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Analysis of pandemic with game methodology and numerical approximation

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We build a mathematical game model of pandemic transmission, including vaccinations of population and budget costs of different acting to eliminate pandemic. We assume the interactions among different groups: vaccinated, susceptible, exposed, infectious, super-spreaders, hospitalized and fatality, defining a system of ordinary differential equations, which describes compartment model of disease and costs of the treatment. The goal of the game is to describe the development disease under different types of treatment, but including costs of them and social restrictions, during the shortest time period. To this effect we construct a dual dynamic programming method to describe open-loop Nash equilibrium for treatment, a group of people having antibodies and budget costs. Next, we calculate numerically an approximate open-loop Nash equilibrium.

Key words: COVID-19, game model of pandemic, approximate dual dynamic programming, sufficient approximate optimality conditions for Nash equilibrium, numerical algorithm.

1. Introduction

Since coronavirus (COVID-19) pandemic outbreak (in 2019, in Wuhan) a huge number of articles devoted to different aspects of COVID-19 pandemic have been written (see e.g. [1,7–9,12,14–16,18,21–23,26–28,30–32] and literature therein). In these papers we find mathematical models of infectious disease transmission dynamics. These models are slightly different in each of that paper, depending on aims it treats, e.g., in [9] the model extends the existing epidemiological models by specifying how a vaccine and its arrival are included in the optimization process. The understanding of development of the diseases is done by analysis and simulating of dynamics of the mathematical models depending on parameters, which the model contains. The parameters are mostly constant, i.e., independent on time and

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chosen suitably, accordingly to the case study e.g. when the system is related to the class of super-spreaders (see e.g. [23]). However, we can meet, lastly, some articles with parameters, which are functions (see e.g. [1, 26, 28]). In some of these papers the authors consider the vaccination problem (see e.g. [8, 12, 23, 26, 28]), also as evolutionary game (see e.g. [2, 9, 16]). Earlier, mainly control and mitigation efforts against COVID-19 are focused on the implementation of non-pharmaceutical interventions, such as lockdown of community, maintaining social distancing, using face masks in public, quarantine, isolation and hospitalization of the confirmed cases, surveillance and serology testing and contact tracing. All papers mentioned above give a good insight into the transmission dynamics and control of COVID-19 infectious. It should be noticed that in most of papers mentioned above the main rate in analysis of the transmission dynamics is so called the basic reproduction number R_0, depending on the parameters of the system. In the case of autonomous systems, R_0 describes stability of the system (see e.g. [29]). If the basic reproduction number is less than 1, then the (system) transmission is stable, if it is greater than 1, then the system can be unstable. However, we should stress that "control" in these papers does not mean that we have at our disposal a set of the control functions and we have some rules to choose the best one and then steer the rules for population to deliberate a goal. Instead, some simulations are done and suitable parameters are selected (see e.g. in [7, 9, 14, 15, 28, 30–32]).

Our approach to investigate the problems related to COVID-19 is different. We do not concentrate on R_0. Instead, we firstly replace most of parameters (constants) by functions (controls) and secondly, we add new variables and controls related to vaccinations and costs of the pandemic, defining also a cost functional, which takes into account costs of the pandemic, basic reproduction number R_0 and number of vaccinated persons. A different cost functional is considered in e.g. [26]. It captures the healthcare costs, which are proportional to the sum of the squares of the number of hospitalized $H(t)$ and the socio-economic costs associated with the implementation of NPIs. As a next step, we observe that not all our controls cooperate, i.e. some of them should cause minimization of our functional, while the other should maximize them. This observation suggests considering COVID-19 problems not as standard optimal control problem (see e.g. [26]), but rather as a kind of a game (compare also [9]). Let us notice that this game is non-cooperative differential game. Therefore, we separate our strategies into two players. The first one wants to minimize costs of the pandemic, as well the basic reproduction number R_0, while the second one wants to maximize number of vaccinated persons plus these, which have antibodies. Thus, the main goal of this paper is to construct a new approach to the treatment coronavirus disease COVID-19 allowing to take into account in making decision the costs of the pandemic as well the number of population having antibodies.

The game method to investigate pandemic problems is known in the literature (see e.g. [2, 4–6, 15, 27]). These papers use the evolutionary games with strategies do not depending on time. The solutions of the games – a kind of Nash equilibrium – are found by discussing the parameters. In [21] closed-loop Nash equilibrium approach is presented.

We assume the vaccination of the population and we are mainly interested in finding the parameters (functions) in system ensuring maximum of the population having antibodies. As the time of duration of the pandemic is essential in making suitable decisions as well relates to different costs of community, we consider as parameter – strategy the time and want to minimize it. The approach described above cause that we construct a new mathematical model of COVID-19. It is a differential game with suitable distinguished players and opponents' players in which we want to find an approximate open-loop Nash equilibrium, in fact, approximate dual open-loop Nash equilibrium. This type of the approach to the pandemic (in general) allows, by choosing suitable strategies, to be more conscious in making decisions. In order to solve mathematical model, we develop a dual dynamic programming methodology for such a game and formulate sufficient conditions for approximate open-loop Nash equilibrium in the form of the verification theorem. It allows to assert that the calculated solution is really ε-Nash equilibrium, accordingly to our mathematical model. Our method can help to find approximate open-loop Nash equilibrium without experiments.

We show in the example a quality of the game theoretic approach. It differs significantly from standard approach to pandemic problems. The cost functional is neither maximal with respect to controls, nor minimal with respect to them. Our game approach is more realistic, because it allows to calculate strategies which are against insight: for strategies of the open-loop Nash equilibrium the value of costs is higher and the number of population having antibodies is lower than using standard approach. The reason is that we have the game with two players, of which the first one using his four strategies wants to maximize our functional, while the second one using his own eight strategies wants to minimize the functional. Thus the one player can not only to maximize the functional as opponent player wants to minimize the functional at the same time.

We would like to stress that we do not concentrate in this paper on analysis of mathematical model and disease transmission. The main aim is to develop a new tools to study such a mathematical model. This is also a reason, why we use so many strategies (controls) in our model. We want to show a richness of possibilities to deal with these new tools, but we can always assume that some strategies are identically zero. Then the model will be simplified, and the theory is still working.

1.1. Model of the infectious disease transmission

In epidemiology the most popular model is the so-called Susceptible-Exposed-Infected-Recovered (SEIR) model. It belongs to the class of compartmental models [10] (compare [11]). In that model the main assumption is: the total population N can be divided into four classes of individuals that are susceptible S, exposed E, infected I, and recovered or dead R (assumed to be not susceptible to reinfection). Moreover, it is assumed:

1. The total population does not vary in time,
2. Susceptible individuals become infected that then can only recover or die,
3. Exposed individuals encountered an infected person but are not yet themselves infectious,
4. Recovered or died individuals are forever immune.

However, the longevity of the antibody response is still unknown, but it is known that antibodies wane over time. Assumption 3 is not well recognized when we apply vaccination to population. Assumption 2 is to strong as we can observe that many infected people suffer long time consequence of infection and require medical care. It is difficult to expect that during the period of half or one year the total population does not vary. The population is not living in hermetic box. The (SEIR) model is presented as:

$$\frac{dS}{dt} = -\lambda S(t)I(t), \quad \frac{dE}{dt} = -\lambda S(t)I(t)-\alpha E(t), \quad \frac{dI}{dt} = \alpha E(t)-\gamma I(t), \quad \frac{dR}{dt} = \gamma I(t).$$

To deal with uncertainties in long-term extrapolations and with the time-dependency of control parameters in [11] the authors introduce a stochastic approach into modeling epidemic making parameters depending on time and adding three more equations.

The different extension of mathematical model of infectious disease transmission dynamics we find in the paper [23] (compare also [13, 21, 28]). We modify further this model to consider vaccination a part of the population and costs of the pandemic, but we continue the ideas of [11] to deal with uncertainties in long-term extrapolations and with the time-dependency of control parameters. We remove recovery class from the model in [23] by adding it just to vaccinated class, assuming that individuals, which are recovered have antibodies similarly as vaccinated individuals have them. The meaning two of them is a little different. We also add a new differential equation, which includes costs of the pandemic (see (9)). Following [11] we change constant parameters to functions to control behavior of the model along time to consider some uncertainties which appear during the time evolution of pandemic. We include into our consideration a goal functional, which depend on costs of pandemic, the basic reproduction number and the number of population having antibodies in final time. It is obvious that we want to have maximum of population with antibodies while the reproduction number should have values less than 1 by minimal budged. This is why we should the goal functional maximize with respect to the strategies (controls) responsible for developing antibodies in population and minimize with respect to strategies generating costs and causing minimizing the reproduction number. That means we want to formulate a mathematical model of infectious disease transmission suitable for a game theoretic approach to control infectious disease by vaccination a part of the population as well as considering the costs of the pandemic. We use the following states and strategies:

States:

N	total population size,
$V(t)$	a part of the population having antibodies,
$S(t)$	susceptible class,
$E(t)$	exposed class,
$I(t)$	symptomatic and infectious class,
$P(t)$	super-spreaders class,
$A(t)$	infectious but asymptomatic class,
$H(t)$	hospitalized class,
$F(t)$	fatality class,
$C(t)$	costs of the pandemic.

Strategies:

$v_1(t)$	control of patients having antibodies,
$v_2(t)$	control of vaccinating the population N,
$s(t)$	control of the human-to-human transmission,
$s_1(t)$	control of susceptible individuals entering the exposed,
$s_2(t)$	control of treated individuals entering the exposed,
$\kappa(t)$	control of an individual leaving the exposed,
$\gamma_i(t)$	recovery control without being hospitalized,
$\gamma_r(t)$	recovery control of hospitalized,
$c(t)$	costs of lockdown,
$c_1(t)$	costs of maintaining social distancing, using face masks in public,
$c_2(t)$	costs of quarantine and isolation of confirmed cases,
$c_3(t)$	costs of hospitalization of confirmed cases,

where $t \in [0, T]$, $T > 0$.

The fraction of population N having antibodies is described as

$$\frac{dV(t)}{dt} = v_2(t)(N - V(t) + C(t)) + \gamma_i(t)(I(t) + P(t) + A(t)) + \gamma_r(t)H(t), \quad t \in [0, T]. \tag{1}$$

The equation for susceptible class S takes a form:

$$\frac{dS(t)}{dt} = \frac{S(t)}{N} - \Lambda - s(t)(I(t) + P(t) + H(t))S(t) - l(t)v_1(t)V(t)S(t), \quad t \in [0, T], \tag{2}$$

where Λ is some constant and $l(t)$, $t \in [0,T]$, quantifies the relative transmissibility of patients having antibodies.

