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Signed networks with Laplacian feedback: Observability and minimal sensor problem

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Abstract. In many multi-agent networks that evolve according to specific dynamical rules, our direct access is often limited to only a small subset of agents, known as "sensors." The remaining agents in the network, whose states are entirely inaccessible, are referred to as "actuators." If it is possible to retrieve the complete states of the actuators from the knowledge of the sensor states, the network is said to be "sensor–actuator observable"; otherwise, it is deemed unobservable. This research explores the analysis of sensor–actuator observability property (i.e., observable or unobservable) in networks where agent communications encompass both cooperation and competition (i.e., signed networks). It is assumed that the network agents continuously update their states within $\mathbb{R}$ under the Laplacian protocol. We simplify the Kalman's and the PBH rank criteria for evaluating sensor–actuator observability property into several verifiable algebraic tests, emphasizing the significance of the system matrices' spectral properties. This property is also examined from the perspective of the network graph topology. Sensor–actuator observability property is significantly influenced by the nature of agent communications and often differs between signed and unsigned networks. However, we demonstrate that for a structurally balanced signed network, with specific set of sensors, its sensor–actuator observability property aligns with that of its unsigned variant. We present a formula utilizing Laplacian spectral information to determine the minimum number of sensors for an ensured sensor–actuator observable network. Applying the formula to path and cycle networks, it is found that paths are observable with one sensor, and cycles require two for observability.

Keywords: multi-agent systems; observability; signed graph.

1. INTRODUCTION

In recent decades, the study of multi-agent systems has garnered attention because of their applications in various fields, including robotics [1–4], population and system biology [5–7], multiple spacecrafts [8–10], communication and sensor networks [11], and power systems [12]. Various cooperative tasks attributed to multi-agent systems include consensus/agreement/rendezvous, flocking, formation control, and distributed average tracking [13]. Rendezvous problems involve reaching an agreement on agent states, flocking problems involve maintaining the equal distances between agents, formation control aims for a specific geometric pattern, and distributed average tracking aims to achieve consensus on the average of the reference signals associated with the agents.

Algorithms considering local neighbor states, like the Laplacian feedback algorithm, have proven effective in solving cooperative tasks in multi-agent systems [14, 15]. Graph theoretic tools and matrix techniques are used to study the models with Laplacian dynamics, with graphs encoding network information and matrices analyzing cooperative tasks algebraically.

Network controllability and observability are fundamental properties of multi-agent systems, and have become hot topics in control and network communities [16]. Broadly, controllability refers to a system's ability to drive all agents from any initial state to any desired state with admissible control inputs. In contrast, observability involves the system's ability to recover the complete internal states of all agents based on the system output. Notably, for systems with linear time-invariant (LTI) dynamics, controllability and observability are dual, unlike systems with nonlinear dynamics [17]. However, due to algebraic simplicity and for developing intuition, many scholars analyze controllability and observability separately, even for systems with LTI dynamics [18–25]. While numerous approaches exist to study the controllability and observability properties in multi-agent systems, the 21st century has seen particular attention on three: the leader–follower approach, the behavioral approach, and the virtual structural approach [10].

Observability in multi-agent systems is of practical significance [26, 27]. Various cooperative tasks in multi-agent systems operating under the Laplacian protocol were examined for observability properties using graph and matrix tools. In [28], the authors focused on the observability of sensor networks. A framework for observability analysis in certain classes of linear multi-agent systems, encompassing both homogeneous and heterogeneous types, is developed in [29]. The authors quantify

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the relative degree of observability. In [30], it is demonstrated that the complexity of network observability can be reduced by examining suitable subgraphs of the original communication graph. Consequently, a sufficient observability condition for a tree network is proposed. Further exploration in [31] results in identifying necessary and sufficient observability conditions, specifically in the context of path and cycle networks, based on the results of number theory.

In a parallel research line, the leader–follower scheme is widely employed for observability analysis. This scheme categorizes network agents into leaders and followers, with followers tracking leader states. In [18], observability analysis is conducted for multi-agent systems under chain and cyclic topologies as well as for the combination of these topologies, using the leader–follower scheme. [19] employs the scheme to propose a decentralized method, utilizing eigenvalues for online observability checking in multi-agent systems. The application is demonstrated with a convoy of vehicles. Additionally, the leader–follower approach is utilized for estimating observability in Laplacian networks with communication topologies represented by grid/torus graphs in [20]. Subsequent research has produced numerous articles addressing observability issues in various types of multi-agent systems through the adoption of the leader–follower framework [21–25, 28, 31–34].

The networks that consist solely of cooperative interactions are referred to as unsigned networks. However, recognizing the significance of both cooperative and competitive interactions within networks is essential [36–39]. Many real-world networked systems, such as:

- social systems (involving friend–enemy or alliance–rivalry relationships),
- engineering systems (involving reinforcing–opposing relationships),
- biological gene systems (involving activation–inhibition relationships),
- management systems (involving profit–loss relationships),
- spin systems in statistical mechanics (involving ferromagnetic–antiferromagnetic relationships between particles)

exhibit both cooperative and competitive interactions. Unsigned graphs are inadequate for modeling such networks, as their exclusively positive-weighted edges represent only cooperative interactions. By allowing the inclusion of both positive and negative weighted edges, unsigned graphs are generalized to signed graphs. Since signed networks represent a more general form of unsigned networks, it is reasonable to expect that observability results for signed networks apply to their corresponding unsigned versions. However, observability results for unsigned networks do not necessarily hold for their corresponding signed versions.

Few studies have explored the observability in signed networks [40, 41]. In [40], network employed the agreement protocol over the discrete-time scale, and observability investigations are carried from both algebraic and graphical points of view. In [41], matrix-weighted networks are considered to manifest the interactions among agents more precisely, and established their controllability and observability properties. This paper contributes to this area by investigating observability in signed networks with Laplacian feedback. Classifying agents into sensors and actuators, akin to the leader–follower approach, offers insights into network observability. The paper introduces necessary and sufficient verifiable observability conditions, delving into observability through spectral information and graph topological properties. The conditions are established under which observability of a signed network become equivalent to that of the corresponding unsigned network. A formula is devised, leveraging spectral knowledge of system matrices, to determine the minimum number of sensors for guaranteed observability. Applying this formula demonstrates that a path network is observable with just one sensor, and a cycle network requires a minimum of two sensors, regardless of communication nature or the number of agents. All results are validated through examples.

The paper structure is as follows. Section 2 covers preliminaries on matrices and graphs. Section 3 discusses the observability problem model and basic facts. Section 4 analyzes observability using spectral properties and the tools from algebraic graph theory. Section 5 introduces the minimal sensor problem and presents a formula for computing the minimum number of sensors for guaranteed observability. Finally, Section 6 draws concluding remarks.

2. PRELIMINARIES

Notation used are standard. Positive integers, real numbers, and complex numbers are denoted by $\mathbb{Z}^+$, $\mathbb{R}$, and $\mathbb{C}$, respectively. The set of first m positive integers is denoted as $\mathscr{I}_m$. The vector space of $(n \times m)$ real (complex) matrices $(n, m \in \mathbb{Z}^+)$ over the real (complex) field under the usual matrix addition and scalar multiplication is denoted by $\mathbb{R}^{n\times m}$ $(\mathbb{C}^{n\times m})$. Specifically, $\mathbb{R}^{n\times 1}$ is denoted as $\mathbb{R}^n$ and $\mathbb{C}^{n\times 1}$ as $\mathbb{C}^n$.

On an interval I, the space $\mathfrak{L}^2(I : \mathbb{R}^n)$ consists of Lebesgue measurable functions $\boldsymbol{f}\colon I \to \mathbb{R}^n$, where $\boldsymbol{f}(t) = \begin{bmatrix} f_1(t) & \dots & f_n(t) \end{bmatrix}^\intercal$ for $t \in I$ (with $^\intercal$ denoting the transpose) satisfies $0 \leqq \int_I \sum_{r=1}^{n} |f_r(t)|^2 \mathrm{d}t < +\infty$. This space is a real Hilbert space under the inner product $\langle\cdot,\cdot\rangle : \mathfrak{L}^2(I : \mathbb{R}^n) \times \mathfrak{L}^2(I : \mathbb{R}^n) \to \mathbb{R}$ defined as $\langle \boldsymbol{f}, \boldsymbol{g}\rangle := \int_I \boldsymbol{g}(t)^\intercal \boldsymbol{f}(t)\mathrm{d}t$ for the elements $\boldsymbol{f}$ and $\boldsymbol{g}$ of $\mathfrak{L}^2(I : \mathbb{R}^n)$.

2.1. Matrix preliminaries

Since $\mathbb{R}^{n\times n} \subsetneqq \mathbb{C}^{n\times n}$, every $\mathbf{A}$ in $\mathbb{R}^{n\times n}$ also belongs to $\mathbb{C}^{n\times n}$. Consequently, the set of all eigenvalues of $\mathbf{A}$, denoted as $\sigma(\mathbf{A})$, is a subset of $\mathbb{C}$, and its eigenspace $\ker(\lambda \mathbf{I}_n - \mathbf{A})$ is a subspace of $\mathbb{C}^n$. Here, $\ker(\cdot)$ represents the kernel, $\mathbf{I}_n$ is the $(n \times n)$ identity matrix, and $\lambda \in \sigma(\mathbf{A})$. All nonzero elements of $\ker(\lambda \mathbf{I}_n - \mathbf{A})$ are the eigenvectors of $\mathbf{A}$ corresponding to the eigenvalue λ. The algebraic multiplicity of $\lambda \in \sigma(\mathbf{A})$, denoted as $\mathrm{AM}(\lambda)$, is the count of times λ appears as an eigenvalue of $\mathbf{A}$, while its geometric multiplicity, denoted as $\mathrm{GM}(\lambda)$, is the dimension of $\ker(\lambda \mathbf{I}_n - \mathbf{A})$. It is worth noting that $\sigma(\mathbf{A}) = \sigma(\mathbf{A}^\intercal)$. A diagonal matrix with diagonal entries $a_1, a_2, \dots, a_n$, in that specific order, is simply represented as $\mathrm{diag}(a_1, a_2, \dots, a_n)$. The orthogonality between $\boldsymbol{v}$ in $\mathbb{R}^n(\mathbb{C}^n)$ and $\boldsymbol{w}$ in $\mathbb{R}^n(\mathbb{C}^n)$ is defined by the condition $\boldsymbol{w}^\intercal \boldsymbol{v} = 0$ ($\boldsymbol{w}^* \boldsymbol{v} = 0$, with * indicating the conjugate transpose).

2.2. Graph preliminaries

Let $\mathscr{V}$ be a nonempty set, and let $\mathscr{E}$ be a subset of $\mathscr{V} \times \mathscr{V}$ such that $(i,i) \notin \mathscr{E}$ for all $i \in \mathscr{V}$. An undirected simple graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is a diagram in a plane, consisting of $|\mathscr{V}|$ points corresponding to the elements of $\mathscr{V}$ and $|\mathscr{E}|$ undirected lines corresponding to the elements of $\mathscr{E}$. In this context, a line e corresponding to (j,i) represents a connection between j and i (and hence from i and j). The sets $\mathscr{V}$ and $\mathscr{E}$ are commonly referred to as the vertex set and edge set of $\mathscr{G}$, respectively. $\mathscr{G}$ is called a finite graph if $\mathscr{V}$ is finite. The neighbor set $\mathscr{N}_i$ of $i \in \mathscr{V}$ contains all $j \in \mathscr{V}$ such that $(j,i) \in \mathscr{E}$; in this context, i and j are considered neighbors to each other. The weight of lines in $\mathscr{G}$ is described by the function $\mathsf{w} : \mathscr{E} \to \{1, -1\}$. Given a finite undirected simple graph $\mathscr{G}$, its adjacency matrix $\mathscr{A}(\mathscr{G})$, which is of size $|\mathscr{V}| \times |\mathscr{V}|$, captures the weights of all lines. The entry in the ith row and jth column of this matrix is $\mathsf{w}(j,i)$ if there exists a line between j and i, or it is zero otherwise. Notably, $\mathscr{A}(\mathscr{G})$ is a symmetric signed matrix with all principal diagonal entries equal to zeros. The graph Laplacian, denoted as $\mathscr{L}(\mathscr{G})$, is defined as $\Delta(\mathscr{G}) - \mathscr{A}(\mathscr{G})$, where $\Delta(\mathscr{G})$ is the valency matrix of $\mathscr{G}$, defined as $\mathrm{diag}(\sum_{j\in\mathscr{N}_i} |\mathsf{w}(j,i)|)_{i\in\mathscr{V}}$. The Laplacian matrix is also symmetric with only real eigenvalues and is diagonalizable over $\mathbb{R}$. Two fundamental types of graphs are paths and cycles, serving as building blocks for a variety of graphs. A path graph $\mathscr{P}_n$ is defined as a graph with n points which can be arranged in a sequence $\{i_1, i_2, i_3, \ldots, i_n\}$ in which every pair of consecutive points are neighbors. (In this case, $\mathscr{P}_n$ represents the path between the points i_1 and i_n.) A cycle graph $\mathscr{C}_n$ is identical to $\mathscr{P}_n$ with the additional constraint that $i_1 = i_n$. A connected graph is the one in which, between every pair of its distinct points, there exists at least one path.

3. PROBLEM DESCRIPTION

Consider a network consisting of $N \geqq 2$ agents, denoted as $1, 2, \ldots, \mathsf{N}$. Within the network, communication between agents is either cooperative or competitive with unit strength. Specifically, 1 denotes the cooperative communication and -1 denotes the competitive communication. The network communications are undirected, meaning the exchange of information between any pair of agents is bi-directional and the nature of communication in both directions is the same. The following additional assumptions are imposed on the network:

(i) The topology of communications is static, meaning no new communication links are established among agents, and existing communication links retain their nature and strength.
(ii) Each agent has a single way to communicate with any other agent.
(iii) Self-communication is not allowed.
(iv) The network remains connected.

With these assumptions in place, the graph modeling the network, referred to as the "interconnection graph," in which the points correspond to agents, lines between points represent communications between agents, and the weights of the lines indicate the strength of the communications. This is a finite undirected simple connected graph. For brevity, denote this network as $\mathfrak{N}$ and its interconnection graph as $\mathscr{G}$. A typical example of multi-agent network is given in Fig. 1.

