BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 71(5), 2023, Article number: e145765
DOI: 10.24425/bpasts.2023.145765

CONTROL, INFORMATICS AND ROBOTICS

Eigenvalues assignment in descriptor linear systems by state and its derivative feedbacks

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Abstract. The eigenvalues assignment problems for descriptor linear systems with state and its derivative feedbacks are considered herein. Necessary and sufficient conditions for the existence of solutions to the problems are established. The Euler and Tustin approximations of the continuous-time systems are analyzed. Procedures for computation of the feedbacks are given and illustrated by numerical examples.

Key words: eigenvalue assignment; descriptor; linear system; state derivative; feedback.

1. INTRODUCTION

The descriptor linear systems (also referred to as singular systems) have been analyzed in many books and papers [1–6, 9–16]. By the use of the widely-known Weierstrass-Kronecker theorem, the analysis of the systems has been reduced to analysis of the standard linear systems [1–5, 12, 16]. The method based on the Drazin inverse matrices is also used to analyze this class of linear systems [4, 6]. The stability of positive descriptor linear systems has been analyzed in [15] and the stabilization of the positive descriptor fractional discrete-time linear systems with two different fractional orders by means of a decentralized controller has been investigated in [14]. To modify the dynamics of the descriptor linear systems, the state or output feedbacks are used [3, 12, 16].

In this paper, the state and its derivative feedbacks will be applied to descriptor linear systems with regular pencils in order to obtain closed-loop systems with the desired dynamics. Application of these feedbacks essentially enlarges the possibility of modification of dynamical properties in the descriptor linear systems.

The paper is organized as follows. In Section 2 the synthesis of the descriptor linear continuous-time systems with state and its derivative feedbacks and desired dynamics are considered. Similar problems for the descriptor linear systems obtained by means of Euler and Tustin approximations are analyzed in Section 4. Concluding remarks are given in Section 5.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n– the $n\times n$ identity matrix.

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Manuscript submitted 2022-05-17, revised 2022-12-20, initially accepted for publication 2023-04-12, published in October 2023.

2. CONTINUOUS-TIME LINEAR SYSTEMS

Let us consider the descriptor linear continuous-time system:

$$E\dot{x} = Ax + Bu, \tag{1}$$

where $x = x(t) \in \Re^n$ is the state vector, $u = u(t) \in \Re^m$ is the input vector and $E, A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$. It is assumed that $\det E = 0$ and:

$$\det[Es - A] \neq 0 \tag{2}$$

for some $s \in C$ (the field of complex numbers). If condition (2) is satisfied, then equation (1) has a solution.

The state and its time-derivative feedback:

$$u = v - K_1 u - K_2 \dot{x} \tag{3}$$

is applied to system (1) (Fig. 1), where $v = v(t) \in \Re^m$ is the new input. Equation (3) describes a PD controller.

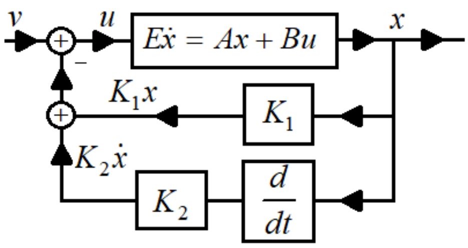

Fig. 1. The system with feedbacks

Substituting (3) into (1), we obtain:

$$E\dot{x} = (A - BK_1)x - BK_2\dot{x} + Bv \tag{4}$$

and

$$(E + BK_2)\dot{x} = (A - BK_1)x + Bv. \tag{5}$$

The matrix K_2 is chosen so that

$$\det[E+BK_2]\neq 0. \tag{6}$$

If condition (6) is satisfied, then from (5) we obtain:

$$\dot{x}=\bar{A}x+\bar{B}v \tag{7a}$$

where:

$$\bar{A}=[E+BK_2]^{-1}(A-BK_1), \qquad \bar{B}=[E+BK_2]^{-1}B. \tag{7b}$$

The problem being considered can be divided into the following two sub-problems.

Sub-problem 1. For the given matrices E, A, B of system (1), find the matrix K_2 such that condition (6) is satisfied.

