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Revision of the formula describing the spectrum of output signals at A/D converters

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Abstract—**This paper crowns efforts, made by its author, aiming in showing and proving that the current formula for calculation of the spectra of output signals at A/D converters requires a correcting factor in it. A number of partial results obtained and published in the last years are referred to here. They paved the way to a fully satisfactory and correct result; it is presented in this work. The corrected formula for spectrum calculation is derived using a description of the output signal of an A/D converter by means of the so-called Dirac comb, however not in a direct form, but with taking into account physical reality. In addition, the paper contains a number of interpretative remarks, comments, and explanations - clarifying those matters that have so far been omitted in analyses of the sampling process, despite the fact that they raised various types of doubts.**

Keywords—**Modelling of output signals of A/D converters for calculation of their spectra and a corrected formula for performing this task**

I. Introduction

THERE are many important formulas that are used in signal processing. And their understanding does not cause troubles, except for a one. This is the highly celebrated formula that determines the spectrum of a sampled signal treated as a signal of a continuous time at the output of an analog-to-digital (A/D) converter. It has the following form:

$$X_{A/D}\left(f\right)=\frac{1}{T}\sum_{k=-\infty}^{\infty}X\left(f-kf_s\right)\,, \qquad (1)$$

where $X\left(f\right)$ and $X_{A/D}\left(f\right)$ stand for the spectra (Fourier transforms) of a band-limited, un-sampled signal $x\left(t\right)$ and, respectively, of its sampled version $x_{A/D}\left(t\right)$ at the output of an A/D converter (in the sense as mentioned above). The continuous frequency and continuous time are denoted here by f and t, accordingly. Furthermore, T means the sampling period, but f_s stands for the sampling frequency; $T=1/f_s$. Moreover, the indices k in (1) belong to the set of integers, i.e. $k\in Z$.

A trouble with the formula (1) lies in the fact that it is a direct consequence of modelling the signal $x_{A/D}\left(t\right)$ with the help of weighted Dirac deltas, i.e. with the use of objects which are not present at the outputs of A/D converters. Consequently, this formula is not fully reliable. Its credibility should be demonstrated in some way.

The author of this paper tried to perform this task by checking various arguments present in the literature, conventional opinions and beliefs of researchers working in the field of signal processing that refer to the description of signals at the output of A/D converters through Dirac deltas and/or so-called non-ideal variants associated with this description.

The works mentioned above have been done in a careful, comprehensive, and thorough manner. The outcomes of these pursuits have been published in a series of papers [1]–[11]. However, in order not to be exposed to the accusation of self-plagiarism, they will not be discussed here. Besides, there is no room for them here, in this short paper; but, see that all this material is readily available on the websites whose addresses are provided in the section References below. Moreover, note that a critical summary of all these efforts along with the reasons of failures of the various ideas discussed in [1]–[11], the false paths leading to nowhere, the lessons learned, and an ordering of all this material can be found in [12].

The correct form for signals appearing at outputs of A/D converters – considered as functions of a continuous time – was derived in [12] and [13]. And, as well known, it is precisely such functions, i.e. functions of a continuous time, that are used in calculations of the spectra of signals at the outputs of A/D converters. The signal form derived therein has the shape of a step function – weighted by samples of an analog signal applied at the input of the A/D converter. Examples provided in [12] and [13] illustrate its use. This form is also exploited here.

The analysis presented in [12] and its outcomes explain why the values of a step function between the successive instants of signal sampling (i.e. step values) should be taken as they appear at the output of an A/D converter. Simply because they then correspond directly with the discrete signal values with which a microprocessor connected to the output of the A/D converter works. That is they are equal to them, and this is the best choice when it comes to the need to take into account any non-idealities of the signal sampling process. Then any possible imperfections of this process are automatically accounted for in both. What do we mean here? Namely, we mean the need to take into account amplitude quantization errors (quantization noise) as well as, at the same time, all other errors associated with an imprecise "signal sampling" performed by a given A/D converter. The latter ones are attempted to be modelled somehow in various ways,

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such as, for example, by an averaging operation, by blurring the sampled signal, or by switching of finite duration, or by some combination of the above-mentioned operations. Moreover, note that the imperfections listed above, excluding quantization noise, are accounted for in the literature under a notion of "nonideal sampling" or "non-idealities of the sampling process".

Expressing this more briefly and having the detailed considerations carried out in [12] and [13] in the background, we can say that a separate consideration of the effect of quantization noise and all other imperfections on the discrete values processed by the microprocessor, as postulated by some researchers, makes no sense from the point of view of digital signal processing. Simply, since these two things overlap in an inseparable way so that a possible separation would be very difficult. And if it were to be done, it is not really known for what purpose?

