BP\bar{B}')^{-1} H_Y|^{-m}, \end{aligned}$$

which is the density of CMACG($BP\bar{B}'$). □

The features stated in Theorem 5 and Corollary 7 let us define a more general class of random matrices with orientations having the CMACG(P) distribution.

Theorem 8. *Assume that an $m \times r$ random complex matrix Z has the density of the form*

$$f_Z(Z) = |P|^{-r} g(\bar{Z}' P^{-1} Z) \tag{14}$$

invariant under right unitary transformation ($Z \to ZQ$, $\bar{Q}'Q = I_r$) with P being an $m \times m$ positive define matrix then its orientation H_Z has the CMACG(P) distribution.

Proof. The proof is a straightforward generalization of the proof of the Theorem 3.2 in Chikuse (1990).
There exists a matrix B such that $P = B\bar{B}'$ with $|B| \neq 0$. Define $U = B^{-1}Z$, so the distribution of U is

$$f_U(U) = |P|^{-r} g(\bar{U}'U)|P|^r = g(\bar{U}'U),$$

which is also invariant under right unitary transformation ($U \to UQ$, $\bar{Q}'Q = I_r$).
According to Theorem 5 the orientation of U is uniformly distributed on $V_{r,m}^{\mathbb{C}}$, i.e. $H_U \sim CMACG(I_m)$.
From Corollary 7 applied to the orientation of the matrix $Z = BU$ we obtain that

$$H_Z \sim CMACG(BI_m\bar{B}') = CMACG(P).$$

□

3 Sampling from the CMACG distribution

As mentioned in the Introduction, one advantage of the CMACG distribution is that it provides an easy way to obtain a pseudo-random sample, which is especially useful for Bayesians. To do so, researchers can use the definition of the considered distribution. Note that Koop, León-González, Strachan (2009) use the same strategy in the real case. Suppose that one needs a sample from the CMACG(P) distribution with the known matrix parameter P. According to Theorem 4 the orientation H_Z of Z – the normally distributed complex random matrix, $Z \sim mN^{\mathbb{C}}(0, I_r, P)$ – has the CMACG(P) distribution. Thus, one draw is generated in two steps:

1. Generate an $m \times r$ matrix Z from $mN^{\mathbb{C}}(0, I_r, P)$.
2. Set $H_Z = Z(\bar{Z}'Z)^{-\frac{1}{2}}$, where $(\bar{Z}'Z)^{-\frac{1}{2}}$ is the inverse of the square root of $\bar{Z}'Z$.

It was mentioned that due to its invariance property CAMCG(P) may be treated as a distribution definite for the elements of the complex Grassmann manifold. Thus, setting $P_Z = H_Z\bar{H}_Z'$ we obtain the projection matrix from the desired distribution.

The above points need additional comments. First, the easiest way to obtain the draw from the complex matrix variate distribution is to use its relationship with the real case. That is, if $Z = Z_R + iZ_I$, where $i = \sqrt{-1}$, then Z has a complex normal distribution, i.e. $Z \sim mN^{\mathbb{C}}(0, I_r, P)$, where $P = P_R + iP_I$ is a Hermitian matrix. This is equivalent to saying that its real and imaginary parts are jointly normally distributed: $\begin{pmatrix} Z_R \\ Z_I \end{pmatrix} | \sim mN\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, I_r, \frac{1}{2}\begin{pmatrix} P_R & -P_I \\ P_I & P_R \end{pmatrix}\right)$. Second, the square root of a complex Hermitan matrix $(\bar{Z}'Z)^{\frac{1}{2}}$, needed in the polar decomposition of Z, can be obtained with the Newton's method, as proposed in Highman (1986).

4 Conclusions

This paper extends the matrix angular central Gaussian distribution proposed by Chikuse (1990) to the complex case. By considering the polar decomposition of a random complex matrix and the appropriate decomposition of measures, we obtained the density function of the matrix's orientation, which is the element of the complex Stiefel manifold. We demonstrate that this new distribution inherits the properties of the MACG distribution. We also discuss the method for obtaining pseudo-random samples from this distribution.
The complex MACG distribution is a valuable tool for the Bayesian analysis of VEC models with complex unit roots, including seasonally and cyclically cointegrated VAR models.

Acknowledgements

I would like to thank the Editor and the anonymous Reviewer for their valuable comments and suggestions, which helped me improve the final manuscript.
The article presents the results of the Project nr 036/EIE/2025/POT financed from the subsidy granted to the Krakow University of Economics.

References

[1] Abramovich Y. I., Besson O., (2013a), Regularized covariance matrix estimation in complex elliptically symmetric distributions using the expected likelihood approach-Part 1: The over-sampled case, *IEEE Transactions on Signal Processing* 61(23), 5807–5818.