The equation for exposed class E takes a form:

$$\frac{dE(t)}{dt} = s_1(t)(I(t)+P(t))S(t) + s_2(t)H(t)S(t) + l(t)v_1(t)V(t)S(t) - \kappa(t)E(t), \quad t \in [0,T]. \tag{3}$$

The equation for infectious class I takes a form:

$$\frac{dI(t)}{dt} = \kappa(t)\rho_1(t)E(t) - (\gamma_a(t)+\gamma_i(t))I(t) - \delta_i(t)I(t), \quad t \in [0,T], \tag{4}$$

where $\rho_1(t)$, $t \in [0,T]$, is a proportion of progression from exposed class E to symptomatic infectious class I, $\gamma_a(t)$, $t \in [0,T]$, is the average rate at which symptomatic and super-spreaders individuals become hospitalized and $\delta_i(t)$, $t \in [0,T]$ is the disease induced death rates due to infected.

The equation for super-spreaders class P takes a form:

$$\frac{dP(t)}{dt} = \kappa(t)\rho_2E(t) - (\gamma_a(t)+\gamma_i(t))P(t) - \delta_pP(t), \quad t \in [0,T], \tag{5}$$

where ρ_2 is a relative very low rate at which exposed individuals become super-spreaders and δ_p is the disease induced death rates due to super-spreaders.

The equation for infectious but asymptomatic class A takes a form:

$$\frac{dA(t)}{dt} = \kappa(t)(1-\rho_1(t)-\rho_2)E(t), \quad t \in [0,T], \tag{6}$$

where $1-\rho_1(t)-\rho_2$, $t \in [0,T]$ is a progression from exposed to asymptomatic class.

The equation for hospitalized class H takes a form:

$$\frac{dH(t)}{dt} = \gamma_a(t)(I(t)+P(t)) - \gamma_r(t)H(t) - \delta_hH(t), \quad t \in [0,T], \tag{7}$$

where δ_h is the disease induced death rates due to hospitalized individuals.

The equation for fatality class F takes a form:

$$\frac{dF(t)}{dt} = \delta_i(t)I(t) + \delta_pP(t) + \delta_hH(t), \quad t \in [0,T]. \tag{8}$$

The equation related to acting influenced costs of government such as: lockdown of community, maintaining social distancing, using face masks in public, quarantine, isolation and hospitalization of confirmed cases, surveillance and serology testing and contact tracing takes a form:

$$\frac{dC(t)}{dt} = c(t)(N - V(t)) + c_1(t)S(t) + c_2(t)(I(t) + P(t)) + c_3(t)H(t), \quad t \in [0,T]. \tag{9}$$

Flow diagram of the model represented by system (1)–(9) is shown in Figure 1 and Figure 2.

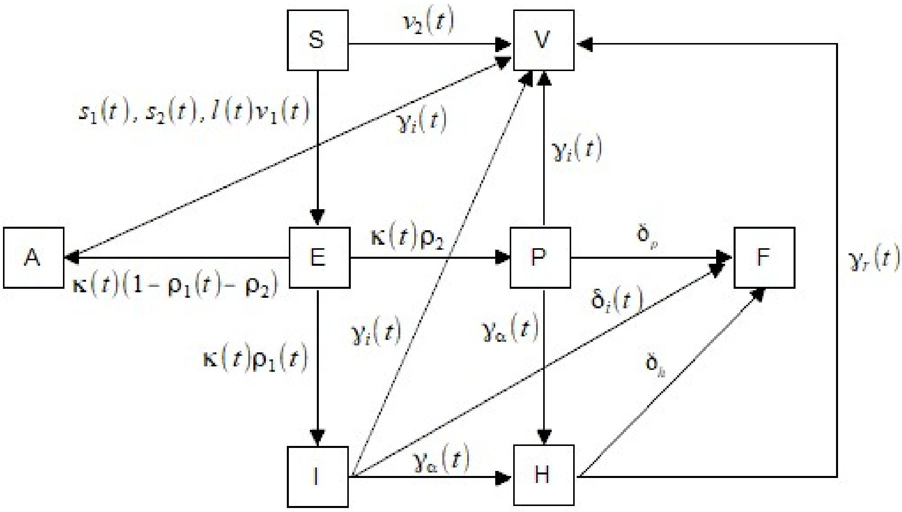

Figure 1: Flow diagram of the model described by (1)–(8)

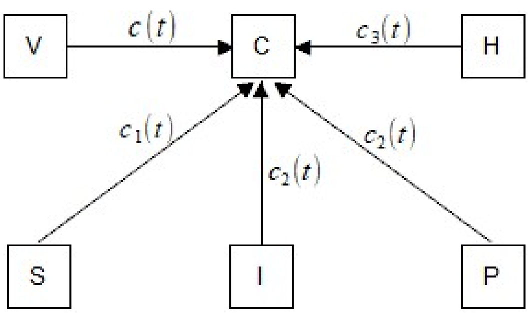

Figure 2: Flow diagram of the model described by (9)

We assume, that all our controls have values in bounded sets of $\mathbb{R}$, i.e. $v_2(t) \in \mathcal{V}_2$, $v_1(t) \in \mathcal{V}_1$, $\gamma_i(t) \in \mathcal{G}_i$, $\gamma_r(t) \in \mathcal{G}_r$, $s(t) \in \mathcal{S}$, $s_1(t) \in \mathcal{S}_1$, $s_2(t) \in \mathcal{S}_2$, $\kappa(t) \in \mathcal{K}$, $c(t) \in C$, $c_1(t) \in C_1$, $c_2(t) \in C_2$, $c_3(t) \in C_3$, $t \in [0,T]$.

Some parameters (not depending on time) in equations (1)–(8) appear. They are not controls, they relate to the disease and particular place it appears. We have to determine them or take from the literature related to the disease. The other parameters are controls (unknown), which we want to determine, considering a suitable cost functional (12) and determined by this game. As a result we calculate a set of twelve strategies, which can help the policymakers to instigate suitable acts.

Simulations made with different parameters (functions) (see Appendix) acknowledge that all mentioned strategies have influence on behavior of all states. Hence the choice of large numbers of strategies as well as proposed system of equations describe more exactly the behavior of the pandemic. However, simulations are not sufficient to study them, we need mathematical tools which help us to infer more correct corollaries. To this effect we develop game theoretic methodology in Sections 1.2 and 2.

One of the ways to control an ongoing outbreak is *basic reproduction number*. It measures of a disease spread in a population. It can be understood as the average number of cases, in which one infected individual infects healthy individuals.

The basic reproduction number R_0 for system (1)–(9) is given by (see in [21] for the method of its calculation)

$$\begin{aligned} R_0(t) = {} & \frac{(\rho_1 s_1(t)v_2(t) + \gamma_i(t) l \rho_1 v_1(t))\bar{\omega}_h + \gamma_a \rho_1 s_2(t) v_2(t) + \gamma_a \gamma_r(t) l \rho_1 v_1(t)}{v_2(t)\bar{\omega}_h \bar{\omega}_i} \\ & + \frac{(\rho_2 s_1(t)v_2(t) + \gamma_i(t) l \rho_2 v_1(t))\bar{\omega}_h + \gamma_a \rho_2 s_2(t) v_2(t) + \gamma_a \gamma_r(t) l \rho_2 v_1(t)}{v_2(t)\bar{\omega}_h \bar{\omega}_p}, \end{aligned} \tag{10}$$

where $\bar{\omega}_i = \gamma_a + \gamma_i + \delta_i$, $\bar{\omega}_p = \gamma_a + \gamma_i + \delta_p$ and $\bar{\omega}_h = \gamma_r + \delta_h$.

In the case of parameters, when they do not depend on time, the basic reproduction number $R_0 > 1$ means that epidemic (or pandemic) will persist and on the other hand $R_0 < 1$ means that virus transmission dies out.

Denote by $x = (V, S, E, I, P, A, H, F, C)$ and $u = (v_2, v_1, \gamma_i, \gamma_r, s, s_1, s_2, \kappa, c, c_1, c_2, c_3)$. In order to be near control theory let us denote the right-hand sides of (1)–(9) by:

$$\begin{aligned} f_1(t, V, I, P, A, H, C, v_2, \gamma_i, \gamma_r) &= v_2(N - V + C) + \gamma_i(I + P + A) + \gamma_r H, \\ f_2(t, V, S, I, P, H, s, v_1) &= \frac{S}{N} - \Lambda - s(I + P + H)S - l v_1 V S, \\ f_3(t, V, S, E, I, P, H, s_1, s_2, v_1, \kappa) &= s_1(I + P)S + s_2 H S + l v_1 V S - \kappa E, \\ f_4(t, E, I, \gamma_i, \kappa) &= \kappa \rho_1 E - (\gamma_a + \gamma_i)I - \delta_i I, \\ f_5(t, E, P, \gamma_i, \kappa) &= \kappa \rho_2 E - (\gamma_a + \gamma_i)P - \delta_p P, \\ f_6(t, E, \kappa) &= \kappa(1 - \rho_1 - \rho_2)E, \end{aligned}$$

$$f_7(t, I, P, H, \gamma_r) = \gamma_a(I + P) - \gamma_r H - \delta_h H,$$
$$f_8(t, I, P, H) = \delta_i I + \delta_p P + \delta_h H,$$
$$f_9(t, V, S, I, P, H, c, c_1, c_2, c_3) = c(N - V) + c_1 S + c_2(I + P) + c_3 H$$

and put:

$$\begin{aligned} f(t, x, u) = (&f_1(t, V, I, P, A, H, C, v_2, \gamma_i, \gamma_r), \\ &f_2(t, V, S, I, P, H, s, v_1), \\ &f_3(t, V, S, E, I, P, H, s_1, s_2, v_1, \kappa), \\ &f_4(t, E, I, \gamma_i, \kappa), \\ &f_5(t, E, P, \gamma_i, \kappa), \\ &f_6(t, E, \kappa), \\ &f_7(t, I, P, H, \gamma_r), \\ &f_8(t, I, P, H), \\ &f_9(t, V, S, I, P, H, c, c_1, c_2, c_3)). \end{aligned}$$

Then, we can rewrite system of (1)–(9) equations as

$$\frac{dx}{dt} = f(t, x, u). \tag{11}$$

1.2. Formulation of a game for COVID-19 problem

The cost functional for our control problem should minimize the government costs of the pandemic $C(t)$ during the whole time and the time of its continuation, the value of basic reproduction number R_0 at final time and maximize the fraction of population N having antibodies $V(t)$ at final time. Thus our goal functional takes the form:

$$J(x, u, T) = \int_0^T (C(t) + 1)\,\mathrm{d}t + (aR_0(T))^n + V(T), \tag{12}$$

where a and n are suitable chosen (sufficiently large) for concrete problem. The values a and n relate to the fact that we minimize J also with respect to T, therefore if there is not a power function then minimum of J could be for some cases with $T = 0$.