In conclusion, we say that in [12] and [13] a model of the signal at the output of A/D converters has been derived, in which the error in the sample amplitude value is treated compactly. Not as a quantization error plus something else, but both taken into account together as, let us call it, a sampling error (in place of a quantization error).

To emphasize this fact, it is worth to illustrate the last conclusion also in a graphic way. So, for this purpose, let us refer to Fig. 2.21 on page 74 in the famous van de Plassche's monograph [14]. Observe that in our model we do not try to render (in one or another form) this whole delicate structure of the signal occurring between the successive sampling points shown in the figure mentioned, but we model it through a step with a constant value. And this is really enough in the digital signal processing.

The description of the signal appearing at the output of A/D converters, whose basic form was derived in [12] and further developed in [13], was also used in the latter article to calculate the spectrum of this signal. For this purpose, the Fourier transform formula was applied directly to the step function. However, in this paper, we carry out this task differently (it is performed in the next section). Namely, by writing first the step function in a form of the convolution of a weighted Dirac comb with a single rectangular pulse, and only then applying the Fourier transform to the resulting function. As we will see in the next section, the latter approach makes it possible to reveal an "internal structure" of the spectrum pattern of the signal at the output of the A/D converter and, consequently, to relate it with the one given by (1).

Section II deals with derivation of the new formula for the spectrum of the signal at the output of an A/D converter, but the next one contains some interpretative remarks and conclusions following from it. The paper ends with collected comments that the author received from people who read it and his other publications written by him and referred to here. These comments are accompanied with relevant and comprehensive explanations.

II. Derivation of the New Formula

It seems best to start this section with discussing a helpful example from everyday life. Let it be a thought experiment we discuss below. And now for its details: let us consider one digital sample of an acoustic signal and a CD where we want to store it and take away after some time, but with a simultaneous deletion on that CD. How can we describe this operation as a function of time? Just as it is shown in Fig. 1.

Observe the step character of the "stay function" shown in Fig. 1. It describes the fact of staying a sample on CD during some time. And this is visualized by just a step function that has the form of a single box (rectangle). Obviously, this cannot be any other function, for instance, a function containing a (single) triangle.

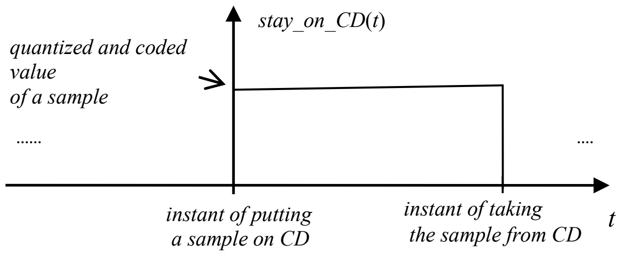

Fig. 1. Illustration to a "stay" function on example of an acoustic signal sample staying some time on a CD

Further, we note that we have to do with a similar situation in the case of the A/D converter output. The signal samples $x(kT)$'s, each of them, step by step, stay only a time interval T at the A/D converter output. Then, they move, step by step, to a buffer, for example, of a digital signal processor. Moreover, note that no dedicated special device as, for example, a so-called sample-and-hold unit is needed here to have to do with the step-wise behavior of an A/D converter output.

Let us now define, for the purposes of our derivations in this section, the "stay" function visualized in Fig. 1 in a formal way as

$$\operatorname{rect}(t) = 1 \text{ for } t \in \langle 0, T) \text{ and } 0 \text{ otherwise.} \tag{2}$$

This is simply a rectangular pulse of the length T. (Mathematicians call it a boxcar function.)

Further, define a Dirac comb and a weighted one, which we denote here by $\delta_T(t)$ and $x_T(t)$, respectively. The first of them is given by

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \tag{3}$$

where $\delta(t-kT)$, $k = .,-1,0,1,.,$ mean the time-shifted Dirac deltas (called also Dirac distributions or Dirac impulses). But the second has the following form:

$$x_T(t) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT) = \delta_T(t)\cdot x(t) \ . \tag{4}$$

and it is a Dirac comb weighted by the corresponding values of the samples of the signal $x(t)$ taken at the successive instants kT, where $k \in \mathrm{Z}$. The most right-hand side of (4) shows also that it can viewed as a signal $x(t)$ modulated by the comb $\delta_T(t)$.

