Central European Journal of Economic Modelling and Econometrics

Type II Exponentiated Half-Logistic-Gompertz Topp-Leone-G Family of Distributions with Applications

Broderick Oluyede* and Thatayaone Moakofi†

Submitted: 15.02.2022, Accepted: 30.10.2022

Abstract

The purpose of this paper is to introduce and study a new generated family of distributions based on the type II transformation which is called the type II exponentiated half-logistic-Gompertz-Topp-Leone-G (TIIEHL-Gom-TL-G) family of distributions. We investigate its general mathematical properties, including, hazard rate function, quantile function, moments, moment generating function, Rényi entropy and order statistics. Parameter estimates of the new family of distributions are obtained based on the maximum likelihood estimation method and their performance is evaluated via a simulation study. For illustration of the applicability of the new family of distributions, four real data sets are analyzed.

Keywords: exponentiated half-logistic, Topp-Leone distribution, Gompertz distribution, maximum likelihood estimation

JEL Classification: C2, C4, C6, C8

*Department of Mathematics and Statistical Sciences, Botswana International University of Science and Technology, Palapye, Botswana; e-mail: oluyedeo@biust.ac.bw; ORCID: 0000-0002-9945-2255

†Department of Mathematics and Statistical Sciences, Botswana International University of Science and Technology, Palapye, Botswana; e-mail: thatayaone.moakofi@studentmail.biust.ac.bw; ORCID: 0000-0002-2676-7694

1 Introduction

Statistical distributions available in the literature often do not adequately describe and model most of the interesting data sets. This motivated researchers to introduce generated families of distributions as an improvement to the usual standard distributions. Several generated families of distributions developed include: the odd power generalized Weibull-G family of distributions by Moakofi et al. (2021b), sine Topp-Leone-G family of distributions by Al-Babtain et al. (2020), the Marshall-Olkin Topp-Leone-G family of distributions by Khaleel et al. (2020), the Zubair-G family of distributions by Ahmad (2020), Topp-Leone odd Fréchet generated family of distributions by Al-Marzouki et al. (2020), a new modified Lehmann type II-G class of distributions by Balogun et al. (2021), the alpha power Marshall-Olkin-G family of distributions by Eghwerido et al. (2021), and a new Kumaraswamy generalized family of distributions by Tahir et al. (2020).
The type II transformations have been applied to several distributions, leading to generalized families of distributions, including type II exponentiated half-logistic-Topp-Leone-G power series class of distributions by Moakofi et al. (2021a), type II Kumaraswamy half-logistic-G family of distributions by El-Sherpieny and Elsehetry (2019), type II general exponential class of distributions by Hamedani et al. (2019), type II Topp-Leone generated family of distributions by Elgarhy et al. (2018) and type II half logistic-G family of distributions by Soliman et al. (2017).
The basic motivations for developing the TIIEHL-Gom-TL-G family of distributions in practice include the following:

i) to produce skewness for symmetrical models;

ii) to define special models with different shapes of hazard rate function;

iii) to construct heavy-tailed distributions for modeling real data;

iv) to provide consistently better fits than other generalized distributions with the same underlying model;

v) to generalize some existing models in the literature.

The rest of the paper is organized as follows. Section 2 contains the new proposed TIIEHL-Gom-TL-G family of distributions and its hazard rate function, sub-families, quantile function and series representation of the density function. In Section 3, probability weighted moments, moments and generating function are derived. In Section 4, Rényi entropy and the distribution of order statistics are presented. In Section 5, the maximum likelihood procedure is used for estimation of the parameters of the TIIEHL-Gom-TL-G family of distributions followed by some special cases. A Monte Carlo simulation study is used to examine the bias and mean square error of the maximum likelihood estimators in Section 6. Four applications to real data sets are given in Section 7, followed by some concluding remarks in Section 8.

2 The model, sub-families, hazard Rate and quantile functions

In this section, we derive the new type II exponentiated half-logistic-Gompertz-Topp-Leone-G (TIIEHL-Gom-TL-G) family of distributions and some of the statistical properties including sub-families, expansion of the density, hazard rate function, and quantile function.