Let each agent state be described by a real number, and the time duration of communication be given by the interval $\tau := [t_\mathrm{i}, t_\mathrm{f}]$, where $0 \leqq t_\mathrm{i} < t_\mathrm{f} < +\infty$. The state of the ith agent ($i = 1, 2, \ldots, \mathsf{N}$) at time $t \in \tau$, denoted by $x_i(t)$, updates using the following protocol:

$$\dot{x}_i(t) = \sum_{j\in\mathscr{N}_i} \left(\mathrm{sgn}(\mathsf{w}(j,i))x_j(t) - x_i(t)\right), \quad t \in \tau. \tag{1}$$

Here, $\mathrm{sgn}(\cdot)$ represents the signum function, defined as $\mathrm{sgn}(a) = 1$ if $a > 0$, $= -1$ if $a < 0$, and $= 0$ if $a = 0$.

By adopting a graph-based notation, the above protocols governing the state updates of all agents is described by the following single equation:

$$\dot{\boldsymbol{x}}(t) = -\mathscr{L}\boldsymbol{x}(t), \quad t \in \tau, \tag{2}$$

where $\boldsymbol{x}(t) := \begin{bmatrix} x_1(t) & x_2(t) & \ldots & x_\mathsf{N}(t) \end{bmatrix}^\mathsf{T} \in \mathbb{R}^N$ denotes the aggregated state and $\mathscr{L}$ is the Laplacian of $\mathscr{G}$. (The usage of $\mathscr{L}$ instead of $\mathscr{L}(\mathscr{G})$ is due to the reason that $\mathscr{L}$ dependency on $\mathscr{G}$ is clear from the context.)

Let there be k sensors ($1 \leqq k \leqq (N-1)$) in $\mathfrak{N}$. Without loss of generality, assume that these sensors correspond to the last k agents in the set $\{1, 2, \ldots, \mathsf{N}\}$. Relabel their states as $z_1(t), z_2(t), \ldots, z_k(t)$, respectively, and their aggregated state as $\boldsymbol{z}(t) := \begin{bmatrix} z_1(t) & z_2(t) & \ldots & z_k(t) \end{bmatrix}^\mathsf{T} \in \mathbb{R}^k$. The remaining $(N-k)$ agents are the actuators, whose states are relabeled as $y_1(t), y_2(t), \ldots, y_{N-k}(t)$, respectively, with their aggregated state being $\boldsymbol{y}(t) := \begin{bmatrix} y_1(t) & y_2(t) & \ldots & y_{N-k}(t) \end{bmatrix}^\mathsf{T} \in \mathbb{R}^{N-k}$. The functions $\boldsymbol{z}$ and $\boldsymbol{y}$ are called the aggregated sensor and actuator functions of $\mathfrak{N}$, respectively. Their admissible spaces are considered to be $\mathfrak{L}^2(\tau : \mathbb{R}^k)$ and $\mathfrak{L}^2(\tau : \mathbb{R}^{N-k})$, respectively.

To derive the dynamics governing the actuator states from (2), partition the matrix $\mathscr{L}$ as follows:

$$\mathscr{L} = \left[\begin{array}{c|c} \mathbf{F} & \mathbf{L} \\ \hline \mathbf{M} & \mathbf{N} \end{array}\right].$$

Here, $\mathbf{F} \in \mathbb{R}^{(N-k)\times(N-k)}$ captures the interactions among actuators, $\mathbf{L} \in \mathbb{R}^{(N-k)\times k}$ captures the influence of sensors on actuators, $\mathbf{M} \in \mathbb{R}^{k\times(N-k)}$ captures the influence of actuators on sensors, and $\mathbf{N} \in \mathbb{R}^{k\times k}$ captures the interactions among sensors. Notably, the off-diagonal matrices $\mathbf{M}$ and $\mathbf{L}$ are related by the equation: $\mathbf{M} = \mathbf{L}^\mathsf{T}$. The dynamics governing the actuators' states is now expressed as:

$$\dot{\boldsymbol{y}}(t) = -\mathbf{F}\boldsymbol{y}(t) - \mathbf{L}\boldsymbol{z}(t), \quad t \in \tau. \tag{3}$$

A formal definition of observability is proposed below.

Definition 1 Observability**.** $\mathfrak{N}$ is termed sensor–actuator observable if, based on the knowledge of the states of the k sensors

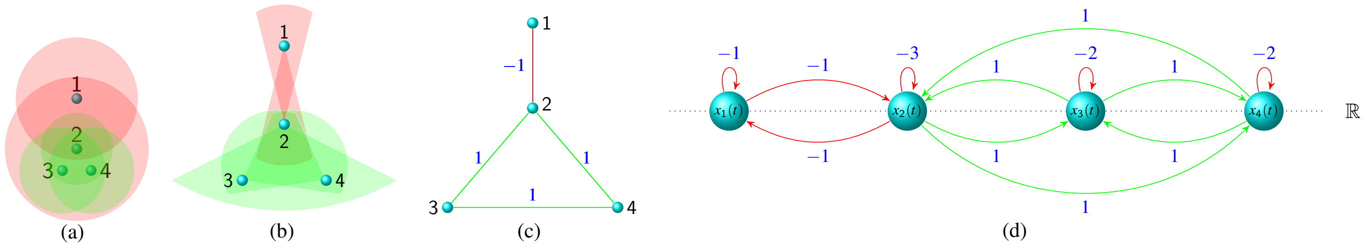


Fig. 1. A typical multi-agent network. (a) Each agent communication range is circular, enabling omnidirectional communication, similar to oral communication. (b) The communication range takes on a $\vee$-shaped geometry, resembling visual communication. In these diagrams, the green region represents the region of cooperative communications, while the red region signifies the region of competitive communications. (c) The corresponding interconnection graph of the network. (d) Presenting the inference diagram of the dynamical system (2) for this network. A directed line from $x_j(t)$ to $x_i(t)$ with an associated factor k indicates that $x_j(t)$ appears in $x_i(t)$'s differential equation with coefficient k, illustrating that monitoring the state of the jth agent allows us to gather information about the state of the ith agent

such that its aggregated function resides in $\mathfrak{L}^2(\tau : \mathbb{R}^k)$, it is feasible to uniquely retrieve the complete states of the $(N-k)$ actuators such that their aggregated function lies in $\mathfrak{L}^2(\tau : \mathbb{R}^{N-k})$ and adhere to the protocol (3).

To ensure a unique retrieval of $\boldsymbol{y}(t)$ throughout τ in a manner that solves system (3), it is essential to first retrieve $\boldsymbol{y}(t_\mathrm{i})$ (the initial value of the function $\boldsymbol{y}$), uniquely in $\mathbb{R}^{N-k}$. In fact, it is sufficient to retrieve a unique $\boldsymbol{y}(t_\mathrm{i})$, as the differential system presented in equation (3) is well-posed, and hence the uniqueness of $\boldsymbol{y}(t_\mathrm{i})$ will determine a complete $\boldsymbol{y}(t)$, uniquely, that solves system (3). Therefore, the above definition can be restated as follows: The network $\mathfrak{N}$ is deemed "sensor–actuator observable" if it is feasible to uniquely retrieve the vector $\boldsymbol{y}(t_\mathrm{i})$ in $\mathbb{R}^{N-k}$ adhering to the protocol (3), based on the knowledge of $\boldsymbol{z}$ in $\mathfrak{L}^2(\tau : \mathbb{R}^k)$.

It is important to highlight that, sensors indirectly influence actuators, as their own states are impacted by actuators through the protocol:

$$\dot{\boldsymbol{z}}(t) + \mathbf{N}\boldsymbol{z}(t) = -\mathbf{M}\boldsymbol{y}(t), \quad t \in \tau.$$

Calling $\dot{\boldsymbol{z}}(t) + \mathbf{N}\boldsymbol{z}(t)$ as $\boldsymbol{\omega}(t)$ for all $t \in \tau$. Now the function $\boldsymbol{\omega}$ can be entirely determined in $\mathfrak{L}^2(\tau : \mathbb{R}^k)$ as the complete knowledge of the function $\boldsymbol{z}$ in $\mathfrak{L}^2(\tau : \mathbb{R}^k)$ is known. The above equation is now written as

$$\boldsymbol{\omega}(t) = -\mathbf{M}\boldsymbol{y}(t), \quad t \in \tau. \tag{4}$$

Evidently, the definition of sensor–actuator observability of $\mathfrak{N}$ is simplified to the following statement: The network $\mathfrak{N}$ is "sensor–actuator observable" if it is feasible to uniquely retrieve the vector $\boldsymbol{y}(t_\mathrm{i})$ in $\mathbb{R}^{N-k}$ adhering to the protocol (3), based on the knowledge of the function $\boldsymbol{\omega}$ in $\mathfrak{L}^2(\tau : \mathbb{R}^k)$. This definition is equivalent to the classical definition of observability of system (3), with system (4) functioning as its output equation, and $\boldsymbol{\omega}$ & $\boldsymbol{z}$ serving as output and input functions to the system (3), respectively. In other words, the goal of this research now boils down to the investigation of the observability of system pair ((3), (4)).

4. OBSERVABILITY ANALYSIS

The literature is rich in the investigation of observability of linear time-invariant (LTI) systems. Given that the pair ((3), (4)) consists of LTI systems, its observability is guaranteed only if the Gramian matrix:

$$\mathrm{G} := \int_{t_\mathrm{i}}^{t_\mathrm{f}} \exp(-\mathbf{F}(t-t_\mathrm{i}))\mathbf{M}^\intercal\mathbf{M}\exp(-\mathbf{F}(t-t_\mathrm{i}))\mathrm{d}t \tag{5}$$

is invertible [Theorem 6.4, [17]]. Furthermore, in that case, the vector $\boldsymbol{y}(t_\mathrm{i})$ can be uniquely retrieved in $\mathbb{R}^{N-k}$ using the formula:

$$\begin{aligned}\boldsymbol{y}(t_\mathrm{i}) = -\mathrm{G}^{-1} \int_{t_\mathrm{i}}^{t_\mathrm{f}} \exp(-\mathbf{F}(t-t_\mathrm{i}))\mathbf{M}^\intercal \\ \times \left(\boldsymbol{\omega}(t) - \mathbf{M}\int_{t_\mathrm{i}}^{t} \exp(-\mathbf{F}(t-s))\mathbf{M}^\intercal \boldsymbol{z}(s)\mathrm{d}s\right)\mathrm{d}t.\end{aligned} \tag{6}$$

Observability testing for the pair ((3), (4)) can also be conducted following Kalman's method [17]. It states that the pair ((3), (4)) is observable if and only if the $k(N-k)\times(N-k)$ matrix:

$$\mathcal{O} := \begin{bmatrix}\mathbf{M}^\intercal & (\mathbf{MF})^\intercal & \ldots & (\mathbf{MF}^{N-k-1})^\intercal\end{bmatrix}^\intercal \tag{7}$$

possesses a full (column) rank equal to $(N-k)$. Another easily applicable observability test for the pair ((3), (4)) is due to Popov-Belevitch-Hautus (PBH) [17], which states that the system pair ((3), (4)) is observable if and only if the $N\times(N-k)$ matrix:

$$\mathcal{P}_\lambda := \begin{bmatrix}\lambda\mathbf{I}_{N-k} + \mathbf{F} & -\mathbf{M}^\intercal\end{bmatrix}^\intercal \tag{8}$$

possesses a full (column) rank equal to $(N-k)$ for all $\lambda \in \sigma(-\mathbf{F})$. It is important to note that, all the above mentioned observability tests are equivalent.

Remark 1. Complete knowledge of the function $\boldsymbol{z}$, and consequently of $\boldsymbol{\omega}$, is crucial when assessing the observability of

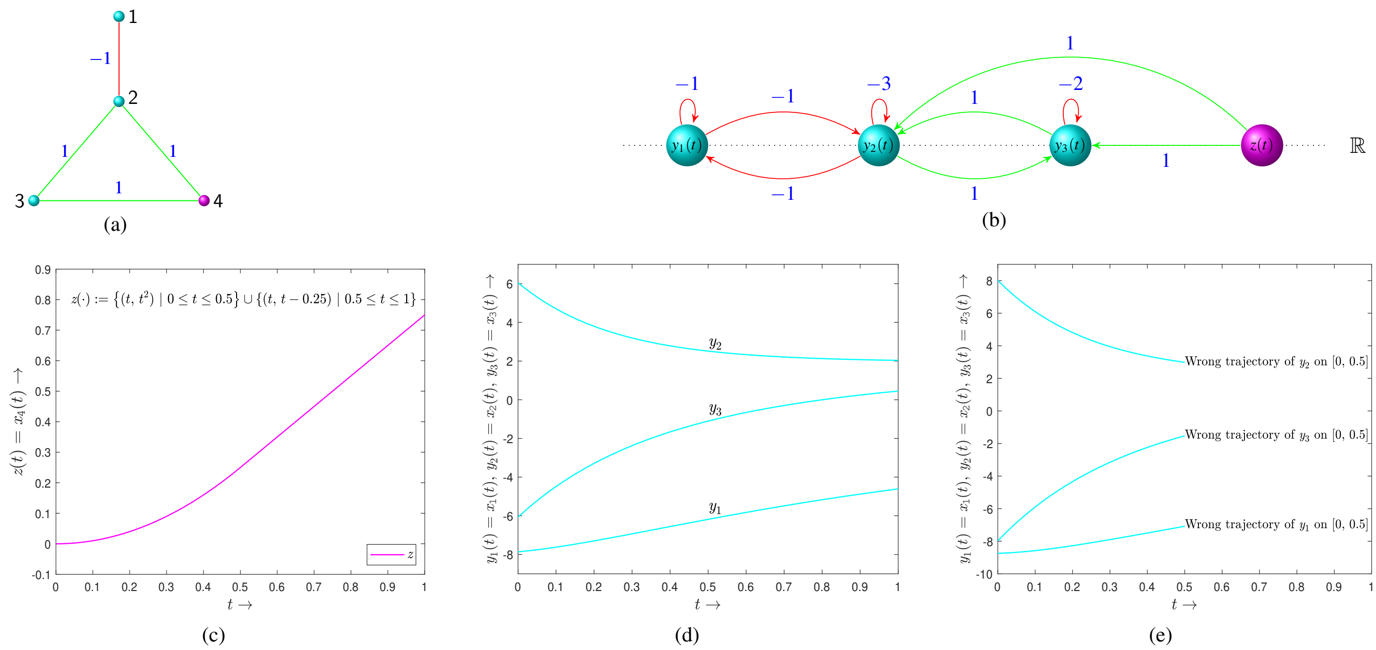


Fig. 2. Reproduction of the network depicted in Fig. 1 after assigning the role of sensor to 4. (a) The interconnection graph. (b) An inference diagram of the corresponding actuator dynamics (3). By comparing this diagram with Fig. 1(d), it becomes clear that, in general, when some of the agents of $\mathfrak{N}$ have been assigned to the role of sensors, the flow of information from the actuator states to the sensor states ceases to exist. Furthermore, there is a termination of information flow among the sensor states (in the case of multiple sensors in $\mathfrak{N}$), and sensor states do not self-interact. However, the information flow from the sensor states to the neighboring actuator states, as well as the flow of information among the actuator states, continues as before (i.e., as before the assignment of sensor roles). (c) For $\tau = [0, 1]$, a priori known trajectory of sensor 4 in $\mathfrak{L}^2([0, 1] : \mathbb{R})$. (d) It is possible here to retrieve $\boldsymbol{y}(0)$ when we use the information of the function $\boldsymbol{z}$ over the entire interval $[0, 1]$. In this case, we obtain $\boldsymbol{y}(0) \approx \begin{bmatrix} -7.863 & 6.050 & -6.070 \end{bmatrix}^\intercal$ allowing the retrieval of the complete trajectories of actuators 1, 2, and 3. (e) The picture illustrates how having knowledge of $\boldsymbol{z}(t)$ only for $0 \leqq t \leqq 0.5$ leads to an incorrect retrieval of $\boldsymbol{y}(0)$ as $\approx \begin{bmatrix} -8.742 & 8.018 & -7.975 \end{bmatrix}^\intercal$, consequently resulting in incorrect trajectories for 1, 2, and 3 over the interval $[0, 0.5]$. This underscores the importance of having knowledge of the sensor states throughout the time interval over which the actuator dynamics is defined to address the observability problem of $\mathfrak{N}$