In the next step, note that the convolution of $x_T(t)$ with a rectangular pulse $\mathrm{rect}(t)$ given by (2) results in

$$\begin{aligned} x_T(t)\otimes\mathrm{rect}(t) &= \left(\sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT)\right)\otimes\mathrm{rect}(t) = \\ &= \sum_{k=-\infty}^{\infty} x(kT)\left(\delta(t-kT)\otimes\mathrm{rect}(t)\right) = \\ &= \sum_{k=-\infty}^{\infty} x(kT)\left(\int_{-\infty}^{\infty}\delta(\tau-kT)\mathrm{rect}(t-\tau)d\tau\right) = \\ &= \sum_{k=-\infty}^{\infty} x(kT)\mathrm{rect}(t-kT) = x_{STEP}(t) \ , \end{aligned} \tag{5}$$

where $x_{STEP}(t)$ means the resulting step function. In view of what has been said above, this function is a proper (i.e. correct) function of a continuous time t, which describes a waveform at the output of an A/D converter.

In what follows, we calculate the spectrum of this waveform. And to this end, we use a classical definition of the Fourier transform. So, then, we get in the successive steps

$$\begin{aligned} X_{STEP}(f) &= \mathrm{F}\ \left(x_{STEP}(t)\right) = \int_{-\infty}^{\infty} x_{STEP}(t)\exp(-j2\pi ft)dt = \\ &= \int_{-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty} x(kT)\left(\delta(t-kT)\otimes\mathrm{rect}(t)\right)\right\}\exp(-j2\pi ft)\,dt = \\ &= \sum_{k=-\infty}^{\infty} x(kT)\mathrm{F}\ \left(\delta(t-kT)\otimes\mathrm{rect}(t)\right) = \\ &= \sum_{k=-\infty}^{\infty} x(kT)\mathrm{F}\ \left(\delta(t-kT)\right)\cdot\mathrm{F}\ \left(\mathrm{rect}(t)\right) = \\ &= \sum_{k=-\infty}^{\infty} x(kT)\frac{\exp(-j2\pi fkT)}{j2\pi f}\left[1-\exp(-j2\pi fT)\right] \ . \end{aligned} \tag{6}$$

In (6), the description of the step function $x_{STEP}(t)$ is intentionally taken as shown in the second row of (5) (not as in the last row there). Further, the change of order of summing and integrating operations has been carried out in (6). Moreover, $\mathrm{F}\ (\cdot)$ stands there for the Fourier transform of the signal or of the Dirac delta; $\mathrm{F}\ \left(x_{STEP}(t)\right)$ is also denoted as $X_{STEP}(f)$ to emphasize its meaning as the spectrum of this waveform. Moreover, $j=\sqrt{-1}$.

And rearranging further the expressions in the last row of (6), we obtain

$$\begin{aligned} X_{STEP}(f) &= \\ &= T\cdot\frac{\left(\exp(j\pi fT)-\exp(-j\pi fT)\right)}{j2\pi fT\exp(j\pi fT)}\cdot\mathrm{DTFT}\left(x(kT)\right) = \\ &= T\cdot\mathrm{DTFT}\left(x(kT)\right)\exp(-j\pi fT)\frac{\sin(\pi fT)}{\pi fT} \ , \end{aligned} \tag{7}$$

where the $\mathrm{DTFT}\left(x(kT)\right)$ means the so-called discrete-time Fourier transform of the sequence $\{x(kT)\}$ and is given by

$$\mathrm{DTFT}\left(x(kT)\right) = \sum_{k=-\infty}^{\infty} x(kT)\exp(-j2\pi fkT) \ . \tag{8}$$

Note that the last row of (7) can be rewritten with the use of the sinc function. So we get then

$$X_{STEP}(f) = T\cdot\mathrm{DTFT}\left(x(kT)\right)\mathrm{sinc}(\pi fT)\exp(-j\pi fT). \tag{9}$$

And this is our final result in the first derivation presented in this section, showing the spectrum of the output signal at an A/D converter via the DTFT of an associated sequence $\{x(kT)\}$. It is identical with the result presented in [13], where, to derive it, a step function as it is (that is in the form seen in the last row of (5)) was used. This fact shows that if, in calculation of the spectrum of some signal, where we start with the classical definition of the Fourier transform and in the description of this signal in the time domain t we encounter Dirac deltas (to which, obviously, we cannot apply the above definition), we "switch", with respect to them, to the definition in the sense of distribution theory. And this leads to correct results.