Sub-problem 2. Knowing the matrices A, B and K_2, find the matrix K_1 such that the matrix $\bar{A}$ has the desired eigenvalues [3, 4, 6, 16].

3. SOLUTION OF THE SUB-PROBLEMS

Note that the matrix K_2 can be chosen so that condition (6) is satisfied if and only if:

$$\operatorname{rank}\begin{bmatrix}E & B\end{bmatrix}=\mathrm{n}. \tag{8}$$

The proof of (8) follows immediately from the equality below:

$$[E+BK_2]=\begin{bmatrix}E & B\end{bmatrix}\begin{bmatrix}I_n\\ K_2\end{bmatrix}. \tag{9}$$

Note that if condition (8) is satisfied, then the desired matrix K_2 can be computed from (9).

It is well-known [1–8, 13, 16] that the matrix K_1 can be chosen so that the matrix $\bar{A}$ has the desired eigenvalues if and only if the following pair:

$$\hat{A}=[E+BK_2]^{-1}A, \quad \hat{B}=[E+BK_2]^{-1}B \tag{10}$$

is completely controllable, i.e.:

$$\operatorname{rank}\begin{bmatrix}\hat{B} & \hat{A}\hat{B} & \dots & \hat{B}\hat{A}^{n-1}\end{bmatrix}=\mathrm{n}. \tag{11}$$

Therefore, the following theorem has been proved.

Theorem 1. Sub-problems 1 and 2 have solutions if and only if condition (8) is satisfied and the pair in (10) is completely controllable.

Example 1. Consider system (1) with the following matrices:

$$E=\begin{bmatrix}1 & -2\\ -2 & 4\end{bmatrix}, \quad A=\begin{bmatrix}1 & 0\\ 2 & -2\end{bmatrix}, \quad B=\begin{bmatrix}1\\ -1\end{bmatrix}. \tag{12}$$

System (1) with (12) satisfies conditions (2) and (8) since:

$$\det[Es-A]=\begin{vmatrix}s-1 & -2s\\ -2s-2 & 4s+2\end{vmatrix}=-6s-2\neq 0, \quad \text{for } s\neq-\frac{1}{3} \tag{13}$$

and

$$\operatorname{rank}\begin{bmatrix}E & B\end{bmatrix}=\operatorname{rank}\begin{bmatrix}1 & -2 & 1\\ -2 & 4 & -1\end{bmatrix}=2. \tag{14}$$

Note that for

$$K_2=\begin{bmatrix}1 & 1\end{bmatrix}, \tag{15}$$

the matrix

$$[E+BK_2]=\begin{bmatrix}1 & -2\\ -2 & 4\end{bmatrix}+\begin{bmatrix}1\\ -1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}=\begin{bmatrix}2 & -1\\ -3 & 3\end{bmatrix} \tag{16}$$

is nonsingular. In this case the matrices in (10) have the forms below:

$$\begin{aligned}\hat{A}&=[E+BK_2]^{-1}A\\ &=\begin{bmatrix}2 & -1\\ -3 & 3\end{bmatrix}^{-1}\begin{bmatrix}1 & 0\\ 2 & -2\end{bmatrix}=\frac{1}{3}\begin{bmatrix}5 & -2\\ 7 & -4\end{bmatrix},\\ \hat{B}&=[E+BK_2]^{-1}B\\ &=\begin{bmatrix}2 & -1\\ -3 & 3\end{bmatrix}^{-1}\begin{bmatrix}1\\ -1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}2\\ 1\end{bmatrix}.\end{aligned} \tag{17}$$

Note that the matrix $\hat{A}$ defined by (17) is unstable since the coefficients of the polynomial

$$\det\left[I_2s-\hat{A}\right]=\begin{vmatrix}s-\frac{5}{3} & \frac{2}{3}\\ -\frac{7}{3} & s+\frac{4}{3}\end{vmatrix}=s^2-\frac{1}{3}s-\frac{2}{3} \tag{18}$$

have different signs. The pair in (17) is completely controllable since

$$\operatorname{rank}\begin{bmatrix}\hat{B} & \hat{A}\hat{B}\end{bmatrix}=\operatorname{rank}\begin{bmatrix}\frac{2}{3} & \frac{8}{9}\\ \frac{1}{3} & \frac{10}{9}\end{bmatrix}=2. \tag{19}$$