the system pair $((3), (4))$. If we only possess partial knowledge of $\boldsymbol{z}$, say in the space $\mathfrak{L}^2([t_\mathrm{i}, t'] : \mathbb{R}^k)$ with $t' < t_\mathrm{f}$, and this knowledge allows us to compute a certain value for $\boldsymbol{y}(t_\mathrm{i})$ using formula (6), it is important to note that having knowledge of $\boldsymbol{z}$ in $\mathfrak{L}^2([t_\mathrm{i}, t''] : \mathbb{R}^k)$ with either $t'' < t' < t_\mathrm{f}$ or $t' < t'' < t_\mathrm{f}$ may yield a different value for $\boldsymbol{y}(t_\mathrm{i})$ through the same formula (6). (It is worth noting that if $\boldsymbol{z}$ follows a single function, rather than piecewise functions over τ, only such $\boldsymbol{z}$ on each subinterval $[t_\mathrm{i}, t']$ of τ can yield the same value for $\boldsymbol{y}(t_\mathrm{i})$ through the formula (6).) Refer to Fig. 2 for insights into this remark.

In the context of our system model, as given by the pair $((3), (4))$, the behavior of the output function is inextricably linked to the characteristics of the input function, which, in turn, influenced by the actuators of $\mathfrak{N}$. This fundamental observation leads to the following theorem.

Theorem 1. The system pair $((3), (4))$ is observable if and only if the condition $\boldsymbol{z}(t) = \mathbf{0}\ \forall t \in \tau$ implies $\boldsymbol{y}(t) = \mathbf{0}\ \forall t \in \tau$.

Proof. Suppose that the pair $((3), (4))$ is observable. Let the condition $\boldsymbol{z}(t) = \mathbf{0}\ \forall t \in \tau$ hold true. It implies that $\boldsymbol{\omega}(t) = \mathbf{0}\ \forall t \in \tau$. This is valid if and only if $\mathbf{M}\boldsymbol{y}(t) = \mathbf{0}\ \forall t \in \tau$, as indicated in equation (4). Equivalently, we have $\mathbf{M}\exp(-\mathbf{F}(t - t_\mathrm{i}))\boldsymbol{y}(t_\mathrm{i}) = \mathbf{0}\ \forall t \in \tau$ after substituting the solution of system (3). Evaluating this equation at $t = t_\mathrm{i}$, we obtain $\mathbf{M}\boldsymbol{y}(t_\mathrm{i}) = \mathbf{0}$, and after its successive differentiation with respect to t, we arrive at $\mathbf{MF}\boldsymbol{y}(t_\mathrm{i}) = \mathbf{MF}^2\boldsymbol{y}(t_\mathrm{i}) = \mathbf{MF}^3\boldsymbol{y}(t_\mathrm{i}) = \ldots = \mathbf{0}$ at $t = t_\mathrm{i}$. These conditions are equivalent to expressing $\begin{bmatrix} \mathbf{M}^\intercal & (\mathbf{MF})^\intercal & \ldots & (\mathbf{MF}^{N-k-1})^\intercal \end{bmatrix}^\intercal \boldsymbol{y}(t_\mathrm{i}) = \mathbf{0}$, or $\mathcal{O}\boldsymbol{y}(t_\mathrm{i}) = \mathbf{0}$, where $\mathcal{O}$ is defined in (7). This holds true if and only if $\boldsymbol{y}(t_\mathrm{i}) = \mathbf{0}$ given that $\mathrm{rank}(\mathcal{O}) = N - k$, and is valid if and only if $\boldsymbol{y}(t) = \mathbf{0}\ \forall t \in \tau$.

The converse can be established by contrapositive. Assume the system pair $((3), (4))$ is unobservable and the condition $\boldsymbol{z}(t) = \mathbf{0}$ holds $\forall t \in \tau$. Following a similar argument as presented in the preceding paragraph we obtain $\mathcal{O}\boldsymbol{y}(t_\mathrm{i}) = \mathbf{0}$. This time, as $\mathrm{rank}(\mathcal{O}) < N - k$, it is not necessary that $\boldsymbol{y}(t_\mathrm{i}) = \mathbf{0}$, and hence the condition $\boldsymbol{y}(t) = \mathbf{0}\ \forall t \in \tau$ does not necessarily hold true. □

Remark 2. Theorem 1 does not hold in a general context for a pair of LTI systems [17, pp. 109]]. This is particularly true when the input function is not influenced by any of the network agents, and it represents an exogenous control function. Specifically, the fact that a pair of LTI systems is observable does not necessarily imply that the condition $\mathbf{z}(t) = \mathbf{0}\ \forall t \in \tau$ implies $\mathbf{y}(t) = \mathbf{0}\ \forall t \in \tau$. However, the validity of the statement that $\mathbf{z}(t) = \mathbf{0}\ \forall t \in \tau$ implies $\mathbf{y}(t) = \mathbf{0}\ \forall t \in \tau$ does necessarily imply that the pair of LTI systems is indeed observable.

Remark 3 Generalization to higher dimensions**.** We introduce Kronecker product to model the problem if the state space of agents of $\mathfrak{N}$ is of higher dimensions $\mathbb{R}^n$ ($n \geqq 2$). In this case, system (2) take the form:

$$\dot{\mathbf{x}}(t) = -(\mathscr{L} \otimes \mathbf{I}_n)\mathbf{x}(t), \quad t \in \tau, \tag{2$'$}$$

and the pair ((3), (4)) transforms into the pair ((3′), (4′)), where system (3′) is:

$$\dot{\mathbf{y}}(t) = -(\mathbf{F} \otimes \mathbf{I}_n)\mathbf{y}(t) - (\mathbf{L} \otimes \mathbf{I}_n)\mathbf{z}(t), \quad t \in \tau \tag{3$'$}$$

and system (4′) is:

$$\boldsymbol{\omega}(t) = -(\mathbf{M} \otimes \mathbf{I}_n)\mathbf{y}(t), \quad t \in \tau. \tag{4$'$}$$

Here
$\mathbf{x}(t) := \begin{bmatrix} \mathbf{x}_1(t)^\intercal & \ldots & \mathbf{x}_N(t)^\intercal \end{bmatrix}^\intercal \in \mathbb{R}^{nN}$,
$\mathbf{y}(t) := \begin{bmatrix} \mathbf{y}_1(t)^\intercal & \ldots & \mathbf{y}_{N-k}(t)^\intercal \end{bmatrix}^\intercal \in \mathbb{R}^{n(N-k)}$,
$\mathbf{z}(t) := \begin{bmatrix} \mathbf{z}_1(t)^\intercal & \ldots & \mathbf{z}_k(t)^\intercal \end{bmatrix}^\intercal \in \mathbb{R}^{nk}$, and $\boldsymbol{\omega}(t) := \dot{\mathbf{z}}(t) + (\mathbf{N} \otimes \mathbf{I}_n)\mathbf{z}(t)$. It can be shown that the observability of the pair ((3′), (4′)) is equivalent to the observability of the pair ((3), (4)). In other words, the observability results presented in this paper remain valid in the higher dimensional case.

4.1. Spectral characterization

In this subsection, we examine how the spectral properties of the matrices associated with the system pair ((3), (4)) influence its observability.

Proposition 1. The system pair ((3), (4)) is observable if and only if none of the eigenvectors of $\mathbf{F}$ in $\mathbb{C}^{N-k}$ are simultaneously orthogonal to all rows of $\mathbf{M}$. In other words, for all $\lambda \in \sigma(-\mathbf{F})$,

$$\ker(\lambda \mathbf{I}_n + \mathbf{F}) \cap \ker(\mathbf{M}) = \{\mathbf{0}\}.$$

This is a direct consequence of the PBH observability test mentioned earlier.

It is evident that the sensor–actuator observability property of $\mathfrak{N}$ depends on the protocols (3) & (4), as well as on the matrices $\mathbf{F}$ and $\mathbf{M}$ (the communications among the actuators and the communications between actuators & sensors). Notably, the observability tests given so far do not incorporate the time factor t due to the static nature of the communication topology in $\mathfrak{N}$. Consequently, we can deduce that the sensor–actuator observability property of $\mathfrak{N}$ remains consistent over any interval $\tau' = [t_\mathrm{i}, t]$, $0 \leqq t_\mathrm{i} < t < +\infty$.

Lemma 1. The system pair ((3), (4)) is observable if and only if the system pair ((3″), (4″)), where system (3″) is

$$\dot{\boldsymbol{\theta}}(t) = -\mathbf{D}\boldsymbol{\theta}(t) - (\mathbf{V}^\intercal\mathbf{L})\mathbf{z}(t), \quad t \in \tau \tag{3$''$}$$

and system (4″) is

$$\boldsymbol{\omega}(t) = -(\mathbf{M}\mathbf{V})\boldsymbol{\theta}(t), \quad t \in \tau, \tag{4$''$}$$

is observable for any of the eigendecomposition of $\mathbf{F} = \mathbf{V}\mathbf{D}\mathbf{V}^\intercal$ over $\mathbb{R}$.

Proof. As a submatrix of a symmetric matrix $\mathscr{L}$, $\mathbf{F}$ is symmetric as well. Therefore, it has a real spectrum and is diagonalizable over $\mathbb{R}$ with a decomposition in the form $\mathbf{V}\mathbf{D}\mathbf{V}^\intercal$, in which the columns of $\mathbf{V}$ are the orthonormal real eigenvectors of $\mathbf{F}$ and the principal diagonal entries of $\mathbf{D}$ are the corresponding eigenvalues of $\mathbf{F}$. Thus, the system pair ((3), (4)) can be written as the pair consisting of

$$\dot{\mathbf{y}}(t) = -(\mathbf{V}\mathbf{D}\mathbf{V}^\intercal)\mathbf{y}(t) - \mathbf{L}\mathbf{z}(t), \quad t \in \tau$$

and

$$\boldsymbol{\omega}(t) = -\mathbf{M}\mathbf{y}(t), \quad t \in \tau.$$

Premultiplying the first equation above by $\mathbf{V}^\intercal$ and defining $\boldsymbol{\theta}(t) := \mathbf{V}^\intercal\mathbf{y}(t)$, we obtain the system pair ((3″), (4″)), thus implying its observability equivalency with the pair ((3), (4)). □

Remark 4. In the eigendecomposition of $\mathbf{F}$ used in Lemma 1, the matrix pair ($\mathbf{V}$, $\mathbf{D}$) may not be unique. In other words, for some other pair ($\tilde{\mathbf{V}}$, $\tilde{\mathbf{D}}$) with the same properties, we may also have $\mathbf{F} = \tilde{\mathbf{V}}\tilde{\mathbf{D}}\tilde{\mathbf{V}}^\intercal$. However, it is not difficult to verify that the observability property of the pair ((3″), (4″)) is the same as that of the pair composed of the following systems:

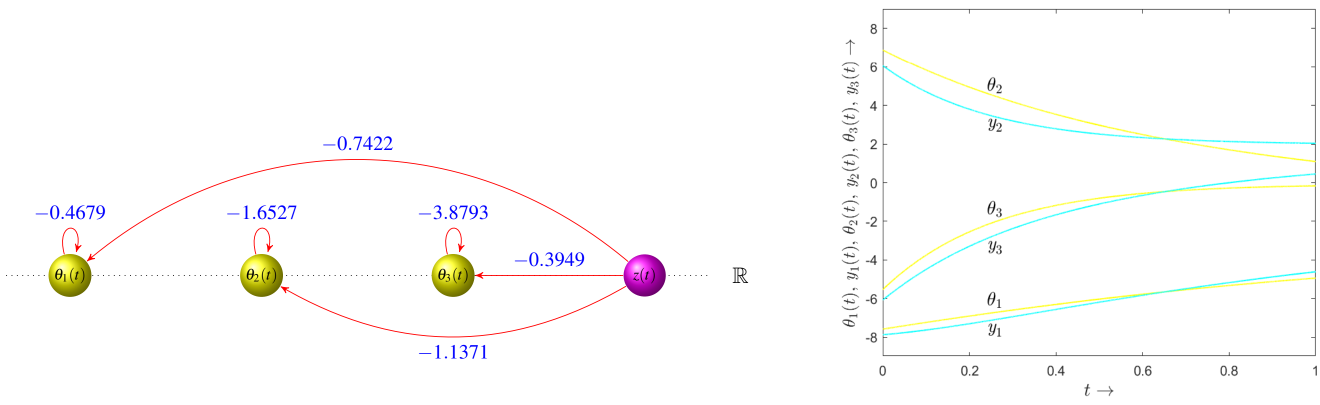
$$\dot{\boldsymbol{\theta}}(t) = -\tilde{\mathbf{D}}\boldsymbol{\theta}(t) - (\tilde{\mathbf{V}}^\intercal\mathbf{L})\mathbf{z}(t), \quad t \in \tau$$

and

$$\boldsymbol{\omega}(t) = -(\mathbf{M}\tilde{\mathbf{V}})\boldsymbol{\theta}(t), \quad t \in \tau.$$

Exploring the observability of $\mathfrak{N}$ through the pair ((3″), (4″)) provides more insights into the structure of $\mathfrak{N}$ (see Fig. 3).