Let us now start derivation of the formula for $X_{STEP}(f)$ slightly differently. So, to this end, note that with the use of (4) and (5) we can write $x_{STEP}(t)$ as

$$x_{STEP}(t) = \left\{\delta_T(t)\cdot x(t)\right\}\otimes\mathrm{rect}(t) \ . \tag{10}$$

Therefore, the Fourier transform of the step function $x_{STEP}(t)$ is given here by

$$\begin{aligned} X_{STEP}(f) &= \mathrm{F}\ \left(\delta_T(t)\cdot x(t)\right)\cdot\mathrm{F}\ \left(\mathrm{rect}(t)\right) = \\ &= \left(\mathrm{F}\ \left(\delta_T(t)\right)\otimes\mathrm{F}\ \left(x(t)\right)\right)\cdot\mathrm{F}\ \left(\mathrm{rect}(t)\right) \ . \end{aligned} \tag{11}$$

Taking into account the fact that $\mathrm{F}\ \left(\delta_T(t)\right)$ in (11) is itself a Dirac comb and can be written as [15]

$$\mathrm{F}\ \left(\delta_T(t)\right) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(2\pi(f-kf_s)\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty}\delta(f-kf_s) \ , \tag{12}$$

and performing the convolution operation shown there, we finally arrive at

$$\begin{aligned} X_{STEP}(f) &= \\ &= \left(\sum_{k=-\infty}^{\infty} X(f-kf_s)\right)\cdot\mathrm{sinc}(\pi fT)\exp(-j\pi fT) \end{aligned} \ . \tag{13}$$

Comparison of (13) with (1) shows that evidently $X_{A/D}(f)$ differs from $X_{STEP}(f)$. Obviously, the correct result is (13), not (1).

III. Some Remarks and Conclusions Following from Comparison of Formulas (1), (9), and (13)

Let us note first that the comparison of (9) with (13) leads to the following:

$$\mathrm{DTFT}\big(x(kT)\big)=\frac{1}{T}\sum_{k=-\infty}^{\infty}X\left(f-kf_s\right), \tag{14}$$

that is to the well-known identity. Further, (14) in connection with (1) shows clearly that $X_{A/D}(f)$ represents only the DTFT of the associated sequence $\{x(kT)\}$, fed to the input of a signal processor – not the spectrum of a real signal of a continuous time t at the output of the A/D converter.

Of these two alternative formulas, (9) and (14), derived above for determining the spectrum of the signal at the output of an A/D converter, the latter seems to be a more useful formula Why? Because it shows what is difficult to guess when looking at the former. It can be said that the latter formula reveals the "internal structure" of the spectrum of the signal mentioned above. It shows that this spectrum can be represented as a sequence of successively shifted on the frequency axis and attenuated spectrum of a certain analog signal. What signal? The above formula, however, does not solve this problem. As we know, there can be a lot of such signals.

The third remark is as follows: It seems that most of students and researchers working in the area of signal processing are aware of the fact that, according to (8), there is only one DTFT for a given sequence $\{x(kT)\}$, but, at the same time, related with a lot of analog signals whose sampling results in that sequence mentioned above. This is called aliasing in the time domain.

In the frequency domain, the picture of the above is what the formulas (1) and (14) represent. And probably only a few pay attention to the fact that they concern many analog signals fulfilling the conditions mentioned above, not just only one of them. Normally, the people focus only on a one signal spectrum to check whether an aliasing in the frequency domain occurs or does not for it.

Next, note also that according to the modified description of the signal at the output of an A/D converter, which we present in this paper, a given sequence $\{x(kT)\}$ is represented by only a one step signal $x_{STEP}(t)$ (and vice versa). Hence, one step signal $x_{STEP}(t)$ stands for a lot of un-sampled analog signals $x(t)$. And therefore, the relationship between this image in the time domain and its equivalent image in the frequency domain is similar as above. That is the formula (13) applies to many of un-sampled analog signals satisfying the conditions mentioned before.

IV. Collected Comments from Readers and Associated Explanations

The manuscript of this paper and other works of its author, which are referred to in it, on understanding the spectrum of a sampled signal at the output of an A/D converter, have been read by many people (scientists, ordinary engineers, as well as by students). They shared with the author of this paper their criticisms, doubts, and objections to the conclusions and interpretations drawn – those most important ones are collected here. And, obviously, all these people mentioned above received answers to their comments, remarks, etc., but it seems worthwhile to share these answers with a wider audience as well. And that is precisely the task of this section.

The discussion on the topic raised in this paper and its predecessors is still ongoing. Hopefully, the explanations provided in this section will help to clarify many contentious and ambiguous issues in modelling of the sampling process of analog signals.