We consider a 9-dimensional dynamical system controlled by twelve strategies over a time. The dynamical system consists of a state variable $x : [0, T] \to \mathbb{R}^9_+$ with nine coordinates $(V, S, E, I, P, A, H, F, C)$ and a profile of strategies

$$u = (v_2, v_1, \gamma_i, \gamma_r, s, s_1, s_2, \kappa, c, c_1, c_2, c_3),$$

where $v_2(t) \in \mathcal{V}_2, \ldots, c_3(t) \in C_3$, $t \in [0,T]$, $U = \mathcal{V}_2 \times \ldots \times C_3$, and all these strategies are measurable functions and state equations are formulated in vector form:

$$\begin{cases} \dot{x}(t) = f\left(t, x(t), u(t)\right), \quad t \in [0,T], \\ x(0) = x_0\,, \\ u(t) \in U, \;\; t \in [0,T], \end{cases} \tag{13}$$

with $f\colon [0,T] \times \mathbb{R}^9 \times \mathbb{R}^{12} \to \mathbb{R}^9$ being a measurable function in t and continuous with respect to (x,u). The initial state of the game is denoted by the vector $x_0 \in \mathbb{R}^9_+$. We are looking for solutions to (13) in space $H^1(0,T;\mathbb{R}^9)$, i.e. absolutely continuous functions with square integrable $\dot{x}(t)$.

We want to maximize the value $V(T)$ in the functional (12) and to minimize the costs of the pandemic $\int\limits_0^T (C(t)+1)dt$, the time T of the last of pandemic as well as the basic reproduction number $R_0(T)$ at final time.

The point of view of developing a disease as a game has some history nn epidemiology, see e.g. [29] where it is considered an evolutionary game. We should have in mind that behavior of people related to vaccinations against disease produces interest in game approaches (evolution games) – see e.g. [4–7, 16]. However, consideration of quarantine and isolation policy or a risk infection took some attention (see e.g. [2, 16]). Evolutionary game theory arose from the game theory by applying the basic concept of Darwinism to compensate for the idea of time evolution, which in the original game theory has not been appeared (as it mainly deals with equilibrium). This enables game players in such models to behave more intelligently and realistically, however, in which the theory predicts that game players should act defectively. In game theory, a non-cooperative game is a game with competition between individual players. Non-cooperative game tries to predict players' individual strategies and payoffs, and to find Nash equilibria. It is also more general, than cooperative games, which can be analyzed using the terms of non-cooperative game theory. Then it is enough to state sufficient assumptions to encompass all the possible strategies players may adopt, in relation to arbitration. We want to consider a non-cooperative game for the problem (12)–(13), in which strategies evaluate in time (are functions) and the payoff is the functional defined on the set of these strategies (functions), in contrary to the articles mentioned earlier, except partially in [16]. The game considered by us in such approach is more general than those investigate in [2, 4–7, 9] or [16].

In our approach to treat pandemic as a game we group our 12 strategies into two players $\nu = (v_2, v_1, \gamma_i, \gamma_r)$ and $\sigma = (s, s_1, s_2, \kappa, c, c_1, c_2, c_3)$. Therefore, player ν wants to use the strategies $v_2, v_1, \gamma_i, \gamma_r$ to maximize (12) and player σ uses the strategies $s, s_1, s_2, \kappa, c, c_1, c_2, c_3$ to minimize (12). We assume that our game is non-cooperative. Thus, to stress dependence of the state variable x for the given player ν, we have on the opponent σ profile of strategies

$$u^{\sim\sigma} = u_\nu(v_2, v_1, \gamma_i, \gamma_r) \tag{14}$$

or for the given player σ

$$u^{\sim\nu} = u_\sigma(s, s_1, s_2, \kappa, c, c_1, c_2, c_3). \tag{15}$$

We write $x^{u^{\sim\nu}}$ for the state variable satisfying (13) for player ν and for the given opponents' strategy $u^{\sim\nu}$, i.e. $x^{u^{\sim\nu}}$ satisfies:

$$\begin{cases} \dot{x}^{u^{\sim\nu}}(t) = f\left(t, x^{u^{\sim\nu}}(t), (u_\nu(t), u^{\sim\nu}(t))\right), \quad t \in [0,T], \\ x^{u^{\sim\nu}}(0) = x_0\,, \\ u_\nu(t) \in U_\nu = \mathcal{V}_2 \times \mathcal{V}_1 \times \mathcal{G}_i \times \mathcal{G}_r, \quad t \in [0,T], \\ u^{\sim\nu}(t) \in U_\sigma = \mathcal{S} \times \mathcal{S}_1 \times \mathcal{S}_2 \times \mathcal{K} \times C \times C_1 \times C_2 \times C_3, \quad t \in [0,T]. \end{cases} \tag{16}$$

Similarly we write $x^{u^{\sim\sigma}}$ for the state variable satisfying (13) for player σ and for the given opponents' strategy $u^{\sim\sigma}$, i.e. $x^{u^{\sim\sigma}}$ satisfies

$$\begin{cases} \dot{x}^{u^{\sim\sigma}}(t) = f\left(t, x^{u^{\sim\sigma}}(t), (u_\sigma(t), u^{\sim\sigma}(t))\right), \quad t \in [0,T], \\ x^{u^{\sim\sigma}}(0) = x_0\,, \\ u^{\sim\sigma}(t) \in U_\nu = \mathcal{V}_2 \times \mathcal{V}_1 \times \mathcal{G}_i \times \mathcal{G}_r, \quad t \in [0,T], \\ u_\sigma(t) \in U_\sigma = \mathcal{S} \times \mathcal{S}_1 \times \mathcal{S}_2 \times \mathcal{K} \times C \times C_1 \times C_2 \times C_3, \quad t \in [0,T]. \end{cases} \tag{17}$$

An admissible process for the game (12) subject to (13) with the strategies of the player ν and the given opponent σ is a trio $(u_\nu, x^{u^{\sim\nu}}, T)$, which belongs to the set

$$Ad_\nu(u^{\sim\nu}) = \left\{(u_\nu, x^{u^{\sim\nu}}, T) : (u_\nu, x^{u^{\sim\nu}}, T) \text{ satisfies (16)}\right\}.$$

An admissible process for the game (12) subject to (13) with the strategies of the player σ and given opponent ν is a trio $(u_\sigma, x^{u^{\sim\sigma}}, T)$ which belongs to the set

$$Ad_\sigma(u^{\sim\sigma}) = \left\{(u_\sigma, x^{u^{\sim\sigma}}, T) : (u_\sigma, x^{u^{\sim\sigma}}, T) \text{ satisfies (17)}\right\}.$$

For the given opponents' strategies $u^{\sim\nu}$ we constitute a differential game in which the goal functional (12) takes the form:

$$J(x^{u^{\sim\nu}}, u_\nu, T) = \int\limits_0^T (C(t) + 1)\,\mathrm{d}t + (aR_0(T))^n + V(T), \tag{18}$$

which we maximize in the set $Ad_\nu(u^{\sim\nu})$ and for the given opponents' strategies $u^{\sim\sigma}$ the goal functional takes the form:

$$J(x^{u^{\sim\sigma}}, u_\sigma, T) = \int_0^T (C(t) + 1)\mathrm{d}t + (aR_0(T))^n + V(T), \tag{19}$$

which we minimize in the set $Ad_\sigma(u^{\sim\sigma})$.

We name a pair of strategies $(\bar{u}_\nu, \bar{u}_\sigma)$ an open-loop Nash equilibrium if process $(\bar{x}^{\bar{u}^{\sim\nu}}, \bar{u}_\nu, \bar{T}) \in Ad_\nu(\bar{u}^{\sim\nu})$ corresponding to $\bar{u}_\nu$ and for all trio $(x^{\bar{u}^{\sim\nu}}, u_\nu, T) \in Ad_\nu(\bar{u}^{\sim\nu})$, $T \leqslant \bar{T}$, satisfies the inequality

$$J\left(\bar{x}^{\bar{u}^{\sim\nu}}, \bar{u}_\nu, \bar{T}\right) \geqslant J\left(x^{\bar{u}^{\sim\nu}}, u_\nu, T\right) \tag{20}$$

and process $(\bar{x}^{\bar{u}^{\sim\sigma}}, \bar{u}_\sigma, \bar{T}) \in Ad_\sigma(\bar{u}^{\sim\sigma})$ corresponding to $\bar{u}_\sigma$ and for all trio $(x^{\bar{u}^{\sim\sigma}}, u_\sigma, T) \in Ad_\sigma(\bar{u}^{\sim\sigma})$, $T \geqslant \bar{T}$, satisfies the inequality

$$J\left(\bar{x}^{\bar{u}^{\sim\sigma}}, \bar{u}_\sigma, \bar{T}\right) \leqslant J\left(x^{\bar{u}^{\sim\sigma}}, u_\sigma, T\right). \tag{21}$$

The state variable $\bar{x}^{\bar{u}^{\sim\nu}}$, corresponding to the strategy $\bar{u}_\nu$, the state variable $\bar{x}^{\bar{u}^{\sim\sigma}}$, corresponding to the strategy $\bar{u}_\sigma$ and time $\bar{T}$ we call *the Nash equilibrium trajectory*. Note, we assume the same $\bar{T}$ in both inequalities (20) and (21).

2. Dual open-loop game

We propose a dual dynamic programming approach to the open-loop game (18) and (19), because classical tools of game theory for our game (18) and (19) are difficult to be applied (see e.g. [3, 16, 29]). We extend dual approach for dynamic programming from [17] and [25]. The method presented here also extends that developed in [20, 24]. The dual approach means that we do not deal directly with a value function, but with some auxiliary function W, defined in a dual set P, satisfying a dual dynamic inequality and then we derive a kind of verification conditions for primal value function. New challenge in this approach to (18) and (19) is that we want to deal with approximate open-loop strategies.