2.1 The model

Al-Shomrani et al. (2016) proposed the Topp-Leone-G (TL-G) family of distributions with the cumulative distribution function (cdf) and probability density function (pdf) given by

$$F_{TL-G}(x;b,\boldsymbol{\xi}) = \left[1-\overline{G}^2(x;\boldsymbol{\xi})\right]^b,$$

and

$$f_{TL-G}(x;b,\boldsymbol{\xi}) = 2b\left[1-\overline{G}^2(x;\boldsymbol{\xi})\right]^{b-1}\overline{G}(x;\boldsymbol{\xi})g(x;\boldsymbol{\xi}),$$

respectively, for $b>0$ and baseline parameter vector $\boldsymbol{\xi}$.
Alizadeh et al. (2017) constructed the Gompertz-G family of distributions with the cdf and pdf given by

$$F_{GOM-G}(x;\theta,\gamma,\boldsymbol{\xi}) = 1-\exp\left(\frac{\theta}{\gamma}\left(1-[1-G(x;\boldsymbol{\xi})]^{-\gamma}\right)\right),$$

and

$$f_{GOM-G}(x;\theta,\gamma,\boldsymbol{\xi}) = \theta\,[1-G(x;\boldsymbol{\xi})]^{-\gamma-1}\exp\left(\frac{\theta}{\gamma}\left(1-[1-G(x;\boldsymbol{\xi})]^{-\gamma}\right)\right)g(x;\boldsymbol{\xi}),$$

respectively, for $\gamma>0$ and baseline parameter vector $\boldsymbol{\xi}$. In this paper, we let $\theta=1$.
Al-Mofleh et al. (2020) introduced the type II exponentiated half-logistic-G (TIIEHL-G) family of distributions with the cdf given by

$$F_{TIIEHL-G}(x;a,\lambda,\boldsymbol{\xi}) = 1-\left[\frac{1-G^{\lambda}(x;\boldsymbol{\xi})}{1+G^{\lambda}(x;\boldsymbol{\xi})}\right]^a,$$

for $a,\lambda>0$ and baseline parameter vector $\boldsymbol{\xi}$. In this paper the parameter λ is taken to be equal to 1.
Using the above families of distributions, the cdf and pdf of the proposed TIIEHL-Gom-TL-G family of distributions are given by

$$F(x;\alpha,\gamma,b,\boldsymbol{\xi}) = 1-\left[\frac{\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)}{1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\right)}\right]^{\alpha} \quad (1)$$

and

$$f(x;\alpha,\gamma,b,\boldsymbol{\xi}) = = 4\alpha b\left(1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\right)\right)^{-(\alpha+1)}\times \times\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma-1}\exp\left(\frac{\alpha}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\times \times\left[1-\overline{G}^2(x;\boldsymbol{\xi})\right]^{b-1}\overline{G}(x;\boldsymbol{\xi})g(x;\boldsymbol{\xi}), \tag{2}$$

respectively, for $\alpha,\gamma,b>0$ and baseline parameter vector $\boldsymbol{\xi}$.

2.2 Hazard rate and quantile functions

In this section, we present the hazard rate function and quantile function of the TIIEHL-Gom-TL-G family of distributions. The hazard rate function of the TIIEHL-Gom-TL-G family of distributions is given by

$$h_F(x;\alpha,\gamma,b,\boldsymbol{\xi}) = 4\alpha b\left(1+\left(\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\right)\right)^{-1}\times \times\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma-1}\left[1-\overline{G}^2(x;\boldsymbol{\xi})\right]^{b-1}\overline{G}(x;\boldsymbol{\xi})g(x;\boldsymbol{\xi}).$$

The quantile function of the TIIEHL-Gom-TL-G family of distributions is obtained by solving the non-linear equation:

$$F(x;\alpha,\gamma,b,\boldsymbol{\xi}) = 1-\left[\frac{\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)}{1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\right)}\right]^{\alpha} = u, \tag{3}$$

for $0\le u\le 1$, that is,

$$G(x;\boldsymbol{\xi}) = 1-\left[1-\left(1-\left[1-\gamma\log\left(\frac{2\,(1-u)^{\frac{1}{\alpha}}}{1+(1-u)^{\frac{1}{\alpha}}}\right)\right]^{-1/\gamma}\right)^{1/b}\right]^{1/2}.$$

Consequently, the quantile function for the TIIEHL-Gom-TL-G family of distributions is given by

$$Q_G(u;\alpha,\gamma,b,\boldsymbol{\xi}) = G^{-1}\left[1-\left[1-\left(1-\left[1-\gamma\log\left(\frac{2\,(1-u)^{\frac{1}{\alpha}}}{1+(1-u)^{\frac{1}{\alpha}}}\right)\right]^{-1/\gamma}\right)^{1/b}\right]^{1/2}\right]. \tag{4}$$

It follows therefore that random numbers can be generated from the TIIEHL-Gom-TL-G family of distributions based on Equation (4), for specified baseline cdf G.