Therefore, by Theorem 1, there exists the feedback matrix K_1 such that the closed-loop system with the matrix $\tilde{A}=\hat{A}-\hat{B}K_1$ is asymptotically stable. To find the desired matrix K_1, we may use one of the well-known eigenvalue assignment procedures [3, 8, 13, 16]. Let $s_1=-2$, $s_2=-3$ be the desired eigenvalues of the matrix $\tilde{A}$ with the characteristic polynomial:

$$\det\left[I_2s-\tilde{A}\right]=(s+2)(s+3)=s^2+5s+6, \tag{20}$$

then it easy to check that the desired matrix $K_1=\begin{bmatrix}7 & 2\end{bmatrix}$.

4. DISCRETE-TIME LINEAR SYSTEMS

4.1. Euler approximation

Consider the discrete-time linear systems obtained from (4) by means of the Euler approximation of the derivative of the state vector:

$$E\frac{x_{i+1}-x_i}{d} = Ax_i - B\left(K_2\frac{x_{i+1}-x_i}{d} + K_1x_i\right) + Bv_i\,, \qquad i = 0,1,\ldots \tag{21}$$

where $x_i = x(t_i)$, $v_i = v(t_i)$, and d is the discretization step. From (21) we obtain:

$$(E+BK_2)\,x_{i+1} = [E+(A-BK_1)\,d + BK_2]\,x_i + Bdv_i\,. \tag{22}$$

If

$$\det\,[E+BK_2] \neq 0 \tag{23}$$

then from (22) we obtain:

$$x_{i+1} = A_{dt}x_i + B_{dt}v_i \tag{24a}$$

where matrices A_{dt}, B_{dt} of the discrete-time linear system can be written as follows:

$$\begin{aligned} F_1 &= E+BK_2\,,\\ A_{dt} &= F_1^{-1}\,[F_1+(A-BK_1)d] = I_n + F_1^{-1}(A-BK_1)d,\\ B_{dt} &= [E+BK_2]^{-1}\,Bd = F_1^{-1}Bd. \end{aligned} \tag{24b}$$

The matrix K_2 can be chosen so that condition (23) is satisfied if and only if (9) holds.

Note that the matrix A_{dt} can be written in the following form:

$$A_{dt} = \hat{A}_{dt} - \hat{B}_{dt}K_1\,, \tag{25a}$$

where:

$$\hat{A}_{dt} = I_n - F_1^{-1}Ad, \quad \hat{B}_{dt} = F_1^{-1}Bd. \tag{25b}$$

The matrix K_1 can be chosen so that the matrix A_{dt} has the desired eigenvalues (is asymptotically stable) if and only if the pair $(\hat{A}_{dt},\ \hat{B}_{dt})$ is completely controllable (satisfies condition (11) [1–4, 8, 13, 16].

Therefore, the following theorem has been proved.

Theorem 2. The matrices K_2 and K_1 can be chosen so that the discrete-time system (24) has the desired asymptotically stable dynamics if and only if condition (23) is satisfied and the pair $(\hat{A}_{dt}, \hat{B}_{dt})$ is completely controllable.

The desired feedback matrices K_2 and K_1 for the asymptotically stable closed-loop system can be computed by the use of the following procedure.

Procedure 1.

Step 1. If condition (8) is satisfied, then find the matrix K_2 such that the matrix $F_1 = E+BK_2$ is nonsingular (condition (23) is satisfied).

Step 2. Knowing K_2, compute the matrices (25b) and check if the pair $(\hat{A}_{dt},\ \hat{B}_{dt})$ is completely controllable (satisfies condition (11)).

Step 3. Knowing the matrices (25b), compute the matrix K_1 such that the matrix A_{dt} defined by (25) has the desired asymptotically stable dynamics.

The procedure will be illustrated by the following simple example.