Let us start with the first remark. Namely, with observation that some people noted that modelling the signal at the output of an A/D converter with the use of a step function, as proposed in [12], and as it is exploited here, corresponds to a more realistic modelling of the sampling operation than the one which uses a weighted Dirac comb.

This assertion is obviously true. Someone pictorially put it this way: The A/D converter works in a similar way to the shutter of a camera, which cannot let the portions of light falling on it pass through in a time equal to zero, because, de facto, zero opening time means no opening time. In other words, it is not possible to speak of pulses of light passing through the shutter onto the camera lens at zero time. That would be illogical. One can only speak of a finite shutter opening time or shutter closure. Besides, portions of light transmitted through the shutter never have infinite energy or intensity. That is, they "do not deliver Dirac deltas". And any A/D converter works in a similar way.

Some who read the manuscript of this paper had impression that its author implies that the formula (1) is erroneous. Not at all: such a conclusion cannot be drawn from its contents. The author of this paper implies only that under an assumption of modelling the A/D output waveforms as step functions this formula needs a minor correction precisely by introducing a correcting factor. Comparison of (1) with (13) shows that this correcting coefficient is equal to $T\cdot\mathrm{sinc}(\pi fT)\exp(-j\pi fT)$. (In this context, note that no one claims, for example, that Newton's theory of gravity is erroneous. His theory was only refined by Einstein. And here we deal with a similar case.)

By the way, note that the author of this paper questioned the correctness of formula (1) in [1], but for a different reason. Namely, for the reason that this formula determines the spectrum of a signal that is not present at the output of A/D converters (that is in form of a weighted Dirac comb). In other words, he criticized the inconsistency existing between the form of the signal spectrum assumed and the time-domain form of a real waveform appearing at the output of an A/D converter. And note that just this inconsistency has been overcome in this paper. An excellent approximation of the real waveform at the output of an A/D converter by means of a step function has just its spectrum in the frequency domain as defined by the formula (13). And the latter differs only slightly from the formula (1), which "we should be happy about, not cry that we did not get something more complicated."

The next remark concerns the assertion of some researchers that the formulas (1) and (13) refer to two different sampling processes. Note however that in reality we have to do with only one process of signal sampling. So, because of this fact, such a reasoning as above must be regarded as erroneous.

But, obviously, we can say that there exist two different descriptions of the sampling process – however not two distinct sampling processes. Every signal sampling process is performed by an A/D converter. So the question is rather the following: which of these descriptions describes the real sampling process better. Talking about two types of sampling processes is a methodological error.

Note that on the above point one can also reason as follows: Let us assume that formulas (1) and (13) actually refer to two distinct sampling processes. So since there is actually only one sampling process, one of these formulas must be completely false. A comparison of (1) with (13) shows however that this is not the case. They differ from each other, as we already said above, only in the coefficient $T\cdot\mathrm{sinc}(\pi fT)\exp(-j\pi fT)$. Hence, a conclusion that must be drawn from this fact is that one of them "approximates reality" better, and the other worse.

Further, some distinguish the so-called "delta Dirac-like" A/D samplers (converters) – and use (1) in this case – from the so-called "sample and hold" A/D samplers (converters), for which they reach for (13). Of course, there are various A/D converters on the market, which are manufactured using different architectures, design philosophies as well as different technologies. But it does not mean at all that we need to distinguish among them "delta Dirac-like" A/D converters or "sample and hold" ones, and maybe also others. They all are characterized by a finite time to hold a given sample value. So, from the above, it follows that the most reasonable way to model this behavior (in an ideal case) – in a continuous time domain – is by assuming a holding operation of sample values all the time (approximately) from one sampling instant to the next.

With regard to the above, we will now show an example of a deceptive reasoning followed by researchers, who admit that apart of "delta Dirac-like" A/D converters there exist also "sample and hold" A/D ones. But they argue, as, for instance [16], [17], that if we even assume that there are many distinct designs of A/D converters we can describe all of them very well through a "delta Dirac-like" model. That is also any "sample and hold" A/D converter. Because according these people the differences between all of the possible models (including the "sample and hold" mentioned above) are minor.