Thus, let us start with the definition of a dual set. Let $P_\nu, P_\sigma \subset \mathbb{R}^{10}$ be sets of the variables (t, p), $p \in \mathbb{R}^9$, $t \in [0, T]$. Denote by $\boldsymbol{P}_\nu, \boldsymbol{P}_\sigma$ their projections on the space of variable p. In practice they can be any sets which simplify our computations. Denote by $W^1(P_\nu)$ and $W^1(P_\sigma)$ the specific Sobolev spaces of functions of the variables (t, p) having the first order weak derivative with respect to t and continuous with respect to the variable p. We need notions of auxiliaries trajectories. However, first we have to introduce a kind of dual Hamilton-Jacobi inequality with an auxiliary pair of functions $y^0_\nu(t)$, $W_\nu(t,p)$, $t \in [0,T]$, $p \in \boldsymbol{P}_\nu$, $y^0_\nu \in L^1(0,T)$, $W_\nu \in W^1(P_\nu)$. Thus let the opponents' strategy $u^{\sim\nu}$ and $\varepsilon > 0$ be given. We assume that the pair $y^0_\nu(t)$, $W_\nu(t,p)$, $t \in [0,T]$, $p \in \boldsymbol{P}_\nu$ satisfies, for some $T > 0$, in $[0,T] \times \boldsymbol{P}_\nu$, the following dual dynamic programming differential inequality:

$$
\begin{aligned}
y^0_\nu(t) \geqslant \sup \big\{ & p(pW_{\nu,t}(t,p)) + pf(t, -pW_\nu(t,p), u^\nu, u^{\sim\nu}(t)) \\
& - p_9 W_\nu(t,p) :\ u^\nu \in U_\nu \big\}
\end{aligned}
\tag{22}
$$

with the initial condition

$$
pW_\nu(0,p) = x_0,\ p \in \boldsymbol{P}_\nu,\ p = (p_1, \dots, p_9).
$$

Then we require that, for some $y_\nu(\cdot) \in L^1(0,T)$, auxiliaries trajectories $p(t)$, $t \in [0,T]$, corresponding to the player ν satisfy for strategies $u_\nu(t) \in U_\nu$

$$
\begin{aligned}
& p(t)(p(t)W_{\nu,t}(t,p(t))) + p(t)f(t, -p(t)W_\nu(t,p(t)), u^\nu(t), u^{\sim\nu}(t)) \\
& \quad -\frac{1}{T}\left(p_1(T)W_\nu(T,p(T)) + (aR^\nu_0(T))^n\right) \geqslant y_\nu(t) - \varepsilon,
\end{aligned}
\tag{23}
$$

where R^ν_0 is R_0 from (10) calculated along strategy u^ν and opponents' strategy u^σ. We assume that auxiliaries trajectories $p(t)$, $t \in [0,T]$, and the function $W_\nu(t,p)$ along it, define our original trajectory for player ν, i.e. $x(t) = -p(t)W_\nu(t,p(t))$. Thus, for a given opponents' strategy $u^{\sim\nu}$ and fixed $p^\nu_0 \in \boldsymbol{P}_\nu$ we define the set:

$$
\begin{aligned}
\mathcal{P}_\nu(u^{\sim\nu}) = \big\{ & p : [0,T] \to \boldsymbol{P}_\nu : p(0) = p^\nu_0,\ \text{exists } u_\nu(t) \in U_\nu, \\
& \text{such that the pair } (u_\nu(\cdot), p(\cdot)) \text{ satisfies (23)} \big\}.
\end{aligned}
$$

Similarly we follow for player σ, i.e. we introduce a dual Hamilton-Jacobi inequality with an auxiliary pair of functions $y^0_\sigma(t)$, $W_\sigma(t,p)$, $t \in [0,T]$, $p \in \boldsymbol{P}_\sigma$, $y^0_\sigma \in L^1(0,T)$, $W_\sigma \in W^1(P_\sigma)$. Then for the given opponents' strategy $u^{\sim\sigma}$ and $\varepsilon > 0$ we assume that the pair $y^0_\sigma(t)$, $W_\sigma(t,p)$, $t \in [0,T]$, $p \in \boldsymbol{P}_\sigma$ satisfies, for some $T > 0$, in $[0,T] \times \boldsymbol{P}_\sigma$ the following dual dynamic programming differential inequality:

$$
\begin{aligned}
-\varepsilon + y^0_\sigma(t) \leqslant \inf \big\{ & p(pW_{\sigma,t}(t,p)) + pf(t, -pW_\sigma(t,p), u^\sigma, u^{\sim\sigma}(t)) \\
& - p_9 W_\sigma(t,p) : u^\sigma \in U_\sigma \big\}
\end{aligned}
\tag{24}
$$

with the initial condition

$$
pW_\sigma(0,p) = x_0,\ p \in \boldsymbol{P}_\sigma,\ p = (p_1, \dots, p_9).
$$

Then we require that, for some $y_\sigma(\cdot) \in L^1(0,T)$, auxiliaries trajectories $p(t)$, $t \in [0,T]$ corresponding to the player σ satisfy for strategies $u_\sigma(t) \in U_\sigma$

$$\begin{aligned} &p(t)(p(t)W_{\sigma,t}(t,p(t))) + p(t)f(t,-p(t)W_\sigma(t,p(t)),u^\sigma(t),u^{\sim\sigma}(t)) \\ &\quad -\frac{1}{T}\left(p_1(T)W_\sigma(T,p(T)) + (aR_0^\sigma(T))^n\right) \leqslant y_\sigma(t), \end{aligned} \tag{25}$$

where R_0^σ is R_0 from (10) calculated along strategy u^σ and opponents' strategy u^ν. We assume that auxiliaries trajectories $p(t)$, $t \in [0,T]$ and the function $W_\sigma(t,p)$ along it, define our original trajectory for player σ, i.e. $x(t) = p(t)W_\sigma(t,p(t))$. Thus for a given opponents' strategy $u^{\sim\sigma}$ and fixed $p_0^\sigma \in \boldsymbol{P}_\sigma$ we define the set:

$$\begin{aligned} \mathcal{P}_\sigma(u^{\sim\sigma}) = \big\{ &p\colon [0,T] \to \boldsymbol{P}_\sigma \colon p(0) = p_0^\sigma, \text{ exists } u_\sigma(t) \in U_\sigma, \\ &\text{such that the pair } (u_\sigma(\cdot), p(\cdot)) \text{ satisfies (25)} \big\}. \end{aligned}$$

It is clear that not for all strategies $u_\nu(t) \in U_\nu$ there exists $p(\cdot)$ such that the pair $(u_\nu(\cdot), p(\cdot))$ satisfies (23). Therefore, we reduce the set $Ad_\nu(u^{\sim\nu})$ to the set

$$Ad_\nu(u^{\sim\nu}, \boldsymbol{P}_\nu) = \left\{ (x^{u^{\sim\nu}}, u_\nu, T) \colon (u_\nu, p) \text{ satisfies (23)} \right\}.$$

An admissible process for the game (19) with the strategies of the player σ and given opponent ν is a trio $(x^{u^{\sim\sigma}}, u_\sigma, T)$ which belongs to the set

$$Ad_\sigma(u^{\sim\sigma}, \boldsymbol{P}_\sigma) = \{(x^{u^{\sim\sigma}}, u_\sigma, T) : (u_\sigma, p) \text{ satisfies (25)}\}.$$

Hence, we must reformulate our definition of the Nash equilibrium: a pair of strategies $(\bar{u}_\nu, \bar{u}_\sigma)$ is the open-loop Nash equilibrium if process $(\bar{x}^{\bar{u}^{\sim\nu}}, \bar{u}_\nu, \bar{T})$ belongs to $Ad_\nu(\bar{u}^{\sim\nu}, \boldsymbol{P}_\nu)$ for corresponding $\bar{u}_\nu$ and for all trio $(x^{\bar{u}^{\sim\nu}}, u_\nu, T) \in Ad_\nu(\bar{u}^{\sim\nu}, \boldsymbol{P}_\nu)$ with $T \leqslant \bar{T}$ satisfies the inequality

$$J(\bar{x}^{\bar{u}^{\sim\nu}}, \bar{u}_\nu, \bar{T}) \geqslant J(x^{\bar{u}^{\sim\nu}}, u_\nu, T) \tag{26}$$

and process $(\bar{x}^{\bar{u}^{\sim\sigma}}, \bar{u}_\sigma, \bar{T}) \in Ad_\sigma(\bar{u}^{\sim\sigma}, \boldsymbol{P}_\sigma)$ corresponding to $\bar{u}_\sigma$ and for all trio $(x^{\bar{u}^{\sim\sigma}}, u_\sigma, T) \in Ad_\sigma(\bar{u}^{\sim\sigma}, \boldsymbol{P}_\sigma)$ with $T \geqslant \bar{T}$ satisfies

$$J(\bar{x}^{\bar{u}^{\sim\sigma}}, \bar{u}_\sigma, \bar{T}) \leqslant J(x^{\bar{u}^{\sim\sigma}}, u_\sigma, T). \tag{27}$$

As we are looking for approximate solutions to our game, we introduce a notion of ε-Nash equilibrium for given ε. Each pair of strategies $(\bar{u}_\nu^\varepsilon, \bar{u}_\sigma^\varepsilon)$ we call an ε-open-loop Nash equilibrium if process $(\bar{x}_\varepsilon^{\bar{u}^{\varepsilon\sim\nu}}, \bar{u}_\nu^\varepsilon, \bar{T}^\varepsilon)$ belongs to $Ad_\nu(\bar{u}^{\varepsilon\sim\nu}, \boldsymbol{P}_\nu)$ for corresponding $\bar{u}_\nu^\varepsilon$ and for all trio $(x^{\bar{u}^{\varepsilon\sim\nu}}, u_\nu, T) \in Ad_\nu(u^{\sim\nu}, \boldsymbol{P}_\nu)$ with $T \leqslant \bar{T}^\varepsilon$ satisfies the inequality

$$J(\bar{x}_\varepsilon^{\bar{u}^{\varepsilon\sim\nu}}, \bar{u}_\nu^\varepsilon, \bar{T}^\varepsilon) \geqslant J(x^{\bar{u}^{\varepsilon\sim\nu}}, u_\nu, T) - 2\varepsilon \tag{28}$$

and process $(\bar{x}_\varepsilon^{\bar{u}^{\varepsilon\sim\sigma}}, \bar{u}_\sigma^\varepsilon, \bar{T}^\varepsilon) \in Ad_\sigma(\bar{u}^{\varepsilon\sim\sigma}, \boldsymbol{P}_\sigma)$ corresponding to $\bar{u}_\sigma^\varepsilon$ and for all trio $(x^{\bar{u}^{\varepsilon\sim\sigma}}, u_\sigma, T) \in Ad_\sigma(\bar{u}^{\varepsilon\sim\sigma}, \boldsymbol{P}_\sigma)$ with $T \geqslant \bar{T}^\varepsilon$ satisfies

$$J(\bar{x}_\varepsilon^{\bar{u}^{\varepsilon\sim\sigma}}, \bar{u}_\sigma^\varepsilon, \bar{T}^\varepsilon) \leqslant J(x^{\bar{u}^{\varepsilon\sim\sigma}}, u_\sigma, T) + 2\varepsilon. \tag{29}$$

3. Verification theorem for approximate dual open-loop Nash equilibrium

Having the notions and notations introduced in the former section we are ready to formulate and prove a kind of a verification theorem allowing to check whether a dual open-loop strategy is candidate to be ε-Nash equilibrium for the game (28)–(29).