2.3 Sub-families

In this subsection, some sub-families of the TIIEHL-Gom-TL-G family of distributions are presented.

i) When $\alpha = 1$, we obtain the type II half-logistic-Gompertz-Topp-Leone-G (TIIHL-Gom-TL-G) family of distributions with the cdf

$$F(x;\gamma,b,\boldsymbol{\xi}) = 1-\left[\frac{\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)}{1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\right)}\right] \tag{5}$$

for $\gamma, b > 0$, and baseline parameter vector $\boldsymbol{\xi}$.

ii) If $b = 1$, we obtain a new family of distributions with the cdf

$$F(x;\alpha,\gamma,\boldsymbol{\xi}) = 1-\left[\frac{\exp\left(\frac{1}{\gamma}\left(1-\left[\overline{G}^2(x;\boldsymbol{\xi})\right]^{-\gamma}\right)\right)}{1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[\overline{G}^2(x;\boldsymbol{\xi})\right]^{-\gamma}\right)\right)\right)}\right]^{\alpha} \tag{6}$$

for $\alpha, \gamma > 0$, and baseline parameter vector $\boldsymbol{\xi}$.

iii) If $\gamma = 1$, we obtain a new family of distributions with the cdf

$$F(x;\alpha,b,\boldsymbol{\xi}) = 1-\left[\frac{\exp\left(\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-1}\right)\right)}{1+\left(1-\exp\left(\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-1}\right)\right)\right)}\right]^{\alpha} \tag{7}$$

for $\alpha, b > 0$, and baseline parameter vector $\boldsymbol{\xi}$.

iv) If $\alpha = b = 1$, we obtain a new family of distributions with the cdf

$$F(x;\gamma,\boldsymbol{\xi}) = 1 - \left[\frac{\exp\left(\frac{1}{\gamma}\left(1-\left[\overline{G}^2(x;\boldsymbol{\xi})\right]^{-\gamma}\right)\right)}{1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[\overline{G}^2(x;\boldsymbol{\xi})\right]^{-\gamma}\right)\right)\right)}\right] \tag{8}$$

for $\gamma > 0$, and baseline parameter vector $\boldsymbol{\xi}$.

v) When $\alpha = \gamma = 1$, we obtain a new family of distributions with the cdf

$$F(x;b,\boldsymbol{\xi}) = 1 - \left[\frac{\exp\left(\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-1}\right)\right)}{1+\left(1-\exp\left(\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-1}\right)\right)\right)}\right] \tag{9}$$

for $b > 0$, and baseline parameter vector $\boldsymbol{\xi}$.

vi) If $\alpha = \gamma = b = 1$, we obtain a new family of distributions with the cdf

$$F(x;\boldsymbol{\xi}) = 1 - \left[\frac{\exp\left(\left(1-\left[\overline{G}^2(x;\boldsymbol{\xi})\right]^{-1}\right)\right)}{1+\left(1-\exp\left(\left(1-\left[\overline{G}^2(x;\boldsymbol{\xi})\right]^{-1}\right)\right)\right)}\right] \tag{10}$$

for baseline parameter vector $\boldsymbol{\xi}$.

2.4 Series representation

In this section, we present the series expansion of the TIIEHL-Gom-TL-G density function. Using the generalized binomial and Taylor series expansions given by

$$(1+z)^{-\beta} = \sum_{k=0}^{\infty}(-1)^k\binom{\beta+k-1}{k}z^k, \quad \text{for} \quad |z|<1, \quad \text{and} \quad e^z = \sum_{i=0}^{\infty}\frac{z^i}{i!},$$

we have

$$f(x;\alpha,\gamma,b,\boldsymbol{\xi}) = 4\alpha b\left(1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\right)\right)^{-(\alpha+1)}\times$$
$$\times\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma-1}\left[1-\overline{G}^2(x;\boldsymbol{\xi})\right]^{b-1}\overline{G}(x;\boldsymbol{\xi})g(x;\boldsymbol{\xi})\times$$

$$\times \exp\left(\frac{\alpha}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)=$$

$$= 4\alpha b \sum_{k,s=0}^{\infty} \binom{\alpha+k}{k}\binom{k}{s}(-1)^{k+s}\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma-1}\times$$
$$\times \exp\left(\frac{s+\alpha}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\overline{G}(x;\boldsymbol{\xi})\times$$
$$\times\left[1-\overline{G}^2(x;\boldsymbol{\xi})\right]^{b-1} g(x;\boldsymbol{\xi})=$$