Example 2. Let us take under consideration a continuous-time linear system from Example 1 with the matrices from (12). Applying the Euler approximation for $d = 0.1$ to the system, we obtain the discrete-time system (24a). Find the desired feedback matrices K_2 and K_1.

Using Procedure 1, we obtain the following.

Step 1. Condition (8) is satisfied since:

$$\operatorname{rank}\begin{bmatrix} E & B \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -1 \end{bmatrix} = 2 \tag{26}$$

and for $K_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}$ we have:

$$\det F_1 = \det\,[E+BK_2] = \det\begin{bmatrix} 2 & -1 \\ -3 & 3 \end{bmatrix} = 3. \tag{27}$$

Step 2. Using (25b), we obtain:

$$\begin{aligned} \hat{A}_{dt} &= I_n - F_1^{-1}Ad = \begin{bmatrix} 1.167 & -0.067 \\ 0.233 & 0.867 \end{bmatrix},\\ \hat{B}_{dt} &= F_1^{-1}Bd = \begin{bmatrix} 0.067 \\ 0.033 \end{bmatrix}. \end{aligned} \tag{28}$$

The pair is completely controllable since:

$$\operatorname{rank}\begin{bmatrix} \hat{B}_{dt} & \hat{A}_{dt}\hat{B}_{dt} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0.067 & 0.076 \\ 0.033 & 0.044 \end{bmatrix} = 2. \tag{29}$$

Step 3. For the matrix A_{dt} with the eigenvalues of $s_1 = -2$, $s_2 = -3$, using the Ackerman method, the matrix K_1 has the following form:

$$K_1 = \begin{bmatrix} -742.25 & 1695.5 \end{bmatrix}. \tag{30}$$

4.2. Tustin approximation

Similarly to Section 4.1, consider the discrete-time system obtained from (4) by the Tustin approximation of the derivative of the state vector:

$$E\frac{x_{i+1}-x_i}{d} = A\frac{x_{i+1}+x_i}{2} + B(K_2\frac{x_{i+1}-x_i}{d} + K_1x_i) + Bv_i, \quad i = 0,1,\ldots \tag{31}$$

From (31) we have:

$$\begin{aligned}(2E-dA-2BK_2)x_{i+1}\\ =(2E+dA+2BdK_1-2BK_2)x_i+2Bdv_i.\end{aligned} \tag{32}$$

If

$$\det[2E-dA-2BK_2]\neq 0 \tag{33}$$

then from (32) we obtain:

$$x_{i+1}=\tilde{A}_{dt}x_i+\tilde{B}_{dt}v_i \tag{34a}$$

where:

$$\begin{aligned}F_2&=2E-dA-2BK_2,\\ \tilde{A}_{dt}&=I_n+2F_2^{-1}(A+BK_1)d,\\ \tilde{B}_{dt}&=2F_2^{-1}Bd.\end{aligned} \tag{34b}$$

The matrix K_2 can be chosen so that condition (33) is satisfied if and only if:

$$\operatorname{rank}\begin{bmatrix}2E-dA & -2B\end{bmatrix}=n. \tag{35}$$

Note that if condition (35) is satisfied, then:

$$2E-dA-2BK_2=\begin{bmatrix}2E-dA & -2B\end{bmatrix}\begin{bmatrix}I_n\\ K_2\end{bmatrix} \tag{36}$$

and there exists the matrix K_2 such that (33) holds.

Remark 1. From comparison of (23) and (33), it follows that condition (33) is less restrictive than condition (23).

The matrix $\tilde{A}_{dt}$ can be written in the form presented below:

$$\tilde{A}_{dt}=\hat{A}_{dt}-\hat{B}_{dt}K_1, \tag{37a}$$

where:

$$\hat{A}_{dt}=I_n+2F_2^{-1}Ad,\quad \hat{B}_{dt}=-2F_2^{-1}Bd. \tag{37b}$$

Note that the matrix K_1 can be chosen so that the matrix $\tilde{A}_{dt}$ has the desired eigenvalues if and only if the pair $(\hat{A}_{dt},\hat{B}_{dt})$ is completely controllable [1–6, 8, 13, 16]. Therefore, the following theorem has been proved.