Let us consider why they think so. They say that the reason is that in the range of frequencies used in practice, the formulas (1) and (13) differ only slightly between themselves. Indeed, note that for small values of the frequency f, compared to the value of the sampling frequency $f_s = 1/T$, that is for $f/f_s \ll 1$, we can assume $fT \cong 0$ in the coefficient $\mathrm{sinc}(\pi fT)\exp(-j\pi fT)$ in (13), and this allows us to rewrite the latter formula in the following approximate form:

$$X_{STEP}(f) \cong \sum_{k=-\infty}^{\infty} X(f-kf_s) = T\cdot \mathrm{DTFT}(x(kT)) . \quad (15)$$

A comparison of (15) with (1) shows that, apart from the constant coefficient $1/T$ and the fact that the former indicates an approximate relationship, they do not differ from each other in any way. Moreover, note also that (15) as well as (1) describe very well spectra of the sampled signals for analog signals whose maximum frequencies of their bandwidths are significantly less than their sampling frequencies. And in practice, this is the choice most often made.

By the way, we explain also the reason for occurrence of the scaling factors $1/T$ or T in equations (1), (14), (9) and (15) above. Their existence follows from the fact that the DTFT (denoted here also by $X_{A/D}(f)$) is a weighted sum of signal samples, while a "real" signal spectrum (denoted also here by $X_{STEP}(f)$) is related to the integration over time.

Now, observe that by performing the inverse transform of DTFT occurring in (15), we get the following:

$$\tilde{x}_{STEP}(t) \cong T\cdot \sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT) = T\cdot(\delta_T(t)\cdot x(t)) , \quad (16)$$

where $\tilde{x}_{STEP}(t)$ means the signal following from inverting the spectrum given by (15). That is we get in this case the (scaled by T) weighted Dirac comb $x_T(t)$, i.e. $T\cdot x_T(t)$. So the shape of this signal (which is a distribution) is completely different – as visualized in many textbooks – from the step function $x_{STEP}(t) = \sum_{k=-\infty}^{\infty} x(kT)\mathrm{rect}(t-kT)$. And this observation concludes demonstration of the deceptiveness of the argumentation presented above.

In addition, it seems to be worth illustrating the nature of the deceptive reasoning presented above with the following example: 2 times 2 equals 4, but 2 plus 2 equals 4, too. However, it does not at all follow from this that they describe the same arithmetic operation. In the first case, we have to do with a multiplication, but in the second, with an addition. This makes a colossal difference.

Let us now consider a fundamental issue of modelling the operation of analog signal sampling that is performed in digital systems by A/D converters. We mean here the concept of an ideal (perfect) sampling. Note that it is generally accepted in the research papers and textbooks on digital signal processing that this is the model we call here the "delta Dirac-like" A/D sampler (converter). And, as we know, this model is defined via (4) (in the time domain) and via (1) (in the frequency domain). As seen, the description (4) uses non-physical objects called Dirac deltas. For this reason, it is called in the literature an ideal sampling. Mathematically, the formula (4) is flawless, but the problem with it is that it cannot be used to model the signal at the output of an A/D converter, even in an idealized case. Why? Because it leads to erroneous results, what was shown (in various ways) in [1]–[13]. No A/D converter produces a Dirac comb (i.e. a sequence of Dirac deltas), nor does it act as a Dirac comb (what, as (4) shows, represents the same thing) – even as an idealized approximation. (Incidentally, a number of authors tried to demonstrate the latter, but something did not work. The reason for the failure was explained in [11].) Standard A/D converters produce waveforms at their outputs similar to that in Fig. 2.21 on page 74 (the first curve from the bottom) in the well known van de Plassche's monograph [14]; we refer to this figure in this paper. And it is really hard to imagine that a piece of the waveform shown in Fig. 2.21 between the adjacent sampling instants is modelled better by a Dirac delta multiplied by a number than by means of a step with a value equal to this number. Or, formulated in the form of a question: which of these two approximations: the one by the former ideal variant or that by the latter variant (which is obviously also ideal but in another sense) is better? In opinion of the author of this paper, everyone will answer that with the help of the latter ideal variant. And why? Because it is more natural in terms of how the sampling operation of an analog signal is implemented in an A/D converter. It provides at regular intervals (in the case of regular sampling) the numbers that are the values of the aforementioned analog signal. Not the weighted Dirac deltas. Besides, no A/D converter has a built-in electronic component that implements the zeroing operation after sampling the signal. There is no reason to do this. The value of the signal sample remains naturally present until the next sampling moment (it is not zeroed). That is, it can be assumed that the A/D converter operates – naturally – in a regime that is called in the literature a "sample-and-hold".