Theorem 1 *Assume that there exists a trio* $y_\nu^0(t)$, $y_\nu(t)$, $W_\nu(t,p)$ *and* $\bar{T}^\varepsilon > 0$, $y_\nu^0(t) - y_\nu(t) \geqslant 0$, $t \in [0,\bar{T}]$, $p \in \boldsymbol{P}_\nu$ *satisfying in* $[0,\bar{T}^\varepsilon]\times\boldsymbol{P}_\nu$ *(22) with opponents' strategy* $\bar{u}^{\varepsilon\sim\nu}$*. Let auxiliary trajectories* $\bar{p}_\nu^\varepsilon \in \mathcal{P}_\nu(\bar{u}^{\varepsilon\sim\nu})$ *together with strategy* $\bar{u}_\nu^\varepsilon$ *satisfy*

$$\begin{aligned} y_\nu^0(t) - \varepsilon \leqslant{} & \bar{p}_\nu^\varepsilon(t)(\bar{p}_\nu^\varepsilon(t) W_{\nu,t}(t,\bar{p}_\nu^\varepsilon(t))) \\ & + \bar{p}_\nu^\varepsilon(t) f(t, -\bar{p}_\nu^\varepsilon(t) W_\nu(t,\bar{p}_\nu^\varepsilon(t)), \bar{u}_\nu^\varepsilon(t), \bar{u}^{\varepsilon\sim\nu}(t)) \\ & - \bar{p}_{9,\nu}^\varepsilon(t) W_\nu(t,\bar{p}_\nu^\varepsilon(t)), \end{aligned} \tag{30}$$

$$\begin{aligned} y_\nu(t) \geqslant{} & \bar{p}_\nu^\varepsilon(t)(\bar{p}_\nu^\varepsilon(t) W_{\nu,t}(t,\bar{p}_\nu^\varepsilon(t))) \\ & + \bar{p}_\nu^\varepsilon(t) f(t, -\bar{p}_\nu^\varepsilon(t) W_\nu(t,\bar{p}_\nu^\varepsilon(t)), \bar{u}_\nu^\varepsilon(t), \bar{u}^{\varepsilon\sim\nu}(t)) \\ & - \frac{1}{\bar{T}^\varepsilon}\left(\bar{p}_{1,\nu}^\varepsilon(\bar{T}^\varepsilon) W_\nu(\bar{T}^\varepsilon, \bar{p}_\nu^\varepsilon(\bar{T}^\varepsilon)) + (aR_0^\nu(\bar{T}^\varepsilon))^n\right). \end{aligned} \tag{31}$$

Assume that there exists a trio $y_\sigma^0(t)$, $y_\sigma(t)$, $W_\sigma(t,p)$, $y_\sigma^0(t) - y_\sigma(t) \geqslant 0$, $t \in [0,T_1]$, $T_1 > \bar{T}^\varepsilon$, $p \in \boldsymbol{P}_\sigma$ *satisfying in* $[0,T_1]\times\boldsymbol{P}_\sigma$ *(24) with opponents' strategy* $\bar{u}^{\varepsilon\sim\sigma}$*. Let auxiliary trajectories* $\bar{p}_\sigma^\varepsilon \in \mathcal{P}_\sigma(\bar{u}^{\varepsilon\sim\sigma})$ *together with strategy* $\bar{u}_\sigma^\varepsilon$ *satisfy in* $[0,\bar{T}^\varepsilon]$

$$\begin{aligned} y_\sigma^0(t) \geqslant{} & \bar{p}_\sigma^\varepsilon(t)(\bar{p}_\sigma^\varepsilon(t) W_{\sigma,t}(t,\bar{p}_\sigma^\varepsilon(t))) \\ & + \bar{p}_\sigma^\varepsilon(t) f(t, -\bar{p}_\sigma^\varepsilon(t) W_\sigma(t,\bar{p}_\sigma^\varepsilon(t)), \bar{u}_\sigma^\varepsilon(t), \bar{u}^{\varepsilon\sim\sigma}(t)) \\ & - \bar{p}_{9,\sigma}^\varepsilon(t) W_\sigma(t,\bar{p}_\sigma^\varepsilon(t), \end{aligned} \tag{32}$$

$$\begin{aligned} -\varepsilon + y_\sigma(t) \leqslant{} & \bar{p}_\sigma^\varepsilon(t)(\bar{p}_\sigma^\varepsilon(t) W_{\sigma,t}(t,\bar{p}_\sigma^\varepsilon(t))) \\ & + \bar{p}_\sigma^\varepsilon(t) f(t, -\bar{p}_\sigma^\varepsilon(t) W_\sigma(t,\bar{p}_\sigma^\varepsilon(t)), \bar{u}_\sigma^\varepsilon(t)\bar{u}^{\varepsilon\sim\sigma}(t)) \\ & - \frac{1}{\bar{T}^\varepsilon}\left(\bar{p}_{1,\sigma}^\varepsilon(\bar{T}^\varepsilon) W_\sigma(\bar{T}^\varepsilon, \bar{p}_\sigma^\varepsilon(\bar{T}^\varepsilon)) + (aR_0^\sigma(\bar{T}^\varepsilon))^n\right). \end{aligned} \tag{33}$$

Then, the dual open-loop strategies $(\bar{u}_\nu^\varepsilon, \bar{u}_\sigma^\varepsilon)$ *are Nash equilibrium for the game (28)–(29).*

Proof. We have to prove that for strategy $\bar{u}_\nu^\varepsilon(\cdot)$ and strategy $\bar{u}_\sigma^\varepsilon(\cdot)$ with corresponding their processes, the inequalities (28), (29) hold for all processes from $Ad_\nu(\bar{u}^{\varepsilon\sim\nu}, \boldsymbol{P}_\nu)$ and $Ad_\sigma(\bar{u}^{\varepsilon\sim\sigma}, \boldsymbol{P}_\sigma)$, respectively. Thus let us take any

$p(\cdot) \in \mathcal{P}_\nu(\bar{u}^{\varepsilon\sim\nu})$ with $T \leqslant \bar{T}^\varepsilon$ and put it in (22) in place of p assuming the pair $y_\nu(t)$, $W_\nu(t,p)$ and T. Then we get, for $t \in [0,T]$

$$y^0_\nu(t) \geqslant p(t)(p(t)W_{\nu,t}(t,p(t))) + p(t)f(t,-p(t)W_\nu(t,p(t)),u_\nu(t),\bar{u}^{\varepsilon\sim\nu}(t)) \\ - \; p_9(t)W_\nu(t,p(t)). \tag{34}$$

Since $p(\cdot) \in \mathcal{P}_\nu(\bar{u}^{\varepsilon\sim\nu})$ then it satisfies (23) and we can transform (34) to

$$\varepsilon + y^0_\nu(t) - y_\nu(t) \geqslant -p_9(t)W_\nu(t,p(t)) \\ - \; \frac{1}{T}(p_1(T)W_\nu(T,p(T)) + (aR^\nu_0(T))^n). \tag{35}$$

Integrating (35) in the interval $[0,T]$ we come to

$$\int\limits_0^T (y^0_\nu(t) - y_\nu(t))\,\mathrm{d}t \geqslant -\int\limits_0^T p_9(t)W_\nu(t,p(t))\,\mathrm{d}t \\ - \; p_1(T)W_\nu(T,p(T)) + (aR^\nu_0(T))^n - \varepsilon. \tag{36}$$

Following similarly as above, but now taking the auxiliary trajectories $\bar{p}^\varepsilon_\nu \in \mathcal{P}_\nu(\bar{u}^{\varepsilon\sim\nu})$ together with strategy $\bar{u}^\varepsilon_\nu$ and inequalities (30), (31) we get the inequality

$$\int\limits_0^{\bar{T}^\varepsilon} (y^0_\nu(t) - y_\nu(t))\,\mathrm{d}t \leqslant -\int\limits_0^{\bar{T}^\varepsilon} \bar{p}^\varepsilon_{9,\nu}(t)W_\nu(t,\bar{p}^\varepsilon_\nu(t))\,\mathrm{d}t \\ - \; \bar{p}^\varepsilon_{1,\nu}(\bar{T}^\varepsilon)W_\nu(\bar{T}^\varepsilon,\bar{p}^\varepsilon_\nu(\bar{T}^\varepsilon)) + (aR^\nu_0(\bar{T}^\varepsilon))^n - \varepsilon. \tag{37}$$

Comparing (36), (37) and taking into account that $x(t) = -p(t)W_\nu(t,p(t))$, $T \leqslant \bar{T}^\varepsilon$ and $y^0_\nu(t) - y_\nu(t) \geqslant 0$, $t \in [0,\bar{T}]$ we get

$$J(\bar{x}^{\bar{u}^{\varepsilon\sim\nu}}_\varepsilon, \bar{u}^\varepsilon_\nu, \bar{T}^\varepsilon) \geqslant J(x^{\bar{u}^{\varepsilon\sim\nu}}, u_\nu, T) - 2\varepsilon$$

i.e. the first inequality in the definition of the ε-dual open-loop Nash equilibrium for our game.

In order to receive the second one we follow analogously, thus we only sketch the proof of it. Let $p(\cdot) \in \mathcal{P}_\sigma(\bar{u}^{\varepsilon\sim\sigma})$ and put it in (24) assuming the pair $y_\sigma(t)$, $W_\sigma(t,p)$ and $T_1 \geqslant T \geqslant \bar{T}^\varepsilon$. Then for $t \in [0,T]$

$$-\varepsilon + y^0_\sigma(t) \leqslant p(t)(p(t)W_{\sigma,t}(t,p(t))) \\ + \; p(t)f(t,-p(t)W_\sigma(t,p(t)),u_\sigma(t),\bar{u}^{\varepsilon\sim\sigma}(t)) - p_9(t)W_\sigma(t,p(t)). \tag{38}$$

Since $p(\cdot) \in \mathcal{P}_\sigma(\bar{u}^{\varepsilon\sim\sigma})$ then we transform (38) to

$$-\varepsilon + y_\sigma^0(t) - y_\sigma(t) \leqslant -p_9(t)W_\sigma(t,p(t)) - \frac{1}{T}\big(p_1(T)W_\sigma(T,p(T)) \\ +\ (aR_0^\sigma(T))^n\big). \tag{39}$$

Integrating (39) we come to

$$\int_0^T (y_\sigma^0(t) - y_\sigma(t))\,\mathrm{d}t \leqslant -\int_0^T p_9(t)W_\sigma(t,p(t))\,\mathrm{d}t \\ -\ p_1(T)W_\sigma(T,p(T)) + (aR_0^\sigma(T))^n + \varepsilon. \tag{40}$$

Next now taking the auxiliary trajectories $\bar{p}_\sigma^\varepsilon \in \mathcal{P}_\sigma(\bar{u}^{\varepsilon\sim\sigma})$ together with strategy $\bar{u}_\sigma^\varepsilon$ and suitably adapted inequalities (32), (33) we get the inequality

$$\int_0^{\bar{T}^\varepsilon} (y_\sigma^0(t) - y_\sigma(t))\,\mathrm{d}t \leqslant -\int_0^{\bar{T}^\varepsilon} \bar{p}_{9,\sigma}^\varepsilon(t)W_\sigma(t,\bar{p}_\sigma^\varepsilon(t))\,\mathrm{d}t \\ -\ \bar{p}_{1,\sigma}^\varepsilon(\bar{T}^\varepsilon)W_\sigma(\bar{T}^\varepsilon,\bar{p}_\sigma^\varepsilon(\bar{T}^\varepsilon)) + (aR_0^\sigma(\bar{T}^\varepsilon))^n + \varepsilon. \tag{41}$$

Thus (40) and (41) imply

$$J(\bar{x}_\varepsilon^{\bar{u}^{\varepsilon\sim\sigma}}, \bar{u}_\sigma^\varepsilon, \bar{T}^\varepsilon) \leqslant J(x^{\bar{u}^{\sim\sigma}}, u_\sigma, T) + 2\varepsilon, \tag{42}$$

i.e. the second inequality of ε-Nash equilibrium (29). □

4. Algorithm

In this section we give numerical algorithm.