Proposition 2. The following statements are equivalent:
1. The pair ((3), (4)) is observable.
2. The matrix
$$\int_{t_\mathrm{i}}^{t_\mathrm{f}} \exp(-\mathbf{D}(t-t_\mathrm{i}))\mathbf{V}^\intercal\mathbf{M}^\intercal\mathbf{M}\mathbf{V}\exp(-\mathbf{D}(t-t_\mathrm{i}))\mathrm{d}t$$
is invertible.
3. The $k(N-k) \times (N-k)$ matrix:
$$\begin{bmatrix} (\mathbf{M}\mathbf{V})^\intercal & (\mathbf{M}\mathbf{V}\mathbf{D})^\intercal & \ldots & (\mathbf{M}\mathbf{V}\mathbf{D}^{N-k-1})^\intercal \end{bmatrix}^\intercal$$
possesses full (column) rank equal to $N-k$.
4. The $N \times (N-k)$ matrix:
$$\begin{bmatrix} \lambda\mathbf{I}_{N-k} + \mathbf{D} & -(\mathbf{M}\mathbf{V})^\intercal \end{bmatrix}^\intercal$$
possesses full (column) rank equal to $N-k$ for each $\lambda \in \sigma(-\mathbf{D})$.



(a) (b)

Fig. 3. The network of Fig. 1 is reproduced again where 4 takes the role of sensor. (a) Inference diagram of the dynamical system: $\dot{\boldsymbol{\theta}}(t) = -\mathbf{D}\boldsymbol{\theta}(t) - (\mathbf{V}^\intercal\mathbf{L})z(t)$, $t \in \tau$, obtained after decoupling system (3). By comparing this diagram with Figs. 1(d) and 2(b), we observe that, in general, when certain agents of $\mathfrak{N}$ have been assigned to the role of sensors and decouple system (3), the flow of information from actuator states to sensor states ceases to exist. Additionally, there is a termination of information flow among the sensor states (in the case of multiple sensors in $\mathfrak{N}$), and sensor states do not self-interact. These observations align with those for the system (3). However, what sets this scenario apart is that, even the information flow among the actuator states ceases to exist, the self-interaction strengths among actuator states change, and perhaps there will be an information flow from sensor states to all actuator states. In this network, due to differences in the self-interaction strengths among all actuator states (indicating that $\mathbf{F}$ possesses $N-k$ distinct eigenvalues) and the flow of information from sensor states to all actuator states (meaning every row in $\mathbf{V}^\intercal\mathbf{L}$ contains at least one nonzero element, i.e., each column of $\mathbf{V}$ is not simultaneously orthogonal to all rows of $\mathbf{L}^\intercal = \mathbf{M}$), the corresponding system pair ((3), (4)) is observable as per Theorem 2. This means that the network in Fig. 1 is observable with 4 serving as its sensor. (b) A comparison of trajectories of actuators with the corresponding state solution trajectories of the decoupled system: $\dot{\boldsymbol{\theta}}(t) = -\mathbf{D}\boldsymbol{\theta}(t) - (\mathbf{V}^\intercal\mathbf{L})z(t)$ over the interval $\tau = [0, 1]$

5. $\ker(\lambda\mathbf{I}_n + \mathbf{D}) \cap \ker(\mathbf{MV}) = \{\mathbf{0}\} \subsetneqq \mathbb{C}^{N-k}$ for each $\lambda \in \sigma(-\mathbf{D})$, which means no eigenvector of $\mathbf{D}$ in $\mathbb{C}^{N-k}$ is simultaneously orthogonal to all rows of $\mathbf{MV}$.

Theorem 2. For $\mathbf{F} = \mathbf{VDV}^\intercal$, a necessary condition for the observability of the pair ((3), (4)) is that "each column of $\mathbf{V}$ is not simultaneously orthogonal to all rows of $\mathbf{M}$." This condition becomes sufficient if $\mathbf{F}$ has $(N-k)$ distinct eigenvalues.

Proof. This theorem can be proven by contradiction.

Necessity of the condition: Let the pair ((3), (4)) be observable, but assuming that the r_0th column '$\boldsymbol{v}_{r_0}$' of $\mathbf{V}$ is simultaneously orthogonal to all rows of $\mathbf{M}$, where $\mathbf{V} = \begin{bmatrix} \boldsymbol{v}_1 & \dots & \boldsymbol{v}_{N-k} \end{bmatrix}$, $\boldsymbol{v}_r \in \mathbb{C}^{N-k}$, $r = 1, \dots, (N-k)$. Then, the r_0th column of $\mathbf{MV} = \begin{bmatrix} \mathbf{M}\boldsymbol{v}_1 & \dots & \mathbf{M}\boldsymbol{v}_{N-k} \end{bmatrix}$ is a zero vector, and hence $\boldsymbol{e}_{r_0} \in \ker(\mathbf{MV})$; $\boldsymbol{e}_{r_0}$ is the standard basis vector of $\mathbb{C}^{N-k}$. This contradicts statement 5) of Proposition 2 on the observability test, as $\boldsymbol{e}_{r_0}$ is always an eigenvector of $\mathbf{D}$ (with the corresponding eigenvalue being the r_0th diagonal entry of $\mathbf{D}$).

Sufficiency of the condition: Assume that the pair ((3), (4)) is unobservable. Because of $(N-k)$ distinct eigenvalues for $\mathbf{F}$, every eigenvector of $\mathbf{D}$ is of the form $\boldsymbol{v} = \begin{bmatrix} \alpha_1\delta_{1r} & \dots & \alpha_{N-k}\delta_{(N-k)r} \end{bmatrix}^\intercal$, where all $\alpha_r \in \mathbb{C}$ are nonzeros, and δ_{qr} is a Kronecker delta function for $q, r = 1, \dots, (N-k)$. Furthermore, because each column of $\mathbf{V}$ is not simultaneously orthogonal to all rows of $\mathbf{M}$, every column of $\mathbf{MV}$ is a nonzero vector. Then $\mathbf{MV}\boldsymbol{v} \neq \mathbf{0}$, which validates statement 5) in Proposition 2. This contradicts our assumption that the system pair ((3), (4)) is unobservable. □

Now, we demonstrate that the proof of the sufficiency condition in Theorem 2 breaks down for a network if we disregard the $(N-k)$ distinct eigenvalues for $\mathbf{F}$.

Example 1. Let us consider a path network $\mathscr{P}_7$ with a central agent as its sensor (Fig. 4(a)). We have:

$$\mathbf{F} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \mathbf{L} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{M}^\intercal,$$

and $\mathbf{N} = \begin{bmatrix} 2 \end{bmatrix}$.

The eigendecomposition of $\mathbf{F}$ over $\mathbb{R}$ gives

$$\mathbf{V} \approx \begin{bmatrix} 0.737 & 0 & 0.591 & 0 & 0 & -0.328 \\ -0.591 & 0 & -0.328 & 0 & 0 & 0.737 \\ -0.328 & 0 & 0.737 & 0 & 0 & 0.591 \\ 0 & -0.328 & 0 & 0.737 & -0.591 & 0 \\ 0 & -0.591 & 0 & 0.328 & 0.737 & 0 \\ 0 & -0.737 & 0 & -0.591 & -0.328 & 0 \end{bmatrix}$$

and

$\mathbf{D} \approx \mathrm{diag}(0.1981, 0.1981, 1.5550, 1.5550, 3.2470, 3.2470)$.

It is worth noting that every column of

$$\mathbf{MV} \approx \begin{bmatrix} 0.328 & 0.328 & -0.737 & -0.737 & 0.591 & -0.591 \end{bmatrix}$$

contains a nonzero number, indicating that every column of $\mathbf{V}$ is not simultaneously orthogonal to all rows of $\mathbf{M}$. However, a quick inspection through the observability test reveals that the corresponding pair ((3), (4)) is unobservable.

Furthermore, to establish the validity of the converse in Theorem 2, the condition that $\mathbf{F}$ has $(N-k)$ distinct eigenvalues is not an absolute requirement. The following example illustrates this point.

Example 2. Let us consider a cycle network $\mathscr{C}_6$ with sensors 5 and 6 (Fig. 4(b)). We compute:

$$\mathbf{F} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \mathbf{L} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix} = \mathbf{M}^\mathsf{T}, \text{ and}$$

$\mathbf{N} = 2\mathbf{I}_2$.

The eigendecomposition of $\mathbf{F}$ over $\mathbb{R}$ gives:

$$\mathbf{V} = \begin{bmatrix} -1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \text{ and}$$

$\mathbf{D} = \mathrm{diag}(1, 1, 3, 3)$.

Furthermore, in $\mathbf{MV} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$, each column contains a nonzero number, indicating that every column of $\mathbf{V}$ is not simultaneously orthogonal to all rows of $\mathbf{M}$, and it can be verified that the corresponding pair ((3), (4)) is observable. However, $\mathbf{F}$ has $2 < 6-2 = N-k$ distinct eigenvalues.

The following observability tests rely solely on the eigenvalues of the system matrices.

Theorem 3. For the observability of the system pair ((3), (4)):

1. A necessary condition is that

$$\max_{\lambda \in \sigma(-\mathbf{F})} \mathrm{AM}(\lambda) \leqq k.$$

2. A sufficient condition is that

$$\sigma(\mathbf{F}) \cap \sigma\left(\left[\begin{array}{c|c} \mathbf{F} & \mathbf{L} \\ \hline \mathbf{M} & \mathbf{N} \end{array}\right]\right) = \emptyset.$$

Proof. Instead of presenting a direct proof, we prove the contrapositive of the above statements.

For the necessity, suppose that there exists a $\lambda_0 \in \sigma(-\mathbf{F})$ such that $\mathrm{AM}(\lambda_0) = k + r$ for some $r \in \{1, 2, \ldots, (N-2k)\}$. In that case, the matrix $(\lambda_0 \mathbf{I}_{N-k} + \mathbf{D})$ will have exactly $(k+r)$ rows filled with only zeros, so $\mathrm{rank}(\lambda_0 \mathbf{I}_{N-k} + \mathbf{D}) = N - 2k - r$. Next, notice that, if we assume $(N-k) \leqq k$, then, as in general $\mathrm{AM}(\lambda_0) \leqq$ the dimension of $\mathbf{F} = N-k$, it implies that $\mathrm{AM}(\lambda_0) \leqq k$, contradicting our hypothesis $\mathrm{AM}(\lambda_0) = k + r$. Thus, $(N-k) > k$, which leads to:

$$\begin{aligned} \mathrm{rank} \begin{bmatrix} \lambda_0 \mathbf{I}_{N-k} + \mathbf{D} & -(\mathbf{MV})^\mathsf{T} \end{bmatrix}^\mathsf{T} \\ &= \mathrm{rank} \begin{bmatrix} \lambda_0 \mathbf{I}_{N-k} + \mathbf{D} & (\mathbf{MV})^\mathsf{T} \end{bmatrix} \\ &\leqq \mathrm{rank}(\lambda_0 \mathbf{I}_{N-k} + \mathbf{D}) + \mathrm{rank}\left((\mathbf{MV})^\mathsf{T}\right) \\ &\leqq (N - 2k - r) + k < (N-k). \end{aligned}$$

This condition implies that the pair ((3), (4)) is unobservable, following statement 4) of Proposition 2.

Now, for sufficiency, assume that the pair ((3), (4)) is unobservable. According to Proposition 1, there exists a $\boldsymbol{v}_0 (\neq \mathbf{0}) \in \mathbb{C}^{N-k}$ and a $\lambda_0 \in \sigma(-\mathbf{F})$ such that $\boldsymbol{v}_0 \in \ker(\lambda_0 \mathbf{I}_n + \mathbf{F}) \cap \ker(\mathbf{M})$. This means:

$$\mathbf{F}\boldsymbol{v}_0 = -\lambda_0 \boldsymbol{v}_0 \text{ and } \mathbf{M}\boldsymbol{v}_0 = \mathbf{0}. \tag{†}$$

Using (†), we deduce that:

$$\left[\begin{array}{c|c} \mathbf{F} & \mathbf{L} \\ \hline \mathbf{M} & \mathbf{N} \end{array}\right] \begin{bmatrix} \boldsymbol{v}_0 \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{F}\boldsymbol{v}_0 \\ \mathbf{M}\boldsymbol{v}_0 \end{bmatrix} = \begin{bmatrix} -\lambda_0 \boldsymbol{v}_0 \\ \mathbf{0} \end{bmatrix} = -\lambda_0 \begin{bmatrix} \boldsymbol{v}_0 \\ \mathbf{0} \end{bmatrix},$$

indicating that $\lambda_0 \in \sigma\left(-\left[\begin{array}{c|c} \mathbf{F} & \mathbf{L} \\ \hline \mathbf{M} & \mathbf{N} \end{array}\right]\right)$, and hence $\sigma(\mathbf{F}) \cap \sigma\left(\left[\begin{array}{c|c} \mathbf{F} & \mathbf{L} \\ \hline \mathbf{M} & \mathbf{N} \end{array}\right]\right) \neq \emptyset$. □

The following results are the direct consequences of Theorem 3.

Corollary 1. If $k = 1$ and $\mathbf{F}$ has fewer than $(N-1)$ distinct eigenvalues, then the pair ((3), (4)) is unobservable.

Corollary 2. If the pair ((3), (4)) is unobservable, there exists at least one eigenvector $\boldsymbol{v}_0 \in \mathbb{C}^{N-k}$ of $\mathbf{F}$ such that $\begin{bmatrix} \boldsymbol{v}_0^\mathsf{T} & \mathbf{0}^\mathsf{T} \end{bmatrix}^\mathsf{T} \in \mathbb{C}^N$ will be one of the eigenvector of $\left[\begin{array}{c|c} \mathbf{F} & \mathbf{L} \\ \hline \mathbf{M} & \mathbf{N} \end{array}\right]$.

The following two examples illustrate that the condition stated in statement 1) of Theorem 3 is not sufficient and the condition in statement 2) of Theorem 3 is not necessary for the observability of the system pair ((3), (4)).

Example 3. Let us consider a cycle network $\mathscr{C}_5$ with 5 as its sensor (Fig. 4(c)). We have:

$$\mathbf{F}=\begin{bmatrix}2 & 1 & 0 & 0\\ 1 & 2 & -1 & 0\\ 0 & -1 & 2 & 1\\ 0 & 0 & 1 & 2\end{bmatrix}, \mathbf{L}=\begin{bmatrix}-1\\ 0\\ 0\\ -1\end{bmatrix}=\mathbf{M}^{\mathsf{T}}, \text{and}\, \mathbf{N}=\begin{bmatrix}2\end{bmatrix}.$$

The matrix $\mathbf{F}$ has four distinct eigenvalues (viz., 0.382, 1.382, 2.618, 3.618), which satisfies

$$\max_{\lambda\in\sigma(-\mathbf{F})} \mathrm{AM}(\lambda)=1=k,$$

as required by the condition stated in statement 1) of Theorem 3. However, the corresponding pair ((3), (4)) is unobservable.