The author of this paper, like everyone else, was deceived by the fact that (4) describes an ideal (perfect) sampling and therefore in all his previous publications [1]–[13] he stuck to this definition exploited in the literature on signal processing as well as at students' lectures. Today, he would not say so any more. Why? Because (4) is not a correct formula, primarily for the reasons outlined in detail in [9]. That is it has no correct physical justification. Therefore, it is difficult to refer to it as a one describing an ideal or non-ideal model. It can be however viewed, as shown in this paper, as a part of a right formula that produces the step waveform. And just the formula which describes the step waveform that follows from an ideal approximation of a one "with turbulences" – like those in Fig. 2.21 on page 74 (the first curve from the bottom) in the van de Plassche's monograph [14] – should be assumed to represent an ideal model of sampling.

It may be worthwhile here, in the context of the above, to refer to a view presented in the textbook [18], where its author assumes a sampling model based on (4) as an ideal model, and calls the other two he mentions (a sample-and-hold one and the second which is based on signal averaging in time segments between the sampling instants) real models. Such an interpretation is incorrect. First of all, because his sample-and-hold model should not be treated as a real version of that based on (4). According to that what was said above, it should be treated as an idealization of that which was presented in the van de Plassche's monograph [14]. Second, his second model he calls a real one is principally false because of the reasons discussed in [9].

Some researchers (for example, S. Zozor [19]) see the function $\mathrm{rect}(t)$ occurring in (5) as a kind of an "interpolation" function that interpolates, but what? Probably, it should be understood in the following way: this function converts signal samples occurring (after S. Zozor [19] and other researchers) in the form of the weighted Dirac deltas into the weighted step impulses. Note however that such a reasoning is subject to the following fundamental error: none of the known converters does produce signals in the form of the weighted Dirac combs at its output. There exists no work that confirms that such physical signals (objects) occur during operation of A/D converters at their outputs. Furthermore, no filtering of the weighted Dirac deltas is performed at the output of any A/D converter (because they simply are not there). At the output of the A/D converter, the waveform as a function of a continuous time, in its idealized form, is simply a step function.

There are also such statements, for example, represented by [19] and [20], that in the formula (5) any other "interpolation" function could be used in place of the rectangular impulse. For instance, a triangular impulse could be applied [20]. Then in the formula (13) on the spectrum of the sampled signal at the output of an A/D converter, we would insert the Fourier transform of the triangular impulse, instead of the function $\mathrm{sinc}(\pi f T)$.

It is not clear whether those who say so as mentioned above are aware of that that in this way they involuntarily or indirectly confirm the thesis of the author of this paper expressed in a number of his publications [1], [3]–[6], [11], that when we begin to define a signal sampling operation using Dirac deltas, we quickly conclude that it is not possible then to unambiguously solve the problem of the sampled signal spectrum, and when trying to solve it we meet a lot of troubles.

We underlie that the considerations presented in this paper show unambiguously that we should not insert in (5) a different function (e.g. a triangular impulse) instead of a rectangular impulse. This is simply forbidden because this model works correctly only with a rectangular impulse. This follows from the fact that in the output waveform of an A/D converter the spaces between the adjacent sampling points are steps parallel to the axis of a continuous time (having values equal to the corresponding values of samples of the analog signal).

Further, it seems that some researchers are mixing interpolation of the signal with the form of the signal at the output of an A/D converter. But we know that the interpolation of the signal from its samples is inherently connected with another converter, namely with the D/A converter. So it is really unclear for what reason they are also looking for the interpolation (or some sort of filtering) at the output of the A/D converter.

The crucial result in this paper is to show that despite modelling the signal at the output of an A/D converter with the use of a step function (correct), but not by means of a weighted Dirac comb (false because no such signal appears at the output of an A/D converter), the key role in the formula for the spectrum of this signal is played precisely by the expression $\sum_{k=-\infty}^{\infty} X(f - kf_s)$. It is, as we know, responsible for the existence of the so-called aliasing and folding effects in the spectrum of the aforementioned signal. These effects really do exist, but as we have just shown here in a way which does not raise any doubts, their modelling with a weighted Dirac comb (via (4)) is not needed. In a paper [1], with a perhaps somewhat perverse title(?): "Spectrum aliasing does not occur in case of ideal signal sampling", the author of this work pointed out first that it is rather inadvisable to use a not correct model (formula (4)) to derive a (partly) correct result (i.e. to demonstrate the existence of the so-called aliasing and folding phenomena in the spectrum of a sampled signal). And secondly, he pointed out that when adopting a more correct model of the sampling operation (with Kronecker's functions), other problems arise. Then, above all, the relativity of the spectrum definition becomes apparent and the need for adoption of its different definition than the Fourier integral becomes apparent, too. And this, as shown, with a certain choice, can lead to an absurd result that the aliasing and folding effects do not occur in the spectrum of the signal at the output of an A/D converter.