1. Let $\varepsilon > 0$ and fix an initial condition $x(0) = x_0$, where $x_0 \in \mathbb{R}^9$.

2.1. Fix $M \times L$ of the fours strategies $u_{\nu j} = \{v_{2j}(t), v_{1j}(t), \gamma_{ij}(t), \gamma_{rj}(t)\}$, $t \in [0,T_k]$, $j = 1,\ldots,M$, $k = 1,\ldots,L$ for player u_ν and fix $1\times L$ of the eight opponents' strategies $u^{\sim\nu} = \{s(t), s_1(t), s_2(t), \kappa(t), c(t), c_1(t), c_2(t), c_3(t)\}$, $t \in [0,T_k]$, $k = 1,\ldots,L$. For each of the $M \times L$ fours strategies u_ν and $1 \times L$ of the eights opponents' strategies $u^{\sim\nu}$, solve $M \times L$-times differential equation (11) in $[0,T_k]$, $k = 1,\ldots,L$, finding C_j^ν and V_j^ν, $j = 1,\ldots,M$, which describe system (1)–(9).

2.2. Fix $M \times L$ of the eights strategies $u_{\sigma j} = \{s_j(t), s_{1j}(t), s_{2j}(t), \kappa_j(t), c_j(t), c_{1j}(t), c_{2j}(t), c_{3j}(t)\}$, $t \in [0, T_k]$, $j = 1, \ldots, M$, $k = 1, \ldots, L$ for player u_σ and fix $1 \times L$ of the four opponents' strategies $u^{\sim\sigma} = \{v_2(t), v_1(t), \gamma_i(t), \gamma_r(t)\}$, $t \in [0, T_k]$, $k = 1, \ldots, L$. For each of the $M \times L$ eights strategies u_σ and $1 \times L$ of the fours opponents' strategies $u^{\sim\sigma}$, solve $M \times L$-times differential equation (11) in $[0, T_k]$, $k = 1, \ldots, L$, finding C_j^σ and V_j^σ, $j = 1, \ldots, M$, which describe system (1)–(9).

3.1. For each of the $M \times L$ fours strategies u_ν and $1 \times L$ of the eights opponents' strategies $u^{\sim\nu}$ calculate $R_{0\,j}^\nu(T_k)$ from (10), $j = 1, \ldots, M$, $k = 1, \ldots, L$.

3.2. For each of the $M \times L$ eights strategies u_σ and $1 \times L$ of the fours opponents' strategies $u^{\sim\sigma}$ calculate $R_{0\,j}^\sigma(T_k)$ from (10), $j = 1, \ldots, M$, $k = 1, \ldots, L$.

4.1. For each of the $M \times L$ fours strategies u_ν, $1 \times L$ of the eights opponents' strategies $u^{\sim\nu}$ and for $M \times L$ functions C_j^ν, $j = 1, \ldots, M$, calculate $\int_0^{T_k} (C_j^\nu(t) + 1)\,dt$, $j = 1, \ldots, M$, $k = 1, \ldots, L$. For so found $M \times L$ integrals, for $R_{0\,j}^\nu(T_k)$, $V_j^\nu(T_k)$, $j = 1, \ldots, M$, $k = 1, \ldots, L$ and for a and n suitable chosen for a concrete problem, find from (18) $M \times L$ values of the functional J. We choose values a and n in such a way that maximal T_k, $k = 1, \ldots, L$ could not decide only on the value of the functional J.

4.2. For each of the $M \times L$ eights strategies u_σ, $1 \times L$ of the fours opponents' strategies $u^{\sim\sigma}$ and for $M \times L$ functions C_j^σ, $j = 1, \ldots, M$, calculate $\int_0^{T_k} (C_j^\sigma(t) + 1)\,dt$, $j = 1, \ldots, M$, $k = 1, \ldots, L$. For so found $M \times L$ integrals, for $R_{0\,j}^\sigma(T_k)$, $V_j^\sigma(T_k)$, $j = 1, \ldots, M$, $k = 1, \ldots, L$ and for a and n found in step 4.1 and such that minimum of the functional J could not be at $T_k = 0$, $k = 1, \ldots, L$, find from (19) $M \times L$ values of the functional J.

5.1. From among of the $M \times L$ values of the functionals (18), choose the one with maximal value and denote suitable strategies, time, class of the vaccinated and costs, which correspond to this value as follows $u_\nu^s = (v_2^s(t), v_1^s(t), \gamma_i^s(t), \gamma_r^s(t))$ and $u_s^{\sim\nu} = (s^s(t), s_1^s(t), s_2^s(t), \kappa^s(t), c^s(t), c_1^s(t), c_2^s(t), c_3^s(t))$, $t \in [0, T^\varepsilon]$, V_ν^ε and C_ν^ε. We name these strategies and time T^ε *suspected optimal*.

5.2. From among of the $M \times L$ values of the functionals (19), choose the one with minimal value and denote suitable strategies, time, class of the vaccinated and costs, which correspond to this value as follows $u^s_\sigma = (s^s(t), s^s_1(t), s^s_2(t), \kappa^s(t), c^s(t), c^s_1(t), c^s_2(t), c^s_3(t))$ and $u^{\sim\sigma}_s = (v^s_2(t), v^s_1(t), \gamma^s_i(t), \gamma^s_r(t))$, $t \in [0, T^\varepsilon]$, V^ε_σ and C^ε_σ. Among $M \times L$ strategies there must be $u_\sigma = u^{\sim\nu}_s$. If values of the functionals found in steps 5.1 and 5.2 are different, we repeat steps 2.1–4.2. Otherwise, we name these strategies, time T^ε and V^ε_σ and C^ε_σ *suspected optimal*.

6.1. Choose a discrete set $\boldsymbol{P_\nu}$ composed of the points $p_i \in \boldsymbol{P_\nu}$, $i = 1, \ldots, K$ and choose dual open-loop functions $u_\nu(t, p_i)$ and $u^{\sim\nu}(t, p_i)$, $(t, p_i) \in [0, T^\varepsilon] \times \boldsymbol{P_\nu}$, $i = 1, \ldots, K$.

6.2. Choose a discrete set $\boldsymbol{P_\sigma}$ composed of the points $p_i \in \boldsymbol{P_\sigma}$, $i = 1, \ldots, K$ and choose dual open-loop functions $u_\sigma(t, p_i)$ and $u^{\sim\sigma}(t, p_i)$, $(t, p_i) \in [0, T^\varepsilon] \times \boldsymbol{P_\sigma}$, $i = 1, \ldots, K$.

7.1. For all $p_i \in \boldsymbol{P_\nu}$, $i = 1, \ldots, K$ solve differential inequality (22) in $[0, T^\varepsilon]$ and find $W_\nu(t, p_i)$ and $y^0_\nu(t)$.

7.2. For all $p_i \in \boldsymbol{P_\sigma}$, $i = 1, \ldots, K$ solve differential inequality (24) in $[0, T^\varepsilon]$ and find $W_\sigma(t, p_i)$ and y^0_σ.

8.1. For dual open-loop strategies $u_\nu(t, p_i)$ and $u^{\sim\nu}(t, p_i)$, for $R^\varepsilon_{0\nu}$, fixed y_ν, for $W_\nu(t, p_i)$, $i = 1, \ldots, K$ found in step 7.1, ε, a, n and T^ε, choose $p_l \in \boldsymbol{P_\nu}$, $l = 1, \ldots, S$, $S \leqslant K$, which satisfy inequality (23).

8.2. For dual open-loop strategies $u_\sigma(t, p_i)$ and $u^{\sim\sigma}(t, p_i)$, for $R^\varepsilon_{0\sigma}$, fixed y_σ, for $W_\sigma(t, p_i)$, $i = 1, \ldots, K$ found in step 7.2, a, n and T^ε, choose $p_l \in \boldsymbol{P_\sigma}$, $l = 1, \ldots, S$, $S \leqslant K$, which satisfy inequality (25).

9.1. For dual open-loop strategies $u_\nu(t, p_l)$ and $u^{\sim\nu}(t, p_l)$, $p_l \in \boldsymbol{P_\nu}$, $l = 1, \ldots, S$, for y_ν, W_ν, C^ε_ν, $R^\varepsilon_{0\nu}$, for fixed y^0_ν, for T^ε, a and n, find auxiliary trajectory among discrete points and on its basis construct continuous trajectory $\bar{p}^\varepsilon_\nu$ so as to (30) and (31) are satisfied.

9.2. For dual open-loop strategies $u_\sigma(t, p_l)$ and $u^{\sim\sigma}(t, p_l)$, $p_l \in \boldsymbol{P_\sigma}$, $l = 1, \ldots, S$, for y_σ, W_σ, C^ε_σ, $R^\varepsilon_{0\sigma}$, for fixed y^0_σ, for T^ε, a, n and ε, find auxiliary trajectory among discrete points and on its basis construct continuous trajectory $\bar{p}^\varepsilon_\sigma$ so as to (32) and (33) are satisfied.

10. If inequalities (30)–(31) and (32)–(33) are satisfied then Verification Theorem guarantees that the dual open-loop strategies $(\bar{u}^\varepsilon_\nu, \bar{u}^\varepsilon_\sigma)$ are ε-Nash equilibrium for the game (28)–(29). Otherwise we repeat steps 2–9.2.

Now we give a numerical example which allow us check whether optimal strategies chosen in some way cause that inequalities (30), (31), (32) and (33) hold.

4.1. The practical example realizing numerical algorithm

We use in this numerical example real data from Poland in May 8, 2021. Total population size $N = 38000000$, which means $N = 100\% = 1$ and

$V(0) = 3579910 + 2563079$ (which constitutes 16.16% of the population) – vaccinated plus healers

$I(0) = 2829196$ (which constitutes 7% of the population) – symptomatic and infectious (taking also super-spreaders)

$H(0) = 17155$ (which constitutes 0.04% of the population) – hospitalized

$F(0) = 69866$ (which constitutes 0.18% of the population) – fatality

We assume the following percentages: 76% for susceptible, 0.18% for exposed, 0.34% for super-spreaders and 0.1% for infectious but asymptomatic.

The other coefficients l, ρ_1, ρ_2, γ_a, δ_i, δ_p and δ_h correspond to the situation in Wuhan (see e.g. [23]), because we were unable to calculate them.