Example 4. Let us consider a wheel network $\mathscr{W}_5$ with 3 and 4 as its sensors (Fig. 4(d)). We obtain the following matrices:

$$\mathbf{F}=\begin{bmatrix}4 & -1 & -1\\ -1 & 3 & -1\\ -1 & -1 & 3\end{bmatrix}, \mathbf{L}=\begin{bmatrix}-1 & -1\\ 0 & -1\\ -1 & 0\end{bmatrix}=\mathbf{M}^{\mathsf{T}}, \text{and}$$

$$\mathbf{N}=\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}.$$

We observe that the corresponding pair ((3), (4)) is observable. However, by examining the eigenvalues we find:

$$\sigma(\mathbf{F})\approx\{1.2679, 4, 4.7321\} \text{ and}$$

$$\sigma\left(\left[\begin{array}{c|c}\mathbf{F} & \mathbf{L}\\ \hline \mathbf{M} & \mathbf{N}\end{array}\right]\right)\approx\{0.5505, 1.5858, 4, 4.4142, 5.4495\},$$

where both sets of eigenvalues share the element 4.

Theorem 4. A necessary condition for the observability of the pair ((3), (4)) is that there exists no eigenvector of $\left[\begin{array}{c|c}\mathbf{F} & \mathbf{L}\\ \hline \mathbf{M} & \mathbf{N}\end{array}\right]$ with zeros in its last k entries.

Proof. This theorem can be proved by contradiction. Let the pair ((3), (4)) be observable, but assuming that $\boldsymbol{\eta}:=\begin{bmatrix}\boldsymbol{v}^{\mathsf{T}} & \mathbf{0}^{\mathsf{T}}\end{bmatrix}^{\mathsf{T}}\in\mathbb{C}^N$ is an eigenvector of $\left[\begin{array}{c|c}\mathbf{F} & \mathbf{L}\\ \hline \mathbf{M} & \mathbf{N}\end{array}\right]$, where its last k entries are zeros. Let the corresponding eigenvalue be "$-\lambda$". Here $\boldsymbol{v}\in\mathbb{C}^{N-k}$ must be a nonzero vector, for otherwise $\boldsymbol{\eta}$ becomes a zero vector, which violates the definition of an eigenvector. Now, as

$$\begin{bmatrix}\mathbf{F}\boldsymbol{v}\\ \mathbf{M}\boldsymbol{v}\end{bmatrix}=\left[\begin{array}{c|c}\mathbf{F} & \mathbf{L}\\ \hline \mathbf{M} & \mathbf{N}\end{array}\right]\begin{bmatrix}\boldsymbol{v}\\ \mathbf{0}\end{bmatrix}=\left[\begin{array}{c|c}\mathbf{F} & \mathbf{L}\\ \hline \mathbf{M} & \mathbf{N}\end{array}\right]\boldsymbol{\eta}=-\lambda\boldsymbol{\eta}=\begin{bmatrix}-\lambda\boldsymbol{v}\\ \mathbf{0}\end{bmatrix},$$

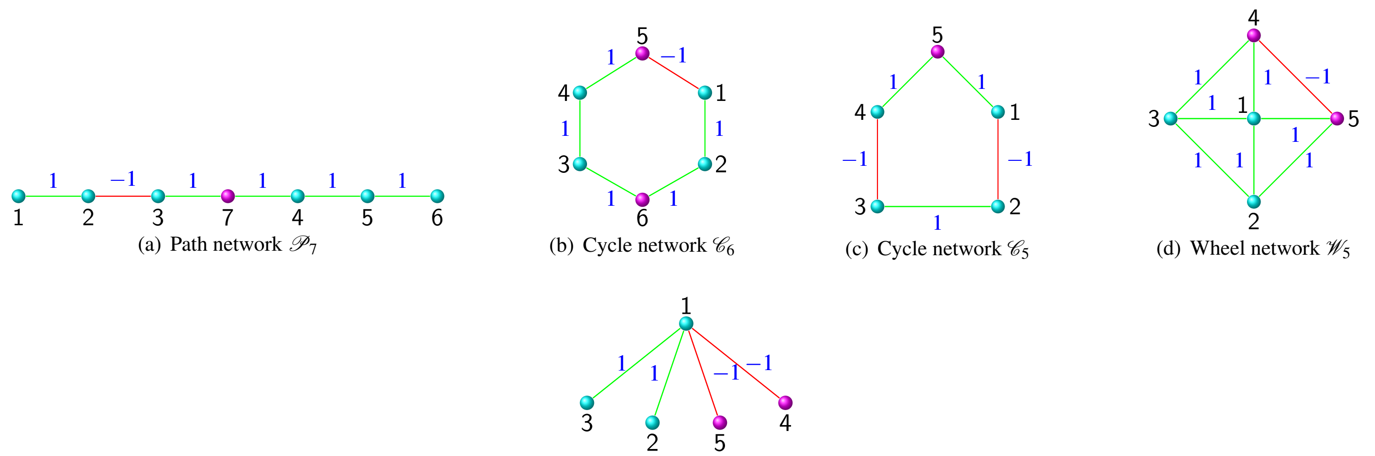
we have $\mathbf{F}\boldsymbol{v}=-\lambda\boldsymbol{v}$ and $\mathbf{M}\boldsymbol{v}=\mathbf{0}$, which implies that $\boldsymbol{v}\in\ker(\lambda\mathbf{I}_n+\mathbf{F})\cap\ker(\mathbf{M})$. This contradicts Proposition 1, implying that the system pair ((3), (4)) is unobservable, which contradicts our hypothesis. □

Now, let us explore an example demonstrating that the observability test provided in Theorem 4 is not sufficient for the observability of the pair ((3), (4)).

Example 5. Consider a star network $\mathscr{S}_5$ with 4 and 5 as its sensors (Fig. 4(e)). We have the following matrices:

$$\mathbf{F}=\begin{bmatrix}4 & -1 & -1\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}, \mathbf{L}=\begin{bmatrix}1 & 1\\ 0 & 0\\ 0 & 0\end{bmatrix}=\mathbf{M}^{\mathsf{T}}, \text{and}\ \mathbf{N}=\mathbf{I}_2.$$

When examining the spectrum, we find that all eigenvalues of $\left[\begin{array}{c|c}\mathbf{F} & \mathbf{L}\\ \hline \mathbf{M} & \mathbf{N}\end{array}\right]$ are 0, 1, 1, 1, 5 with corresponding eigenvectors as



(a) Path network $\mathscr{P}_7$ (b) Cycle network $\mathscr{C}_6$ (c) Cycle network $\mathscr{C}_5$ (d) Wheel network $\mathscr{W}_5$ (e) Star network $\mathscr{S}_5$

Fig. 4. Interconnection graphs of the considered networks in Examples 1–5

described below:

$$\boldsymbol{\eta}_1 \approx \alpha_1 \begin{bmatrix} 1 & 1 & 1 & -1 & -1 \end{bmatrix}^\mathsf{T},$$
$$\boldsymbol{\eta}_2 \approx \alpha_2 \begin{bmatrix} 0 & -1 & 2 & 1 & 0 \end{bmatrix}^\mathsf{T},$$
$$\boldsymbol{\eta}_3 \approx \alpha_3 \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \end{bmatrix}^\mathsf{T},$$
$$\boldsymbol{\eta}_4 \approx \alpha_4 \begin{bmatrix} 0 & 1 & 1 & -1 & 3 \end{bmatrix}^\mathsf{T}, \text{ and}$$
$$\boldsymbol{\eta}_5 \approx \alpha_5 \begin{bmatrix} 4 & -1 & -1 & 1 & 1 \end{bmatrix}^\mathsf{T},$$

where the coefficients α_r are nonzero complex numbers for $r = 1, 2, 3, 4, 5$. Importantly, the last two entries in none of the above eigenvectors are zeros. However, we observe that the corresponding pair ((3), (4)) is unobservable.

4.2. Topological characterization

In this subsection, we explore how the topological structure of $\mathfrak{N}$ influences its sensor–actuator observability.

Proposition 3. The sensor–actuator observability property of $\mathfrak{N}$ remains invariant under any of the following cases:

1. The inclusion of new sensor-to-sensor communications, whether cooperative or competitive.
2. The removal of existing sensor-to-sensor communications.
3. A change in the type of communication among the sensors, such as switching from cooperative communication to competitive, or vice versa.

Proof. It is important to note that any of the three situations mentioned can only affect the entries of $\mathbf{N}$, while all the entries in $\mathbf{F}$ and $\mathbf{M}$ remain unchanged. Since the sensor–actuator observability of $\mathfrak{N}$ is solely determined by the matrices $\mathbf{F}$ and $\mathbf{M}$, the conclusion follows. Refer to Fig. 5 for an illustration of this proposition. □

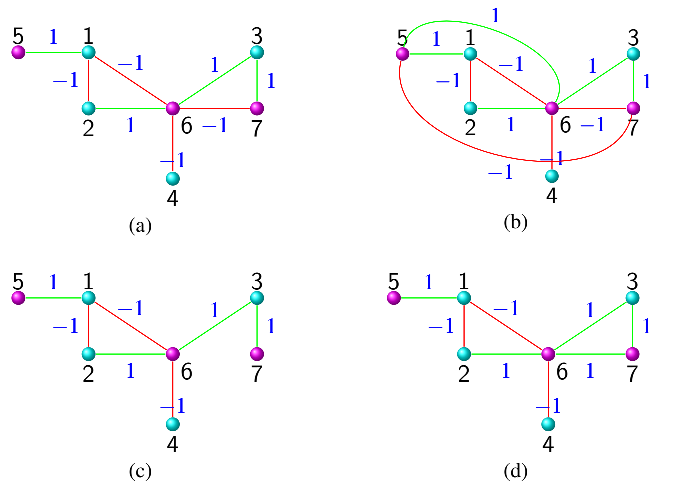


Fig. 5. Example to illustrate Proposition 3. The figure depicts the interconnection graphs of different networks. It can be verified that the network shown in Fig. 5(a) is observable with the sensor set $\{5, 6, 7\}$. It is evident that the networks in Fig. 5(b), 5(c), and 5(d) are also observable with the same sensor set $\{5, 6, 7\}$, as these networks are derived from Fig. 5(a) following the guidelines of Proposition 3

By applying a union operation to multiple networks, a new network is formed. This is obviously disconnected. To determine the observability of this composite network, we examine the observability of its constituent networks. Specifically, for each $q \in \mathscr{I}_m$, $\mathfrak{N}^{(q)}$ represents a network with N_q number of agents. Let $\mathscr{G}^{(q)} = (\mathscr{V}^{(q)}, \mathscr{E}^{(q)})$ be the corresponding interconnection graph with adjacency matrix $\mathscr{A}^{(q)}$. Let the sensor and actuator sets of $\mathfrak{N}^{(q)}$ be $\mathscr{V}_s^{(q)}$ and $\mathscr{V}_a^{(q)}$ with k_q and $N_q - k_q$ elements, respectively. Let $\mathbf{F}^{(q)} \in \mathbb{R}^{(N_q-k_q)\times(N_q-k_q)}$, $\mathbf{L}^{(q)} \in \mathbb{R}^{(N_q-k_q)\times k_q}$, $\mathbf{M}^{(q)} \in \mathbb{R}^{k_q\times(N_q-k_q)}$, and $\mathbf{N}^{(q)} \in \mathbb{R}^{k_q\times k_q}$ be the submatrices obtained from $\mathscr{L}^{(q)} = \mathscr{L}(\mathscr{G}^{(q)})$ by following the similar partitions as described earlier.

Lemma 2. Suppose the composite network, with the interconnection graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, is formed by the union of $\mathfrak{N}^{(q)}$s, as proposed below:

(i) $\mathscr{V} := \bigcup_{q=1}^m \mathscr{V}^{(q)}$ and $\mathscr{E} := \bigcup_{q=1}^m \mathscr{E}^{(q)}$;

(ii) The sensor set of the composite network is $\mathscr{V}_s := \bigcup_{q=1}^m \mathscr{V}_s^{(q)} = \{1, \ldots, (k_1 + \cdots + k_m)\}$, where the first k_1 elements are the re-indexed elements of $\mathscr{V}_s^{(1)}$, the next k_2 elements (viz., $(k_1+1), \ldots, (k_1+k_2)$) are the re-indexed elements of $\mathscr{V}_s^{(2)}$, and so on. The last k_m elements (viz., $(k_1 + \ldots + k_{m-1} + 1), \ldots, (k_1 + \ldots + k_{m-1} + k_m)$) are the re-indexed elements of $\mathscr{V}_s^{(m)}$. The actuator set of the composite network, $\mathscr{V}_a := \bigcup_{q=1}^m \mathscr{V}_a^{(q)} = \{(k_1 + \ldots + k_m + 1), \ldots (N_1 + \ldots + N_m)\}$, where the first $N_1 - k_1$ elements (viz., $(k_1 + \ldots + k_m + 1), \ldots, (k_2 + \ldots + k_m + N_1)$) are the re-indexed elements of $\mathscr{V}_a^{(1)}$, the next $N_2 - k_2$ elements (viz., $(k_2 + \ldots + k_m + N_1 + 1), \ldots, (k_3 + \ldots + k_m + N_1 + N_2)$) are the re-indexed elements of $\mathscr{V}_a^{(2)}$, and so on. The last $N_m - k_m$ elements (viz., $(k_m + N_1 + \ldots + N_{m-1} + 1), \ldots, (N_1 + \ldots + N_m)$) are the re-indexed elements of $\mathscr{V}_a^{(m)}$.

Then, the composite network is sensor–actuator observable if and only if each $\mathfrak{N}^{(q)}$ is sensor–actuator observable.