Theorem 3. The matrix K_2 can be chosen so that condition (33) is satisfied and the matrix K_1 so that the matrix $\tilde{A}_{dt}$ has the desired eigenvalues if and only if the pair $(\hat{A}_{dt},\hat{B}_{dt})$ is completely controllable.

The desired matrices K_2 and K_1 for the asymptotically stable closed-loop system can be computed by the use of the following procedure.

Procedure 2.

Step 1. Check condition (35). If it is satisfied, chose the matrix K_2 such that $\det F_2\neq 0$.

Step 2. Compute the matrices $\hat{A}_{dt}$, $\hat{B}_{dt}$ defined by (37b) and check the controllability of the pair $(\hat{A}_{dt},\hat{B}_{dt})$.

Step 3. Compute the matrix K_1 such that the matrix $\tilde{A}_{dt}$ has the desired eigenvalues (is asymptotically stable).

The procedure will be illustrated by the following simple example.

Example 3. Similarly to Example 2, consider the continuous-time linear system with the matrices from (12). Applying the Tustin approximation for $d=0.1$ to the system, we obtain the discrete-time system (34a). Find the desired feedback matrices K_2 and K_1.

Using Procedure 2, we obtain:

Step 1. Condition (35) is satisfied since:

$$\operatorname{rank}\begin{bmatrix}2E-dA & 2B\end{bmatrix}=\operatorname{rank}\begin{bmatrix}1.9 & -4 & 2\\ -4.2 & 8.2 & -2\end{bmatrix}=2. \tag{38}$$

For $K_2=\begin{bmatrix}1 & 1\end{bmatrix}$ we have:

$$\begin{aligned}\det F_2&=\det[2E-dA-2BK_2]\\ &=\det\begin{bmatrix}-0.1 & -6\\ -2.2 & 10.2\end{bmatrix}=-14.22.\end{aligned} \tag{39}$$

Step 2. In this case using (37b) and (39), we obtain:

$$\begin{aligned}\hat{A}_{dt}&=I_n+2F_2^{-1}Ad=\begin{bmatrix}0.688 & 0.169\\ -0.281 & 0.997\end{bmatrix},\\ \hat{B}_{dt}&=-2F_2^{-1}Bd=\begin{bmatrix}0.059\\ 0.032\end{bmatrix}.\end{aligned} \tag{40}$$

The pair in (40) is controllable since:

$$\operatorname{rank}\begin{bmatrix}\hat{B}_{dt} & \hat{A}_{dt}\hat{B}_{dt}\end{bmatrix}=\operatorname{rank}\begin{bmatrix}0.059 & 0.046\\ 0.032 & 0.031\end{bmatrix}=2. \tag{41}$$

Step 3. Let the desired eigenvalues of the matrix $\tilde{A}_{dt}$ be $s_1=-2$, $s_2=-3$. In this case, using one of the well-known eigenvalue procedures, we obtain:

$$K_1=\begin{bmatrix}-1047.77 & 2119.78\end{bmatrix}. \tag{42}$$

From comparison of (4) and (24a), it follows that the Tustin approach gives better approximation of the continuous-time system than the Euler approach.

5. CONCLUSIONS

The eigenvalue assignment problems for descriptor linear systems with state and its derivative feedbacks have been analyzed. Necessary and sufficient conditions for the existence of solutions to the problems have been established (Theorems 1, 2, 3). The Euler and Tustin approximations of the continuous-time systems have been investigated. Procedures for computation of the feedbacks have been given and illustrated by numerical examples. It has been shown that the application of the feedbacks of the state vector and its derivative essentially enlarge the possibility of modifications of the dynamical properties of the descriptor linear systems. This approach can be applied, for example, to analysis of the dynamical properties of descriptor linear electrical circuits. The considerations can be further extended to descriptor linear fractional order systems.

ACKNOWLEDGEMENTS

This work was carried out within the framework of study No. WZ/WE – IA/6/2020 and financed from the funds earmarked for research by the Polish Ministry of Education and Science.

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