[2] Besson O., Abramovich Y. I., (2013b), Regularized covariance matrix estimation in complex elliptically symmetric distributions using the expected likelihood approach-Part 2: The under-sampled case, *IEEE Transactions on Signal Processing* 61(23), 5819–5829.

[3] Arteche J., Robinson P. M., (1999), Seasonal and cyclical long memory, [in:] *Asymptotics, Nonparametrics and Time Series*, [ed.:] S, Ghosh, Marcel Dekker Inc., New York, 115–148.

[4] Chikuse Y., (1990), The matrix angular central Gaussian distribution, *Journal of Multivariate Analysis* 33(2), 265–274.

[5] Chikuse Y., (2003), *Statistics on special manifolds, Lecture Notes in Statistics*, vol. 174, Springer Science & Business Media.

[6] Constantine A. G., (1963), Some Non-Central Distribution Problems in Multivariate Analysis, *The Annals of Mathematical Statistics* 34(4), 1270–1285.

[7] Cubadda G., Omtzigt P., (2005), Small-sample improvements in the statistical analysis of seasonally cointegrated systems, *Computational statistics & data analysis* 49(2), 333–348.

[8] Díaz-García J. A., Gutiérrez-Jáimez R., (2011), On Wishart distribution: Some extensions, *Linear Algebra and its Applications* 435, 1296–1310.

[9] Dryden I. L., Mardia K. V., (2016), *Statistical shape analysis: with applications in R*, John Wiley & Sons.

[10] Gregoir S., (1999a), Multivariate time series with various hidden unit roots, Part I: Integral Operator Algebra and Representation Theory, *Econometric Theory* 15(4), 435–468.

[11] Gregoir S., (1999b), Multivariate time series with various hidden unit roots, Part II: Estimation and testing, *Econometric Theory* 15(4), 469–518.

[12] Gross K. I., Richards D. S. P., (1987), Special functions of matrix argument, I. Algebraic induction, zonal polynomials, and hypergeometric functions, *Transactions of the American Mathematical Society* 301(2), 781–811.

[13] Higham N. J., (1986), Newton's method for the matrix square root, *Mathematics of Computation* 46(174), 537–549.

[14] Hylleberg S., Engle R. F., Granger C. W. J., Yoo B. S., (1990), Seasonal integration and cointegration, *Journal of Econometrics* 44(1-2), 215–238.

[15] James A. T., (1964), Distributions of matrix variates and latent roots derived from normal samples, *The Annals of Mathematical Statistics* 35(2), 475–501.

[16] Johansen S., Schaumburg E., (1999), Likelihood analysis of seasonal cointegration, *Journal of Econometrics* 88(2), 301–339.

[17] Kent J. T., (1997), Data analysis for shapes and images, *Journal of Statistical Planning and Inference* 57(2), 181–193.

[18] Koop G., León-González R., Strachan R. W., (2009), Efficient posterior simulation for cointegrated models with priors on the cointegration space, *Econometric Reviews* 29(2), 224–242.

[19] Mardia K. V., Jupp P. E., (2009), *Directional statistics*, John Wiley & Sons.

[20] Micheas A. C., Dey D. K., Mardia K. V., (2006), Complex elliptical distributions with application to shape analysis, *Journal of statistical planning and inference* 136(9), 2961–2982.

[21] Ollila E., Tyler D. E., Koivunen V., Poor H. V., (2012), Complex elliptically symmetric distributions: Survey, new results and applications, *IEEE Transactions on Signal Processing* 60(11), 5597–5625.

[22] Polcari J., (2017), Jacobian Determinants of Basic Matrix Transformations, Working Paper, Rev 1, March, 2017, available at `https://drive.google.com/file/d/0B1JunjKxazANYS1LcWc4LVY3cnc`.

[23] Shimizu K., Hashiguchi H., (2021), Heterogeneous hypergeometric functions with two matrix arguments and the exact distribution of the largest eigenvalue of a singular beta-Wishart matrix, *Journal of Multivariate Analysis* 183, 104714.

[24] Wróblewska J., (2011), Bayesian analysis of weak form reduced rank structure in VEC models, *Central European Journal of Economic Modelling and Econometrics* 3(3), 169–186.

[25] Wróblewska J., (2012), Bayesian analysis of weak form polynomial reduced rank structures in VEC models, *Central European Journal of Economic Modelling and Econometrics* 4(4), 253–267.

[26] Wróblewska J., (2015), Common Trends and Common Cycles-Bayesian Approach, *Central European Journal of Economic Modelling and Econometrics* 7(2), 91–110.

[27] Wróblewska J., (2025), Bayesian analysis of seasonally cointegrated VAR models, *Econometrics and Statistics* 35, 55–70.