Proof. For simplicity, we demonstrate this lemma for the case of two networks, with the general case following from this example. Let $\mathbf{y}^{(q)}(t) \in \mathbb{R}^{N_q-k_q}$, $\mathbf{z}^{(q)}(t) \in \mathbb{R}^{k_q}$, and $\boldsymbol{\omega}^{(q)}(t) \in \mathbb{R}^{k_q}$ represent the stacked system states of actuators, sensors, and the output vector of network $\mathfrak{N}^{(q)}$, respectively, for $q = 1, 2$. Under the selected sensor configuration, the observability system pair for $\mathfrak{N}^{(1)}$ is as follows:

$$\dot{\mathbf{y}}^{(1)}(t) = -\mathbf{F}^{(1)}\mathbf{y}^{(1)}(t) - \mathbf{L}^{(1)}\mathbf{z}^{(1)}(t), \quad t \in \tau \tag{9}$$

and

$$\boldsymbol{\omega}^{(1)}(t) = -\mathbf{M}^{(1)}\mathbf{y}^{(1)}(t), \quad t \in \tau, \tag{10}$$

where $\boldsymbol{\omega}^{(1)}(t) := \dot{\mathbf{z}}^{(1)}(t) + \mathbf{N}^{(1)}\mathbf{z}^{(1)}(t)$, $t \in \tau$. Similarly, the observability system pair for $\mathfrak{N}^{(2)}$ is as follows:

$$\dot{\mathbf{y}}^{(2)}(t) = -\mathbf{F}^{(2)}\mathbf{y}^{(2)}(t) - \mathbf{L}^{(2)}\mathbf{z}^{(2)}(t), \quad t \in \tau \tag{11}$$

and

$$\boldsymbol{\omega}^{(2)}(t) = -\mathbf{M}^{(2)}\mathbf{y}^{(2)}(t), \quad t \in \tau, \tag{12}$$

where $\boldsymbol{\omega}^{(2)}(t) := \dot{\boldsymbol{z}}^{(2)}(t) + \mathbf{N}^{(2)}\boldsymbol{z}^{(2)}(t)$, $t \in \tau$. Like this, in the composite network, if we define the stacked system states of all actuators, sensors, and the output of the system, respectively as:

$$\boldsymbol{y}(t) := \begin{bmatrix}\boldsymbol{y}^{(1)\top}(t) & \boldsymbol{y}^{(2)\top}(t)\end{bmatrix}^\top \in \mathbb{R}^{N_1-k_1+N_2-k_2},$$
$$\boldsymbol{z}(t) := \begin{bmatrix}\boldsymbol{z}^{(1)\top}(t) & \boldsymbol{z}^{(2)\top}(t)\end{bmatrix}^\top \in \mathbb{R}^{k_1+k_2}, \text{ and}$$
$$\boldsymbol{\omega}(t) := \begin{bmatrix}\boldsymbol{\omega}^{(1)\top}(t) & \boldsymbol{\omega}^{(2)\top}(t)\end{bmatrix} \in \mathbb{R}^{k_1+k_2},$$

the observability system pair for the composite network will become:

$$\dot{\boldsymbol{y}}(t) = -\mathbf{F}\boldsymbol{y}(t) - \mathbf{L}\boldsymbol{z}(t), \quad t \in \tau \tag{13}$$

and

$$\boldsymbol{\omega}(t) = -\mathbf{M}\boldsymbol{y}(t), \quad t \in \tau, \tag{14}$$

where $\boldsymbol{\omega}(t) := \dot{\boldsymbol{z}}(t) + \mathbf{N}\boldsymbol{z}(t)$, $t \in \tau$. Here, $\mathbf{F} \in \mathbb{R}^{(N_1-k_1+N_2-k_2)\times(N_1-k_1+N_2-k_2)}$ subsumes only $\mathbf{F}^{(1)}$ & $\mathbf{F}^{(2)}$, given by $\mathbf{F} = \begin{bmatrix}\mathbf{F}^{(1)} & \mathbf{O} \\ \mathbf{O} & \mathbf{F}^{(2)}\end{bmatrix}$, $\mathbf{L} \in \mathbb{R}^{(N_1-k_1+N_2-k_2)\times(k_1+k_2)}$ subsumes only $\mathbf{L}^{(1)}$ & $\mathbf{L}^{(2)}$, given by $\mathbf{L} = \begin{bmatrix}\mathbf{L}^{(1)} & \mathbf{O} \\ \mathbf{O} & \mathbf{L}^{(2)}\end{bmatrix}$, $\mathbf{M} \in \mathbb{R}^{(k_1+k_2)\times(N_1-k_1+N_2-k_2)}$ subsumes only $\mathbf{M}^{(1)}$ & $\mathbf{M}^{(2)}$, given by $\mathbf{M} = \begin{bmatrix}\mathbf{M}^{(1)} & \mathbf{O} \\ \mathbf{O} & \mathbf{M}^{(2)}\end{bmatrix}$, and $\mathbf{N} \in \mathbb{R}^{(k_1+k_2)\times(k_1+k_2)}$ also subsumes only $\mathbf{N}^{(1)}$ & $\mathbf{N}^{(2)}$, given by $\mathbf{N} = \begin{bmatrix}\mathbf{N}^{(1)} & \mathbf{O} \\ \mathbf{O} & \mathbf{N}^{(2)}\end{bmatrix}$. It is worth noting that $\sigma(\mathbf{F}) = \sigma(\mathbf{F}^{(1)}) \cup \sigma(\mathbf{F}^{(2)})$, and the arbitrary eigenvector $\boldsymbol{v}$ of $\mathbf{F}$ is a vector in $\mathbb{C}^{N_1-k_1+N_2-k_2}$ given by the following:

(i) For each $\mu \in \sigma(\mathbf{F}^{(1)}) \setminus \sigma(\mathbf{F}^{(2)})$, all eigenvectors of $\mathbf{F}$ are of the form $\boldsymbol{v} = \begin{bmatrix}\boldsymbol{v}^{(1)\top} & \mathbf{0}^\top\end{bmatrix}^\top$, where $\boldsymbol{v}^{(1)} \in \mathbb{C}^{N_1-k_1}$ is any eigenvector of $\mathbf{F}^{(1)}$ corresponding to its eigenvalue μ.

(ii) For each $\mu \in \sigma(\mathbf{F}^{(2)}) \setminus \sigma(\mathbf{F}^{(1)})$, all eigenvectors of $\mathbf{F}$ are of the form $\boldsymbol{v} = \begin{bmatrix}\mathbf{0}^\top & \boldsymbol{v}^{(2)\top}\end{bmatrix}^\top$, where $\boldsymbol{v}^{(2)} \in \mathbb{C}^{N_2-k_2}$ is any eigenvector of $\mathbf{F}^{(2)}$ corresponding to its eigenvalue μ.

(iii) For each $\mu \in \sigma(\mathbf{F}^{(1)}) \cap \sigma(\mathbf{F}^{(2)})$, the eigenvectors of $\mathbf{F}$ are all of the forms:

$$\boldsymbol{v} = \begin{bmatrix}\boldsymbol{v}^{(1)\top} & \mathbf{0}^\top\end{bmatrix}^\top, \begin{bmatrix}\mathbf{0}^\top & \boldsymbol{v}^{(2)\top}\end{bmatrix}^\top, \begin{bmatrix}\boldsymbol{v}^{(1)\top} & \boldsymbol{v}^{(2)\top}\end{bmatrix}^\top,$$

where $\boldsymbol{v}^{(1)} \in \mathbb{C}^{N_1-k_1}$ is any eigenvector of $\mathbf{F}^{(1)}$ corresponding to its eigenvalue μ and $\boldsymbol{v}^{(2)} \in \mathbb{C}^{N_2-k_2}$ is any eigenvector of $\mathbf{F}^{(2)}$ corresponding to its eigenvalue μ.

Now, the proposed necessity and sufficient observability condition is proved.

First, sufficiency. Let both $\mathfrak{N}^{(1)}$ and $\mathfrak{N}^{(2)}$ be sensor–actuator observable. This means, the pairs ((9), (10)) and ((11), (12)) are observable. Take any $\mu \in \sigma(\mathbf{F})$. In the case where $\mu \in \sigma(\mathbf{F}^{(1)}) \setminus \sigma(\mathbf{F}^{(2)})$, consider any corresponding eigenvector $\boldsymbol{v} = \begin{bmatrix}\boldsymbol{v}^{(1)\top} & \mathbf{0}^\top\end{bmatrix}^\top$ of $\mathbf{F}$. Then, $\mathbf{M}\boldsymbol{v} = \begin{bmatrix}(\mathbf{M}^{(1)}\boldsymbol{v}^{(1)})^\top & \mathbf{0}^\top\end{bmatrix}^\top \neq \mathbf{0}$, as every eigenvector $\boldsymbol{v}^{(1)}$ of $\mathbf{F}^{(1)}$ satisfies $\mathbf{M}^{(1)}\boldsymbol{v}^{(1)} \neq \mathbf{0}$ by Proposition 1. In the case $\mu \in \sigma(\mathbf{F}^{(2)}) \setminus \sigma(\mathbf{F}^{(1)})$, consider any corresponding eigenvector $\boldsymbol{v} = \begin{bmatrix}\mathbf{0}^\top & \boldsymbol{v}^{(2)\top}\end{bmatrix}^\top$ of $\mathbf{F}$. Then, $\mathbf{M}\boldsymbol{v} = \begin{bmatrix}\mathbf{0}^\top & (\mathbf{M}^{(2)}\boldsymbol{v}^{(2)})^\top\end{bmatrix}^\top \neq \mathbf{0}$, as every eigenvector $\boldsymbol{v}^{(2)}$ of $\mathbf{F}^{(2)}$ satisfies $\mathbf{M}^{(2)}\boldsymbol{v}^{(2)} \neq \mathbf{0}$ by Proposition 1. Finally, in the case $\mu \in \sigma(\mathbf{F}^{(1)}) \cap \sigma(\mathbf{F}^{(2)})$, consider any corresponding eigenvector $\boldsymbol{v} = \begin{bmatrix}\boldsymbol{v}^{(1)\top} & \mathbf{0}^\top\end{bmatrix}^\top$ or $\begin{bmatrix}\mathbf{0}^\top & \boldsymbol{v}^{(2)\top}\end{bmatrix}^\top$ or $\begin{bmatrix}\boldsymbol{v}^{(1)\top} & \boldsymbol{v}^{(2)\top}\end{bmatrix}^\top$ of $\mathbf{F}$. Then we have $\mathbf{M}\boldsymbol{v} \neq \mathbf{0}$, as every eigenvector $\boldsymbol{v}^{(1)}$ of $\mathbf{F}^{(1)}$ and every eigenvector $\boldsymbol{v}^{(2)}$ of $\mathbf{F}^{(2)}$ satisfies $\mathbf{M}^{(1)}\boldsymbol{v}^{(1)} \neq \mathbf{0} \neq \mathbf{M}^{(2)}\boldsymbol{v}^{(2)}$. In any case, we have $\mathbf{M}\boldsymbol{v} \neq \mathbf{0}$ for any eigenvector $\boldsymbol{v}$ of $\mathbf{F}$ corresponding to any of its eigenvalue. This implies that the pair ((13), (14)) is observable by Proposition 1. In other words, the composite network is sensor–actuator observable.

Now, necessity. This is proven by contradiction. Let the composite network be sensor–actuator observable. Assume that at least one of the constituent networks, say $\mathfrak{N}^{(1)}$, is sensor–actuator unobservable. This makes the pair ((9), (10)) is unobservable, and Proposition 1 applies to show that there exists a $\mu_0 \in \sigma(\mathbf{F}^{(1)})$ with a corresponding eigenvector $\boldsymbol{v}_0^{(1)}$ of $\mathbf{F}^{(1)}$ such that $\mathbf{M}^{(1)}\boldsymbol{v}_0^{(1)} = \mathbf{0}$. Now, $\mu_0 \in \sigma(\mathbf{F})$, and for this μ_0, the vector $\boldsymbol{v}_0 = \begin{bmatrix}\boldsymbol{v}_0^{(1)\top} & \mathbf{0}^\top\end{bmatrix}^\top$ is definitely one of the eigenvectors of $\mathbf{F}$. This vector satisfies $\mathbf{M}\boldsymbol{v}_0 = \mathbf{M}^{(1)}\boldsymbol{v}_0^{(1)} = \mathbf{0}$. Proposition 1 applies now to show that the pair ((13), (14)) is unobservable, or the composite network is sensor–actuator unobservable. This contradicts the hypothesis. A similar contradiction arises if we assume $\mathfrak{N}^{(2)}$ is sensor–actuator unobservable. Therefore, we conclude that both $\mathfrak{N}^{(1)}$ and $\mathfrak{N}^{(2)}$ must be sensor–actuator observable. □

Theorem 5. Suppose a composite network with the interconnection graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is formed from $\mathfrak{N}^{(q)}$s, as described below.

(i) $\mathscr{V} := \bigcup_{q=1}^m \mathscr{V}^{(q)}$ and $\mathscr{E} := \bigcup_{q=1}^m \mathscr{E}^{(q)} \cup \mathscr{E}'$; where $\mathscr{E}' \subset \bigcup_{p\neq r,\, p\in\mathscr{I}_m,\, r\in\mathscr{I}_m} \mathscr{V}_s^{(p)} \times \mathscr{V}_s^{(r)}$ is nonempty, which contains lines with weights 1 or −1;

(ii) The sensor set of the composite network, $\mathscr{V}_s := \bigcup_{q=1}^m \mathscr{V}_s^{(q)} = \{1, \ldots, (k_1 + \cdots + k_m)\}$, where the first k_1 elements are the re-indexed elements of $\mathscr{V}_s^{(1)}$, the next k_2 elements (viz., $(k_1+1), \ldots, (k_1+k_2)$) are the re-indexed elements of $\mathscr{V}_s^{(2)}$, and so on. The last k_m elements (viz., $(k_1 + \ldots + k_{m-1} + 1), \ldots, (k_1 + \ldots + k_{m-1} + k_m)$) are the re-indexed elements of $\mathscr{V}_s^{(m)}$. The actuator set of the composite network, $\mathscr{V}_a := \bigcup_{q=1}^m \mathscr{V}_a^{(q)} = \{(k_1 + \ldots + k_m + 1), \ldots (N_1 + \ldots + N_m)\}$, where the first $N_1 - k_1$ elements (viz., $(k_1+\ldots+k_m+1), \ldots, (k_2+\ldots+k_m+N_1)$) are the re-indexed elements of $\mathscr{V}_a^{(1)}$, the next $N_2 - k_2$ elements (viz., $(k_2 + \ldots + k_m + N_1 + 1), \ldots, (k_3 + \ldots + k_m + N_1 + N_2)$) are the re-indexed elements of $\mathscr{V}_a^{(2)}$, and so on. The last $N_m - k_m$ elements (viz., $(k_m + N_1 + \ldots + N_{m-1} + 1), \ldots, (N_1 + \ldots + N_m)$) are the re-indexed elements of $\mathscr{V}_a^{(m)}$.