There are also such researchers (for example [19]), who say that the factor $\mathrm{sinc}(\pi fT)$ in (13) is a consequence of the (most elementary) "reading" or reconstruction process. Of course, such an explanation of reality is false since the signal at the output of an A/D converter we are talking here about (that is about this continuous time waveform) does not arise as a result of some hypothetical "reading" or reconstruction (of what?) Whereas it is clear that the "reading" of signal sample values takes place at the input of a signal processor connected to the A/D converter, while the reconstruction of the signal from its samples is performed in a D/A converter.

Although there are researchers (such as, for example, [21], [22]), who recognize that the signal at the output of an A/D converter can be perceived differently by the device behind it, which receives that signal for further processing or analysis – but it turns out that they do not quite have this problem "under control". Always, it should be clearly seen that if a signal processor is connected to the output of an A/D converter, it receives this signal "discretely", that is, it receives samples of the analog signal applied to the input of the A/D converter, but at a certain time rhythm (at specified intervals). However, if the signal from the output of this A/D converter is fed to the input of an analog oscilloscope, one will see on its screen an analog waveform similar to a step function. And by applying it at the input of a spectrum analyzer, one will perform a spectral analysis of this continuous time signal.

Note that what was said above can be also expressed, in other words, as follows. To this end, observe that an A/D converter having physically one output port offers the user to work in two variants: to work with continuous time t waveforms (functions) and to work with waveforms of the so-called discrete time k. In the first case (in the case of an ideal modelling) these are the step waveforms (functions) $x_{STEP}(t)$, while in the second case, these are the sequences of numbers $x(k)$ (i.e. of ordered elements, which in the ideal case correspond exactly with the samples of an analog signal at the input of the A/D converter). In the case, a user wants to work with continuous-time waveforms at the output of the A/D converter, this means that she/he will work with waveforms $x_{STEP}(t)$'s (for example, watching them on a standard analog oscilloscope). But if, on the other hand, a user wants to work with the so-called discrete-time signals at the output of the A/D converter, it means that she/he will work with the sequences $x(k)$ (for example, by feeding them into the input of a signal processor and processing them further in that device according to some digital processing algorithm). That is she/he will process the functions of the discrete variable k. And expressing the above in mathematical terms, we can say so: in the first case one will work with functions of a continuous variable (belonging to the set of reals), while in the second case with functions of a discrete variable (belonging to the set of integers).

Therefore, the above model of operation can be also viewed as a model with two virtual outputs: one which is analog and the second that is digital. And this is in principle what M. Vetterli et al. do in [21]. In their model (shown in Fig. 1 in [21]), the virtual analog output of the A/D converter is indicated by the signal $y_s(t)$ (being a counterpart of $x_{STEP}(t)$ in this paper), while the virtual digital output is a discrete function $y(nT)$ there (being a counterpart of the sequence of $x(kT)$'s here).

Of course, a different definition of the spectrum applies to the functions $y_s(t)$ and $x_{STEP}(t)$ (which belong to the space of continuous time functions) than that used for the discrete functions $y(nT)$'s and $x(kT)$'s (that belong to the space of functions of a discrete variable). In the former case, it will be the ordinary Fourier integral (if it exists, of course), and in the latter case the so-called Discrete-Time Fourier Transform (DTFT). Obviously, here and in [1]–[13], the problems associated with the signals at virtual analog outputs of A/D converters as well as some problems associated with calculation of their spectra are considered. An additional issue discussed there is the relationship of these spectra to the DTFT spectra of signals coming from the corresponding virtual digital outputs of A/D converters.

As an aside, it is also worth noting that the question of a relationship between the spectra mentioned above makes sense only because the discrete time axis kT is "immersed" in the continuous time axis t. And this is how the spectrum problem differs from any digital processing in a signal processor. In a signal processor, we can treat the signal samples as numbers $x(k)$'s when analyzing some processing algorithm; the only important matter then is their ordering (that is, which sample occurs first, which occurs second, etc.) It is worth being aware of the above. And, in this context, note also that this theme is discussed in more detail in [10].

V. Final Conclusion

This paper summarizes efforts made so far by his author in correcting the formula (1) used in the literature for the sampled signal spectrum. He hopes that all the doubts raised by researchers so far have been clarified here. The corrected formula (13) has been derived. Its relationship with the formula (1) has been explained in detail. And finally, any hitherto incomprehensible complexities in modelling of the signal form at the output terminals of A/D converters have been discussed.

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