We take percentages only for the acceleration of the numerical calculations. Hence we assume the following initial conditions: $V(0) = 0.1616$, $S(0) = 0.76$, $E(0) = 0.0018$, $I(0) = 0.07$, $P(0) = 0.0034$, $A(0) = 0.001$, $H(0) = 0.0004$, $F(0) = 0.0018$ and $C(0) = 0$, where $V(0) + S(0) + E(0) + I(0) + P(0) + A(0) + H(0) + F(0) + C(0) = N = 1$.

After several sets of calculation, for simplicity, we decided that our time T is fixed and approximately $T^{\varepsilon} = 0.5$. We assumed that the interval with the length 0.1 corresponds to one month and so the interval on which we study our example $[0, 0.5]$ corresponds to five months. That implies also that we do not use any units for these parameters. They are simply constant during one month and dimensionless.

1. Take $\varepsilon = 0.1$, $N = 1$, $\Lambda = 1$, $l = 1.56$, $\rho_1 = 0.58$, $\rho_2 = 0.001$, $\gamma_a = 0.94$, $\delta_i = 3.5$, $\delta_p = 1$, $\delta_h = 0.3$ and fix initial condition $x_0 = (V(0), S(0), E(0), I(0), P(0), A(0), H(0), F(0), C(0))$.

2.1. Fix $M \times L = 6$ fours strategies u_ν and $L = 1$ of the eight opponents' strategies $u^{\sim\nu}$ (see Appendix).

2.2. Fix $M \times L = 6$ eights strategies u_σ and $L = 1$ of the four opponents's strategies $u^{\sim\sigma}$ (see Appendix).

3.1. Calculate $R^{\nu}_{0\,j}(T_k)$ from (10), $j = 1, \ldots, 6$, $k = 1, \ldots, 5$ (see Appendix).

3.2. Calculate $R^{\sigma}_{0\,j}(T_k)$ from (10), $j = 1, \ldots, 6$, $k = 1, \ldots, 5$ (see Appendix).

4.1. For 6 integrals calculated from (12), for $R^{\nu}_{0\,j}(T_k)$, $V^{\nu}_{j}(T_k)$, $j = 1, \ldots, 6$, $k = 1, \ldots, 5$ and for $a = 1$ and $n = 1$, find from (18) $M \times L = 6$ values of the functional J (see Appendix).

4.2. For 6 integrals calculated from (12), for $R^{\sigma}_{0\,j}(T_k)$, $V^{\sigma}_{j}(T_k)$, $j = 1, \ldots, 6$, $k = 1, \ldots, 5$ and for $a = 1$ and $n = 1$, find from (19) $M \times L = 6$ values of the functional J (see Appendix).

5.1. From among of the $M \times L = 6$ values of the functionals (18), the one with maximal value is $J = 2.5979$. For this functional denote suitable strategies and time ($t \in [0, T^\varepsilon]$, $T^\varepsilon = 0.5$ – see Table 1), which correspond to this value as follows (see all the tested strategies in Appendix):

Table 1: Suspected strategies for player ν

	Intervals				
Strategies	$[0, 0.1)$	$[0.1, 0.2)$	$[0.2, 0.3)$	$[0.3, 0.4)$	$[0.4, 0.5]$
$v_1^s(t)$	1.9000	1.9500	2.0000	2.0500	2.1000
$v_2^s(t)$	0.7000	0.7250	0.7500	0.7750	0.8000
$\gamma_i^s(t)$	0.4000	0.4250	0.4500	0.4750	0.5000
$\gamma_r^s(t)$	0.8000	0.8250	0.8500	0.8750	0.9000
$s^s(t)$	0.0200	0.0175	0.0150	0.0125	0.0100
$s_1^s(t)$	0.0300	0.0275	0.0250	0.0225	0.0200
$s_2^s(t)$	0.0200	0.0175	0.0150	0.0125	0.0100
$\kappa^s(t)$	0.1000	0.0950	0.0900	0.0850	0.0800
$c^s(t)$	3.0000	2.8000	2.6000	2.4000	2.2000
$c_1^s(t)$	0.0500	0.0475	0.0450	0.0425	0.0400
$c_2^s(t)$	0.5000	0.4800	0.4600	0.4400	0.4200
$c_3^s(t)$	1.0000	0.9750	0.9500	0.9250	0.9000

These strategies and time T^ε are suspected optimal.

5.2. From among of the $M \times L = 6$ values of the functionals (19), the one with minimal value is $J = 2.5979$. For this functional denote suitable strategies and time ($t \in [0, T^\varepsilon]$, $T^\varepsilon = 0.5$ – see Table 2), which correspond to this value as follows (see all the tested strategies in Appendix):

Table 2: Suspected strategies for player σ

	Intervals				
Strategies	$[0, 0.1)$	$[0.1, 0.2)$	$[0.2, 0.3)$	$[0.3, 0.4)$	$[0.4, 0.5]$
$v_1^s(t)$	1.9000	1.9500	2.0000	2.0500	2.1000
$v_2^s(t)$	0.7000	0.7250	0.7500	0.7750	0.8000
$\gamma_i^s(t)$	0.4000	0.4250	0.4500	0.4750	0.5000
$\gamma_r^s(t)$	0.8000	0.8250	0.8500	0.8750	0.9000
$s^s(t)$	0.0200	0.0175	0.0150	0.0125	0.0100
$s_1^s(t)$	0.0300	0.0275	0.0250	0.0225	0.0200
$s_2^s(t)$	0.0200	0.0175	0.0150	0.0125	0.0100
$\kappa^s(t)$	0.1000	0.0950	0.0900	0.0850	0.0800
$c^s(t)$	3.0000	2.8000	2.6000	2.4000	2.2000
$c_1^s(t)$	0.0500	0.0475	0.0450	0.0425	0.0400
$c_2^s(t)$	0.5000	0.4800	0.4600	0.4400	0.4200
$c_3^s(t)$	1.0000	0.9750	0.9500	0.9250	0.9000

As we see, strategies and values of the functionals found in step 5.1 for player u_ν and strategies and values of the functionals found in step 5.2 for player u_σ are the same. These strategies and time T^ε are suspected optimal.

In the Figure 3 we present values of the basic reproduction number R_0.

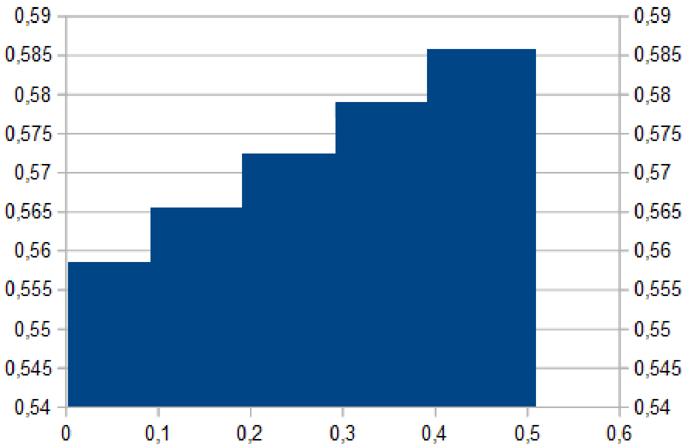

Figure 3: Basic reproduction number R_0 for the suspected optimal strategies

In the Figure 4 given below we see part of the population V having antibodies.

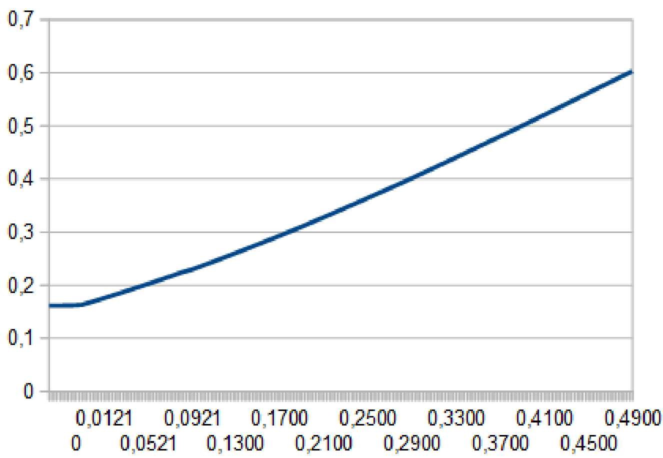

Figure 4: A part of the population having antibodies for the suspected optimal strategies

In the Figure 5 given below we see costs C of the pandemic.

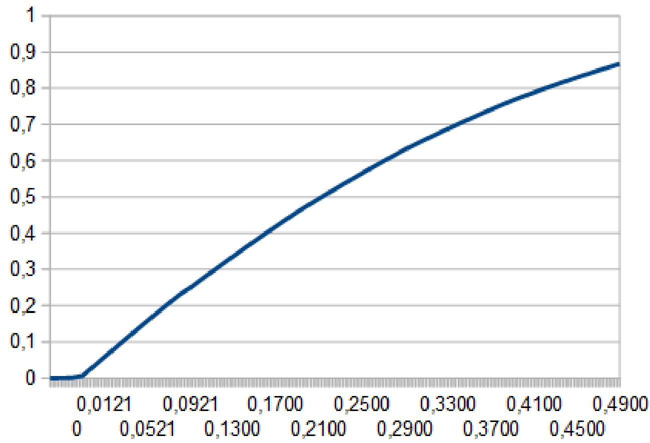

Figure 5: Costs of the pandemic for the suspected optimal strategies

6.1. Take the following vectors:

$p_1 = (5, 5, 5, 5, 5, 5, 5, 5, 5)$,
$p_2 = (50, 50, 50, 50, 50, 50, 50, 50, 50)$,
$p_3 = (20, 50, 30, 40, 90, 70, 10, 80, 60)$,
$p_4 = (150, 130, 180, 190, 170, 120, 140, 110, 160)$.

Having suspected strategies from step 5.1 and vectors $p_i, i = 1, \ldots, 4$, build dual open-loop functions

$$u_\nu(t, p_i) = \{v_2^s(t)p_i^1 p_i^2, v_1^s(t)p_i^2 p_i^3, \gamma_i^s(t)p_i^3 p_i^4, \gamma_r^s(t)p_i^4 p_i^5\} \quad \text{and}$$
$$u^{\sim\nu}(t, p_i) = \{s^s(t)p_i^5 p_i^6, s_1^s(t)p_i^6 p_i^7, s_2^s(t)p_i^7 p_i^8, \kappa^s(t)p_i^8 p_i^9, c^s(t)p_i^9 p_i^1, c_1^s(t)p_i^8 p_i^2, c_2^s(t)p_i^7 p_i^3, c_3^s(t)p_i^6 p_i^4\},$$

where $p_i^1, \ldots, p_i^9$ are coordinates of the vectors $p_i, i = 1, \ldots, 4$ given above.