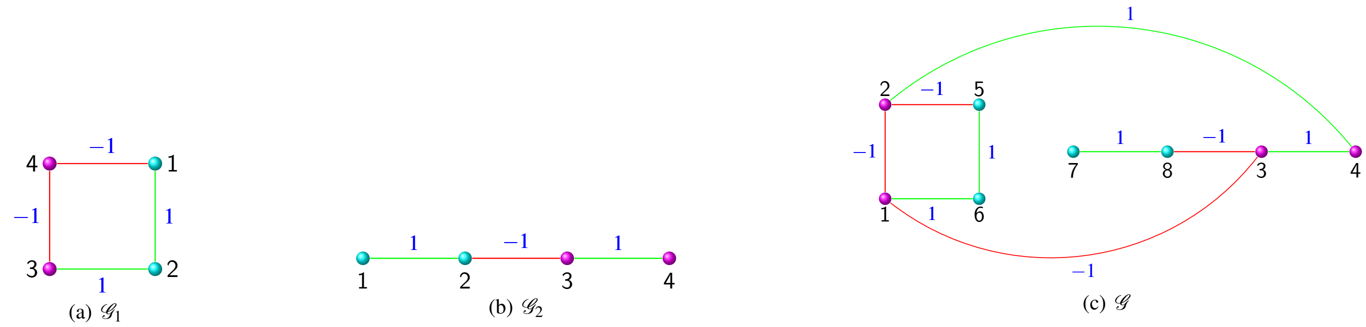


Fig. 6. Example illustrating Theorem 5. (a) Interconnection graph of $\mathfrak{N}^{(1)}$ with $\{3,4\}$ as the sensor set. (b) Interconnection graph of $\mathfrak{N}^{(2)}$ with $\{3,4\}$ as the sensor set. Both networks are sensor–actuator observable. (c) Interconnection graph of the composite network formed by the union of $\mathfrak{N}^{(1)}$ and $\mathfrak{N}^{(2)}$ with additional communications only among the sensors. New communications include sensor 3 of $\mathfrak{N}^{(1)}$ to sensor 3 of $\mathfrak{N}^{(2)}$ (with weight -1) and sensor 4 of $\mathfrak{N}^{(1)}$ to sensor 4 of $\mathfrak{N}^{(2)}$ (with weight 1). This composite network remains sensor–actuator observable as it satisfies the conditions of Theorem 5

Then the necessary and sufficient condition for the sensor–actuator observability of the composite network is that, each $\mathfrak{N}^{(q)}$ is sensor–actuator observable.

Proof. The sufficiency follows by Lemma 2 and then applying Proposition 3, while necessity follows by applying Proposition 3 first and then Lemma 2. □

The networks considered in Fig. 6 illustrate the application of this theorem.

Remark 5. In the composite network formed by the union of several networks (Lemma 2), the total number of agents is $\sum_{q=1}^{m} N_q$, and the total number of communications is $\sum_{q=1}^{m} |\mathscr{E}^{(q)}|$. Although other operations may exist for evolving different composite networks with more agents and communications while preserving observability property with their factor networks, these are not explicitly discussed here.

Remark 6. Combining Lemma 2 and Theorem 5, the composite network is sensor–actuator unobservable if and only if at least one of its constituent networks is sensor–actuator unobservable. Our hypothesis for this result is that the sensor set of the composite network is the union of the sensor sets of all its constituent networks. With a different choice of sensors in the composite network, it may be observable even if none of the constituent networks are observable.

In general, the observability property of a network with a specific choice of sensors may not be equivalent to the observability property of the corresponding unsigned network, even with the same choice of sensors (refer to Fig. 7 for an example). However, if the network is structurally balanced, this equivalency holds true for a special set of sensors in the network.

Definition 2 Structural balance. A graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is structurally balanced if $\mathscr{V}$ can be partitioned into $\mathscr{V}_1$ and $\mathscr{V}_2$ with the following conditions:
(i) $\mathscr{V}_1 \cup \mathscr{V}_2 = \mathscr{V}$ and $\mathscr{V}_1 \cap \mathscr{V}_2 = \emptyset$;
(ii) Intra-group lines within $\mathscr{V}_1$ and $\mathscr{V}_2$ (if they exist) have a weight of 1, while inter-group lines between points in $\mathscr{V}_1$ and $\mathscr{V}_2$ (if they exist) have a weight of -1.

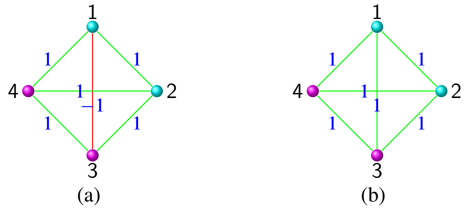


Fig. 7. (a) Interconnection graph of a signed network. (b) Interconnection graph of the corresponding unsigned network. It is confirmed that, with sensors 3 and 4, the signed network is observable, while its unsigned variant is unobservable with the same sensors

A network is structurally balanced if and only if its interconnection graph is structurally balanced.

Proposition 4. The observability property of $\mathfrak{N}$ with a particular choice of sensors is equivalent to the observability property of the corresponding unsigned network with the same choice of sensors, under the following conditions:
1. $\mathfrak{N}$ is structurally balanced. This implies the existence of $\mathscr{V}_1$ and $\mathscr{V}_2$, satisfying $\mathscr{V}_1 \cup \mathscr{V}_2 = \mathscr{V}(\mathscr{G})$, $\mathscr{V}_1 \cap \mathscr{V}_2 = \emptyset$, and adhering to condition (ii) in Definition 2.
2. All sensors in $\mathfrak{N}$ must belong to either $\mathscr{V}_1$ or $\mathscr{V}_2$.

Proof. For notational simplicity, let us denote the unsigned version of network $\mathfrak{N}$ by $\mathfrak{N}^+$ and the corresponding unsigned version of interconnection graph $\mathscr{G}$ by $\mathscr{G}^+$. We observe that the adjacency matrix $\mathscr{A}^+$ (of $\mathscr{G}^+$) is related to the adjacency matrix $\mathscr{A}$ (of $\mathscr{G}$) by $\mathscr{A}^+ = \mathscr{H}\mathscr{A}\mathscr{H}$, where $\mathscr{H} := \mathrm{diag}(h_{ii})_{i\in\mathscr{I}_N}$, with $h_{ii} = 1$ if $i \in \mathscr{V}_1$ and $h_{ii} = -1$ if $i \in \mathscr{V}_2$ (the values of h_{ii} can also be taken in reverse order). As the degree matrices of $\mathscr{G}$ and $\mathscr{G}^+$ are the same, denoted by Δ, we obtain $\mathscr{L}^+ := \Delta - \mathscr{A}^+ = \mathscr{H}(\Delta - \mathscr{A})\mathscr{H} = \mathscr{H}\mathscr{L}\mathscr{H}$, where $\mathscr{L}^+$ is the Laplacian of $\mathscr{G}^+$.

The dynamics of the agents in $\mathfrak{N}^+$ are given by $\dot{\boldsymbol{x}}^+(t) = -\mathscr{L}^+\boldsymbol{x}^+(t)$ for $t \in \tau$. When partitioned, we obtain

$$\begin{bmatrix} \dot{\mathbf{y}}^+(t) \\ \dot{\mathbf{z}}^+(t) \end{bmatrix} = -\left[\begin{array}{c|c} \mathbf{F}^+ & \mathbf{L}^+ \\ \hline \mathbf{M}^+ & \mathbf{N}^+ \end{array}\right] \begin{bmatrix} \mathbf{y}^+(t) \\ \mathbf{z}^+(t) \end{bmatrix}, \quad t \in \tau.$$

Here, one can verify that $\mathbf{F}^+ = \mathscr{H}_a\mathbf{F}\mathscr{H}_a$, $\mathbf{L}^+ = \mathscr{H}_a\mathbf{L}\mathscr{H}_s$, $\mathbf{M}^+ = \mathscr{H}_s\mathbf{M}\mathscr{H}_a$, and $\mathbf{N}^+ = \mathscr{H}_s\mathbf{N}\mathscr{H}_s$, where $\mathscr{H}_a \in \mathbb{R}^{(N-k)\times(N-k)}$ and $\mathscr{H}_s \in \mathbb{R}^{k\times k}$ are derived from $\mathscr{H}$. $\mathscr{H}_a$ retains only those rows and columns corresponding to actuators, while $\mathscr{H}_s$ retains only those corresponding to sensors. Hence, the dynamics of all actuators become

$$\dot{\mathbf{y}}^+(t) = -(\mathscr{H}_a\mathbf{F}\mathscr{H}_a)\mathbf{y}^+(t) - (\mathscr{H}_a\mathbf{L}\mathscr{H}_s)\mathbf{z}^+(t), \quad t \in \tau, \tag{15}$$

and its output equation is

$$\boldsymbol{\omega}^+(t) = -(\mathscr{H}_s\mathbf{M}\mathscr{H}_a)\mathbf{y}^+(t), \quad t \in \tau, \tag{16}$$

where $\boldsymbol{\omega}^+(t) := \dot{\mathbf{z}}^+(t) + (\mathscr{H}_s\mathbf{N}\mathscr{H}_s)\mathbf{z}^+(t)$, $t \in \tau$. Clearly, $\mathfrak{N}^+$ is observable with the given choice of sensors if and only if the pair ((15), (16)) is observable, and this is possible if and only if

$$\operatorname{rank}\Big[(\mathscr{H}_s\mathbf{M}\mathscr{H}_a)^\intercal \quad (\mathscr{H}_s\mathbf{M}\mathscr{H}_a\cdot\mathscr{H}_a\mathbf{F}\mathscr{H}_a)^\intercal \quad \dots \quad (\mathscr{H}_s\mathbf{M}\mathscr{H}_a\cdot(\mathscr{H}_a\mathbf{F}\mathscr{H}_a)^{N-k-1})^\intercal\Big]^\intercal = N-k$$

according to Kalman's method of observability checking. Since all sensors of $\mathfrak{N}$ belong to either $\mathscr{V}_1$ or $\mathscr{V}_2$, we have $\mathscr{H}_a\mathscr{H}_a = \mathbf{I}_{N-k}$ and $\mathscr{H}_s\mathbf{M} = \pm\mathbf{M}$. Therefore, the above condition becomes

$$\operatorname{rank}\Big[\pm(\mathbf{M}\mathscr{H}_a)^\intercal \quad \pm(\mathbf{MF}\mathscr{H}_a)^\intercal \quad \dots \quad \pm(\mathbf{MF}^{N-k-1}\mathscr{H}_a)^\intercal\Big]^\intercal = N-k,$$

which gives

$$\begin{aligned}\operatorname{rank}&\Big[\mathbf{M}^\intercal \quad (\mathbf{MF})^\intercal \quad \dots \quad (\mathbf{MF}^{N-k-1})^\intercal\Big]^\intercal \\ &= \operatorname{rank}\Big(\pm\Big[\mathbf{M}^\intercal \quad (\mathbf{MF})^\intercal \quad \dots \quad (\mathbf{MF}^{N-k-1})^\intercal\Big]^\intercal\mathscr{H}_a\Big) \\ &= \operatorname{rank}\Big[\pm(\mathbf{M}\mathscr{H}_a)^\intercal \quad \pm(\mathbf{MF}\mathscr{H}_a)^\intercal \quad \dots \quad \pm(\mathbf{MF}^{N-k-1}\mathscr{H}_a)^\intercal\Big]^\intercal \\ &= N-k.\end{aligned}$$

This means that the pair ((3), (4)) is observable, and hence $\mathfrak{N}$ is observable with the same choice of sensors as those in $\mathfrak{N}^+$. □

The networks in Figs. 8 and 9 show the applications of Proposition 4.

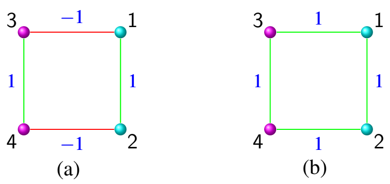


Fig. 8. The network in (a) is observable under the chosen sensor set $\{3, 4\}$, as we can check. This network is structurally balanced. Its point set can be partitioned as $\mathscr{V} = \mathscr{V}_1 \cup \mathscr{V}_2$ with $\mathscr{V}_1 = \{1, 2\}$ and $\mathscr{V}_2 = \{3, 4\}$. Since both sensors 3 and 4 belong to $\mathscr{V}_2$, so the corresponding unsigned network in (b) is also observable with sensors 3 and 4 in accordance with Proposition 4

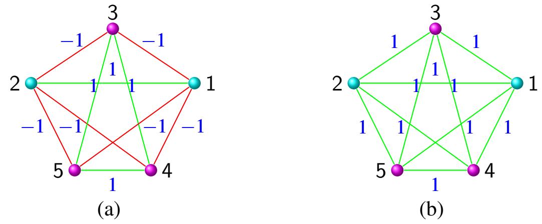


Fig. 9. The network in (a) is not observable under the chosen sensor set $\{3, 4, 5\}$, as we can check. This network is structurally balanced. Its point set can be partitioned as $\mathscr{V} = \mathscr{V}_1 \cup \mathscr{V}_2$ with $\mathscr{V}_1 = \{1, 2\}$ and $\mathscr{V}_2 = \{3, 4, 5\}$. Since all sensors 3, 4, and 5 belong to $\mathscr{V}_2$ alone, so the corresponding unsigned network in (b) is also unobservable with sensors 3, 4, and 5 in accordance with Proposition 4

5. MINIMAL SENSOR PROBLEM

Motivated by the formula [equation (5.2) in [42]] for determining the minimum number of leaders for guaranteed controllability of unsigned networks, this section establishes a formula for computing the minimum number of sensors to ensure sensor–actuator observable $\mathfrak{N}$. From the PBH observability test stated in Section 4, if $\mathfrak{N}$ has $(N-1)$ sensors, then $\mathcal{P}_\lambda$ possesses full (column) rank equal to 1 for every $\lambda \in \sigma(-\mathbf{F})$, making the pair ((3), (4)) observable. While this ensures observability, it demands complete knowledge of $(N-1)$ agents' states to reconstruct a unique initial state of just one agent, which is generally undesirable. Here we address the question: "From the full knowledge of the states of what minimum number of agents in $\mathfrak{N}$ it is possible to uniquely reconstruct the initial states of the rest of the agents?" This is equivalent to asking for the least number of sensors that ensure the sensor–actuator observable $\mathfrak{N}$, referred to as the minimal observability problem. We present a formula that utilizes the spectral information of $\mathscr{L}$ to determine the minimum value k, denoted as $k_{\min}$, resulting in the sensor–actuator observable $\mathfrak{N}$.

Let an arbitrary sensor set with which $\mathfrak{N}$ is observable be $\{i_1, \dots, i_k\} \subsetneqq \{1, \dots, \mathrm{N}\}$. Since there is an equivalence between the observability of the pair ((3), (4)) under the sensor set $\{i_1, \dots, i_k\}$ and the observability of system (2) under the output equation $\boldsymbol{w}(t) = \mathbf{B}_k\boldsymbol{x}(t)$, $t \in \tau$, where $\mathbf{B}_k := \Big[\boldsymbol{e}_{i_1} \quad \dots \quad \boldsymbol{e}_{i_k}\Big]^\intercal$ is a $k \times N$ permutation matrix, the minimal observability problem reduces to:

For what minimal value of k, $\operatorname{rank}\Big[\lambda\mathbf{I}_N + \mathscr{L} \quad \mathbf{B}_k^\intercal\Big] = N$ for all $\lambda \in \sigma(-\mathscr{L})$?

Since, in general $\operatorname{rank}(\lambda\mathbf{I}_N + \mathscr{L}) < N$ for all $\lambda \in \sigma(-\mathscr{L})$, the validity of the rank condition at every $\lambda \in \sigma(-\mathscr{L})$ requires $\operatorname{rank}(\mathbf{B}_k) \geqq N - \min_{\lambda\in\sigma(-\mathscr{L})}\{\operatorname{rank}(\lambda\mathbf{I}_N + \mathscr{L})\}$, i.e., the total number of linearly independent rows in $\mathbf{B}_k$ must be $\geqq N - \min_{\lambda\in\sigma(-\mathscr{L})}\{\operatorname{rank}(\lambda\mathbf{I}_N + \mathscr{L})\}$, or $k \geqq N - \min_{\lambda\in\sigma(-\mathscr{L})}\{\operatorname{rank}(\lambda\mathbf{I}_N + \mathscr{L})\}$, as $\mathbf{B}_k$ contains k linearly independent rows. Hence, the minimum value of k ensuring $\operatorname{rank}\Big[\lambda\mathbf{I}_N + \mathscr{L} \quad \mathbf{B}_k^\intercal\Big] = N$ for all $\lambda \in \sigma(-\mathscr{L})$ is given by:

$$k_{\min} = N - \min_{\lambda\in\sigma(-\mathscr{L})}\{\operatorname{rank}(\lambda\mathbf{I}_N + \mathscr{L})\}.$$

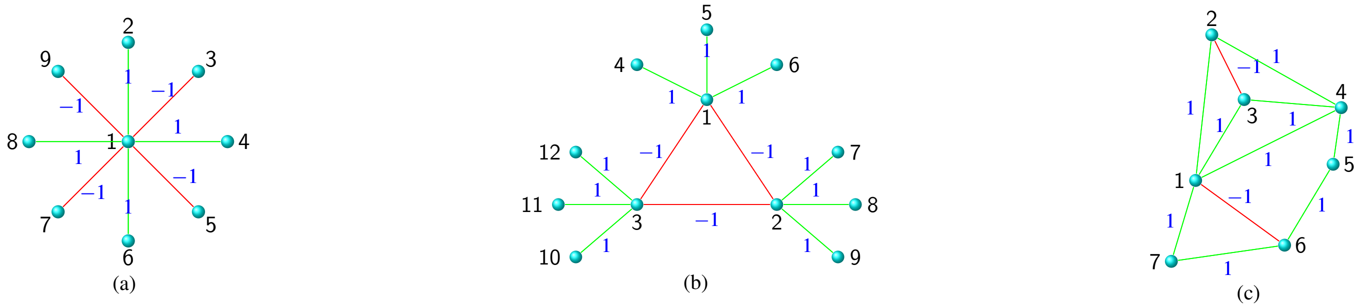


Fig. 10. Determination of $k_{\min}$ by using the formula given in (17): (a) $k_{\min} = 7$. (b) $k_{\min} = 6$. (c) $k_{\min} = 1$

The right-hand side in the above equation equals $\max_{\lambda \in \sigma(-\mathscr{L})}\{\mathrm{GM}(\lambda)\}$ or $\max_{\mu \in \sigma(\mathscr{L})}\{\mathrm{GM}(\mu)\}$. As $\mathscr{L}$ is diagonalizable, this number equals the maximum algebraic multiplicities of eigenvalues of $\mathscr{L}$:

$$k_{\min} = \max_{\mu \in \sigma(\mathscr{L})} \{\mathrm{AM}(\mu)\}. \tag{17}$$

For some of the networks (in Fig. 10), $k_{\min}$ is computed, and its value shows that the interconnection topology of the network has a profound role in the solution to the minimal sensor problem.

Above, we addressed the minimal sensor problem for the network $\mathfrak{N}$ (i.e., we proposed a formula to compute the minimum number of sensors required to ensure that $\mathfrak{N}$ is sensor–actuator observable) from an algebraic point-of-view, since the formula in (17) is developed using the PBH observability test and involves spectral knowledge (eigenvalue information) of the underlying Laplacian.

In contrast to this algebraic approach, various graphical approaches have been developed in [26, 43–47] to solve the minimal sensor problem but in the context of structural networks. (Structural networks are those in which the weights of the links are not known precisely but are known within a certain tolerance and can thus be chosen as independent free parameters.) The lack of knowledge about the precise weights for links in such networks provides flexibility in developing graphical approaches to efficiently solve the minimal sensor problem. In [26], a graphical approach was developed to predict the minimal sensor set in polynomial time to ensure the observability of networks with nonlinear dynamics, using a biochemical reaction network to demonstrate the approach application. A maximum matching concept from graph theory can also be used to solve the minimal sensor problem in polynomial time [43]. By considering structural cyclic networks in [44], a polynomial-time algorithmic solution was obtained to determine both the number and locations of sensors while minimizing the global cost associated with sensors in tracking the unknown states in the network. A polynomial-time algorithmic solution was also proposed in [45] to solve the minimal sensor problem for networks with general nonlinear dynamics. Furthermore, [46] solved the minimal sensor problem in Kronecker composite networks based on the S-rank and strong-connectivity of their constituent networks, while in [47], the minimal sensor set in the Cartesian product network was obtained based on the knowledge of minimal sensor set in its constituent networks.

Note that the independent free parameter assumption is very strong. When the exact weights of the network links are known, as in our network $\mathfrak{N}$ (where the link weights are precisely known as $+1$ or -1), the structural observability theory developed via graph-theoretic approaches could yield misleading results for solving the minimal sensor problem [48]. In other words, the minimal sensor set that ensures observability in structural networks does not necessarily match the minimal sensor set that ensures observability in networks with precisely known weighted links. (See Example 5.2, p. 3171 and 3178 in [42] in the context of the minimal input problem to understand this difference. A similar example can be constructed for our minimal sensor problem to illustrate this distinction, as controllability and observability properties are dual to each other in multi-agent systems with LTI dynamics.) Solving the minimal sensor problem in networks with precisely known weighted links is crucial for achieving exact observability (as opposed to structural observability), especially in man-made engineered networked systems and sparse real-world networks. For these networks, our algebraic test given in equation (17) can be applied efficiently to precisely compute the minimum number of sensors needed to ensure the network observability in its sensor–actuator framework.

Moving ahead, since the Laplacians of all paths and cycles follow specific patterns, as shown in the following sections, the formula in (17) reveals that for any path, a single sensor (chosen appropriately within the path) ensures its observability, and for any cycle, minimum of two sensors (chosen appropriately within the cycle) ensures its observability.

5.1. Path network

Let $\mathscr{P}_N$ denote the path network with N agents, and let $\mathscr{L}$ be its Laplacian. In [49], it is noted that $\sigma(\mathscr{L}) = \sigma(\mathscr{L}^+)$, where $\mathscr{L}^+$ represents the Laplacian of $\mathscr{P}_N^+$. Furthermore, according to [Proposition 4.10, [50]],

$$\sigma(\mathscr{L}) = \{\mu_r \,|\, r = 0, 1, 2, \ldots, (N-1)\},$$

where $\mu_r = 4\sin^2\left(r \times \frac{\pi}{2N}\right)$. Since all these μ_r values are distinct for $r = 0, 1, 2, \ldots, (N-1)$, $\sigma(\mathscr{L})$ contains N distinct elements.

Therefore, $k_{\min} = 1$, indicating that a single sensor is sufficient to ensure the observable $\mathscr{P}_N$.

5.2. Cycle network

Let $\mathscr{C}_N$ denote the cycle network with N agents, and let $\mathscr{L}$ be its Laplacian. We examine two scenarios:

1. Suppose $\mathscr{C}_N$ contains an even number of competitive communications. By [Corollary 1.23, [51]], we have

$$\sigma(\mathscr{L}) = \{\mu_r \,|\, r = 0, 1, 2, \dots, (N-1)\},$$

   where $\mu_r = 4\sin^2\left(2r \times \frac{\pi}{2N}\right)$. We inspect the algebraic multiplicities of elements in $\sigma(\mathscr{L})$ under two cases:

   (a) For even N, the observations are as follows: $|\sigma(\mathscr{L})| = (N+2)/2$, $\min(\sigma(\mathscr{L})) = \mu_0 = 0$ with $\mathrm{AM}(\mu_0) = 1$, $\max(\sigma(\mathscr{L})) = \mu_{N/2} = 4$ with $\mathrm{AM}(\mu_{N/2}) = 1$. The remaining $(N-2)/2$ eigenvalues have AM $= 2$ each, with $\mu_{\left(\frac{N}{2}\right)-p} = \mu_{\left(\frac{N}{2}\right)+p}$ for all $p \in \{1, 2, 3, \dots, (N-2)/2\}$.

   (b) For odd N, we get: $|\sigma(\mathscr{L})| = (N+1)/2$, $\min(\sigma(\mathscr{L})) = \mu_0 = 0$ with $\mathrm{AM}(\mu_0) = 1$. The remaining $(N-1)/2$ eigenvalues have AM $= 2$ each, satisfying $\mu_{\left(\frac{N+1}{2}\right)-p} = \mu_{\left(\frac{N-1}{2}\right)+p}$ for all $p \in \{1, 2, 3, \dots, (N-1)/2\}$.

   In either case, we have $\max_{\mu\in\sigma(\mathscr{L})}\{\mathrm{AM}(\mu)\} = 2$, implying $k_{\min} = 2$. Thus, a minimum of two sensors ensures the observable $\mathscr{C}_N$ when there is an even number of competitive communications.

2. Suppose $\mathscr{C}_N$ contains an odd number of competitive communications. Again, by [Corollary 1.23, [51]], we have

$$\sigma(\mathscr{L}) = \{\mu_r \,|\, r = 0, 1, 2, \dots, (N-1)\},$$

   where $\mu_r = 4\sin^2\left((2r+1) \times \frac{\pi}{2N}\right)$. Here also, we inspect the algebraic multiplicities of elements in $\sigma(\mathscr{L})$ under two cases:

   (a) For even N, we obtain: $|\sigma(\mathscr{L})| = N/2$, with all $N/2$ distinct eigenvalues having AM $= 2$ each, such that $\mu_{\left(\frac{N}{2}\right)-p} = \mu_{\left(\frac{N}{2}\right)+p-1}$ for all $p \in \{1, 2, 3, \dots, N/2\}$.

   (b) For odd N, we get the following: $|\sigma(\mathscr{L})| = (N+1)/2$, $\max(\sigma(\mathscr{L})) = \mu_{(N-1)/2} = 4$ with $\mathrm{AM}(\mu_{(N-1)/2}) = 1$. The remaining $(N-1)/2$ eigenvalues have AM $= 2$ each, such that $\mu_{\left(\frac{N-1}{2}\right)-p} = \mu_{\left(\frac{N-1}{2}\right)+p}$ for all $p \in \{1, 2, 3, \dots, (N-1)/2\}$.

   In either case, $\max_{\mu\in\sigma(\mathscr{L})}\{\mathrm{AM}(\mu)\} = 2$, leading to $k_{\min} = 2$. Therefore, a minimum of two sensors ensures the observable $\mathscr{C}_N$ when there is an odd number of competitive communications.

Formula in (17) provides $k_{\min}$ for any network. Notably, the value of $k_{\min}$ remains independent of the size and type of communications in the case of paths and cycles. However, in arbitrary networks, where paths and cycles act as building blocks, $k_{\min}$ varies unpredictably and significantly depends on the type of communications (cooperative/competitive). Furthermore, formula in (17) does not provide information about the location of the $k_{\min}$ sensors in the given network. For instance, in signed path $\mathscr{P}_8$, any node can be chosen as a sensor to ensure observability, while in signed path $\mathscr{P}_7$, any single sensor can be chosen anywhere except at the center to ensure observability.

Given an arbitrary (signed) network, solving the minimal sensor problem – i.e., determining $k_{\min}$ using the formula in (17) – poses a major challenge in computing the eigenvalues of the underlying Laplacian. To efficiently determine these eigenvalues, which are required by the general maximum multiplicity theory for quantifying the sensor–actuator observability of $\mathfrak{N}$, a singular value decomposition (SVD)-based algorithm can be designed [52]. After computing $k_{\min}$, we can consider designing an algorithm that traces all possible sensor locations within the network that solve this minimal sensor problem. However, in this research, our primary focus was on developing the theoretical framework for solving the minimal sensor problem. To locate such sensor positions within the network, we can use the Gaussian elimination method that comes with a computation complexity in $O(N^2(\ln(N))^2)$ [52]. Additionally, we have not addressed the minimization of the global cost associated with the sensors in solving the minimal sensor problem. These aspects present potential directions for future research.

6. CONCLUSIONS

This work focuses on signed multi-agent networks, where agents communicate through Laplacian feedback. Employing a geometrical approach, we illustrate, through an example, that partial knowledge of sensors results in inaccurate retrieval of actuator states, highlighting the necessity of complete sensor information to address the observability problem effectively. We investigate network observability by leveraging the spectral properties of the system matrices. Verifiable algebraic tests, elucidating the role of eigenvalue–eigenvector pairs in determining network observability, are established and validated through numerical examples. Additionally, we perform observability analysis based on the graph topological properties of networks, revealing that the union of networks is observable if and only if each factor network is observable. For structurally balanced networks, we find that, under a specific set of sensors, the observability of signed networks is equivalent to the corresponding unsigned variant of networks. Utilizing matrix algebra tools, we derive a formula to determine the minimum number of sensors required for ensured observability of the network. Consequently, we conclude that all paths are observable with just one sensor, and all cycles are observable with a minimum of two sensors. The presented results are general, applying not only to signed networks but also to unsigned networks, as unsigned networks are a particular case of signed networks.

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