6.2. Take the following vectors:

$p_1 = (-0.005, -0.008, -0.002, -0.006, -0.004, -0.001, -0.003, -0.007, -0.001)$,
$p_2 = (-0.005, -0.0002, -0.0001, -0.008, -0.003, -0.0007, -0.0004, -0.006, -0.003)$,
$p_3 = (-0.01, -0.01, -0.01, -0.01, -0.01, -0.01, -0.01, -0.01, -0.01)$,
$p_4 = (-0.001, -0.00001, -0.00001, -0.001, -0.00001, -0.00001, -0.001, -0.001, -0.00001)$.

Having suspected strategies from step 5.2 and vectors $p_i, i = 1, \ldots, 4$, build dual open-loop functions

$$u_\sigma(t, p_i) = \{s^s(t)p_i^5 p_i^6, s_1^s(t)p_i^6 p_i^7, s_2^s(t)p_i^7 p_i^8, \kappa^s(t)p_i^8 p_i^9, c^s(t)p_i^9 p_i^1, c_1^s(t)p_i^8 p_i^2, c_2^s(t)p_i^7 p_i^3, c_3^s(t)p_i^6 p_i^4\} \quad \text{and}$$

$$u^{\sim\sigma}(t, p_i) = \{v_2^s(t)p_i^1 p_i^2, v_1^s(t)p_i^2 p_i^3, \gamma_i^s(t)p_i^3 p_i^4, \gamma_r^s(t)p_i^4 p_i^5\},$$

where $p_i^1, \ldots, p_i^9$ are coordinates of the vectors $p_i, i = 1, \ldots, 4$ given above.

7.1. Find W_ν for all vectors p_i, $i = 1, \ldots, 4$ given above, solving differential inequality (22).

7.2. Find W_σ for all vectors p_i, $i = 1, \ldots, 4$ given above, solving differential inequality (24).

8.1. For dual open-loop strategies $u_\nu(t, p_i)$ and $u^{\sim\nu}(t, p_i)$, $i = 1 \ldots, 4$ found in step 6.1, for R_0 given in Appendix, $y_\nu = 1$, for W_ν found in step 7.1, $\varepsilon = 0.1$, $a = 1$, $n = 1$ and $T^\varepsilon = 0.5$, choose the following p_l, $l = l, \ldots, 4$, which satisfy inequality (23):

$p_1 = (5, 5, 5, 5, 5, 5, 5, 5, 5)$,

$p_2 = (50, 50, 50, 50, 50, 50, 50, 50, 50)$,

$p_3 = (20, 50, 30, 40, 90, 70, 10, 80, 60)$,

$p_4 = (150, 130, 180, 190, 170, 120, 140, 110, 160)$.

8.2. For dual open-loop strategies $u_\sigma(t, p_i)$ and $u^{\sim\sigma}(t, p_i)$, $i = 1, \ldots, 4$ found in step 6.2, for R_0 given in Appendix, $y_\sigma = 1$, for W_σ found in step 7.2, $a = 1$, $n = 1$ and $T^\varepsilon = 0.5$, choose the following p_l, $l = 1, \ldots, 4$, which satisfy inequality (25):

$p_1 = (-0.005, -0.008, -0.002, -0.006, -0.004, -0.001, -0.003, -0.007, -0.001)$,

$p_2 = (-0.005, -0.0002, -0.0001, -0.008, -0.003, -0.0007, -0.0004, -0.006, -0.003)$,

$p_3 = (-0.01, -0.01, -0.01, -0.01, -0.01, -0.01, -0.01, -0.01, -0.01)$,

$p_4 = (-0.001, -0.00001, -0.00001, -0.001, -0.00001, -0.00001, -0.001, -0.001, -0.00001)$.

9.1. Find W_ν for the optimal vector $\bar{p}_\nu^\varepsilon$ – see Figure 6.

9.2. Find W_σ for the optimal vector $\bar{p}_\sigma^\varepsilon$ – see Figure 7.

10. Because (30)–(31) and (32)–(33) are satisfied then Verification Theorem guarantees that the dual open-loop strategies $(\bar{u}_\nu^\varepsilon, \bar{u}_\sigma^\varepsilon)$ are ε-Nash equilibrium for the game (28)–(29).

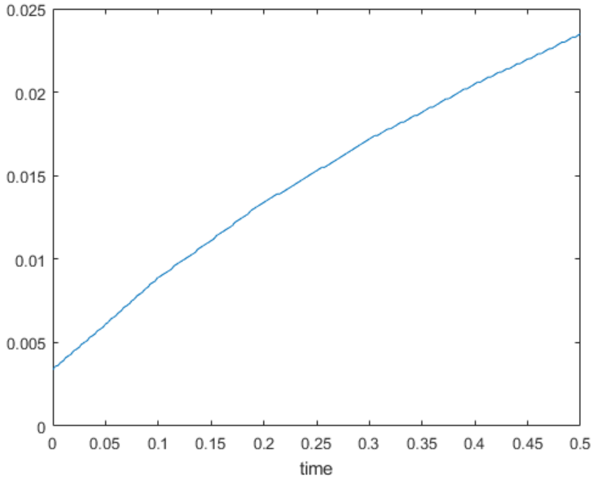

Figure 6: W_{ν} for the optimal vector $\bar{p}^{\varepsilon}_{\nu}$

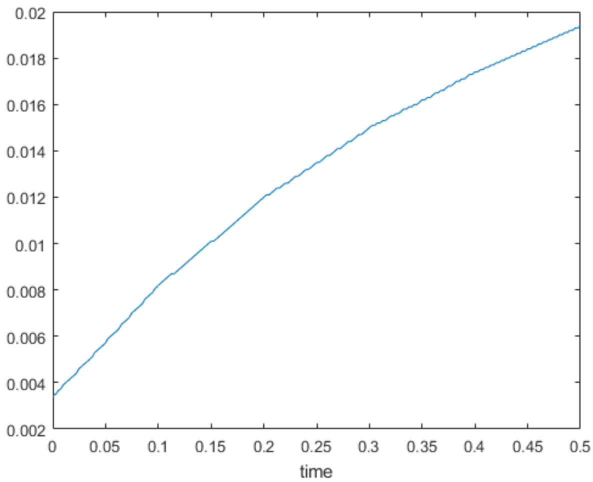

Figure 7: W_{σ} for the optimal vector $\bar{p}^{\varepsilon}_{\sigma}$

Remark 1 *Looking carefully on Tables from Appendix one can wonder why we do not choose strategies from Table* 1*, where we have better* R_0*, lesser costs and almost* 100% *of the population vaccinated. We would like to stress that we have the game with players of which four wants to maximize our functional, but eight of them wants to minimize the functional. Thus the player* ν *can not only to maximize the functional as opponent player* σ *wants to minimize the functional. The resulting strategies – approximate Nash equilibrium, are described* *in Table 6, where a kind of "compromise" between players is chosen and as a result mainly interesting us quantities R_0 and J are higher than in Table 1 and vaccinated population is smaller! We would like to stress that changing the choice of strategies for players ν and σ will change the resulting Nash equilibrium. The same influence for the Nash equilibrium has the choice of the functional.*

5. Interpretation of the results

We can draw the following conclusions based on the multiple calculations in Matlab. We are interested in minimizing basic reproduction number R_0, part of the population being hospitalized H, number of deaths F and costs of the pandemic C. On the other hand, we try to maximize part of the population having antibodies V.

Decreasing the strategies $v_1(t), v_2(t), \gamma_i(t), \gamma_r(t), s_1(t)$ or $s_2(t)$ causes smaller R_0. The strategies $s(t)$, $\kappa(t)$, $c(t)$, $c_1(t)$, $c_2(t)$ and $c_3(t)$ have no effect on R_0, because they are not included in the formula (10), which allow us computing the basic reproduction number.

The greater values $v_1(t), v_2(t), \gamma_r(t), s_1(t), \kappa(t), c(t), c_1(t), c_2(t), c_3(t)$ or the smaller values $\gamma_i(t)$ cause maximizing part of the population having antibodies. Changing the strategies $s(t)$ and $s_1(t)$ have no effect on V.

Decreasing strategies $v_1(t)$ or $\kappa(t)$ or increasing $\gamma_i(t)$ or $\gamma_r(t)$ causes decreasing hospitalized. All other strategies have no effect on H.

Only two strategies cause decreasing fatality cases. It happens so for greater values of $\gamma_i(t)$ or smaller values of $\kappa(t)$.

The pandemic costs are greater for greater values of strategies $v_2(t)$ or $\gamma_i(t)$ or for smaller values of $c(t)$, $c_1(t)$, $c_2(t)$ or $c_3(t)$.

We can enclose the above considerations in the simple Table 3.

Table 3: Conclusions

	$v_1(t)$	$v_2(t)$	$\gamma_i(t)$	$\gamma_r(t)$	$s(t)$	$s_1(t)$	$s_2(t)$	$\kappa(t)$	$c(t)$	$c_1(t)$	$c_2(t)$	$c_3(t)$
$R_0 \searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\times$	$\searrow$	$\searrow$	$\times$	$\times$	$\times$	$\times$	$\times$
$V \nearrow$	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\times$	$\nearrow$	$\times$	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$
$H \searrow$	$\searrow$	$\times$	$\nearrow$	$\nearrow$	$\times$	$\times$	$\times$	$\searrow$	$\times$	$\times$	$\times$	$\times$
$F \searrow$	$\times$	$\times$	$\nearrow$	$\times$	$\times$	$\times$	$\times$	$\searrow$	$\times$	$\times$	$\times$	$\times$
$C \searrow$	$\times$	$\nearrow$	$\nearrow$	$\times$	$\times$	$\times$	$\times$	$\times$	$\searrow$	$\searrow$	$\searrow$	$\searrow$

6. Conclusion

Decision-making, supported with knowledge, is particular important, especially when strong uncertainties appear. We present the game model of COVID-19, where players want to choose different strategies considering vaccination, costs of the pandemic and controls (parameters) depending on time. We construct for such a model a game with functional depending on basic reproduction number, costs of the pandemic and number of the population having antibodies at final and changing time. We build a non-cooperative differential game with two main players of which one disposes with four strategies and the second one with eight strategies. The first one looks for strategies to maximize the number of the population having antibodies, while the second one looks for the strategies to minimize the costs of the pandemic as well the basic reproduction number at final time. We found as a result approximate open-loop Nash equilibrium (dual) for our game and proved for it a verification theorem. Then, using this verification theorem we present a numerical algorithm allowing to calculate approximate open-loop Nash equilibrium. The approximate optimal dual open-loop strategies allow to make optimal decision at each step time of the pandemic assuming suitable initial conditions and fixed parameters for our model. We show in the example a quality of such an approach, which significantly differs from standard approach to pandemic problems. The cost functional is neither maximal with respect to controls nor minimal with respect to them. However using the game approach we find more realistic strategies in spite that they are again insight result, related to the value of costs $\int_0^T C(t)dt$ and $R_0(T)$ (are greater), $V(T)$ is lower.

Appendix

We have six tables with fours strategies for player u_ν and one of the eight fixed opponents' strategies $u^{\sim\nu}$. These results are in website https://wmii.uni.lodz.pl/~radmat/article/1.

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