$$= 4\alpha b \sum_{k,s,i=0}^{\infty} \binom{\alpha+k}{k}\binom{k}{s}(-1)^{k+s}\left(\frac{s+\alpha}{\gamma}\right)^i\times$$
$$\times\frac{\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)^i}{i!}\overline{G}(x;\boldsymbol{\xi})g(x;\boldsymbol{\xi})\times$$
$$\times\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma-1}\left[1-\overline{G}^2(x;\boldsymbol{\xi})\right]^{b-1}=$$

$$= 4\alpha b \sum_{k,i,s,p=0}^{\infty} \binom{\alpha+k}{k}\binom{k}{s}\binom{i}{p}\frac{(-1)^{k+s+p}\left(\frac{k+\alpha}{\gamma}\right)^i}{i!}\left[1-\overline{G}^2(x;\boldsymbol{\xi})\right]^{b-1}\times$$
$$\times\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma(p+1)-1}\overline{G}(x;\boldsymbol{\xi})g(x;\boldsymbol{\xi})=$$

$$= 4\alpha b \sum_{k,i,s,p,q=0}^{\infty} \binom{\alpha+k}{k}\binom{k}{s}\binom{i}{p}\binom{\gamma(p+1)+q}{q}\times$$
$$\times\frac{(-1)^{k+s+p+q}\left(\frac{k+\alpha}{\gamma}\right)^i}{i!}\left[1-\overline{G}^2(x;\boldsymbol{\xi})\right]^{b(q+1)-1}\overline{G}(x;\boldsymbol{\xi})g(x;\boldsymbol{\xi})=$$

$$= 4\alpha b \sum_{k,i,s,p,q,w=0}^{\infty} \binom{\alpha+k}{k}\binom{k}{s}\binom{i}{p}\binom{\gamma(p+1)+q}{q}\binom{b(q+1)-1}{w}\times$$
$$\times\frac{(-1)^{k+s+p+q+w}\left(\frac{k+\alpha}{\gamma}\right)^i}{i!}\overline{G}^{2w+1}(x;\boldsymbol{\xi})g(x;\boldsymbol{\xi})=$$

$$= 4\alpha b \sum_{k,i,s,p,q,w,u=0}^{\infty} \binom{\alpha+k}{k}\binom{k}{s}\binom{i}{p}\binom{\gamma(p+1)+q}{q}\binom{b(q+1)-1}{w} \times$$

$$\times \binom{2w+1}{u} \frac{(-1)^{k+s+p+q+w+u}\left(\frac{k+\alpha}{\gamma}\right)^{i}}{i!} G^{u}(x;\boldsymbol{\xi})g(x;\boldsymbol{\xi}) =$$

$$= \sum_{u=0}^{\infty} b_{u+1} g_{u+1}(x;\boldsymbol{\xi}), \tag{11}$$

where $g_{u+1}(x;\boldsymbol{\xi}) = (u+1)[G(x;\boldsymbol{\xi})]^{u} g(x;\boldsymbol{\xi})$ is the exponentiated-G (exp-G) pdf with the power parameter $(u+1)$ and baseline parameter vector $\boldsymbol{\xi}$, and

$$b_{u+1} = \sum_{k,i,s,p,q,w=0}^{\infty} \binom{\alpha+k}{k}\binom{k}{s}\binom{i}{p}\binom{\gamma(p+1)+q}{q}\binom{b(q+1)-1}{w}\binom{2w+1}{u} \times$$

$$\times \frac{(-1)^{k+s+p+q+w+u}\left(\frac{k+\alpha}{\gamma}\right)^{i}}{i!} \left(\frac{4\alpha b}{w+1}\right). \tag{12}$$

Consequently, the mathematical and statistical properties of the TIIEHL-Gom-TL-G family of distributions follow directly from those of the exponentiated-G (exp-G) family of distributions.

3 Probability weighted moments and generating function

In this section, probability weighted moments (PWMs), moments and moment generating function for the TIIEHL-Gom-TL-G family of distributions are presented.

3.1 Probability weighted moments (PWMs)

The primary use of probability weighted moments is in the estimation of parameters for a probability distribution. They are sometimes used when maximum likelihood estimates are unavailable or difficult to compute. The PWMs of a random variable X is defined by

$$\eta_{m,r} = E\left(X^{m}\left(F(X)^{r}\right)\right) = \int_{-\infty}^{\infty} x^{m}\left(F(x)\right)^{r} f(x)dx.$$

We note that

$$F(x)^r f(x) = \left(1 - \left[\frac{\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)}{1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\right)}\right]^{\alpha}\right)^r \times$$
$$\times 4\alpha b\left(1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\right)\right)^{-(\alpha+1)} \times$$
$$\times \left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma-1} \exp\left(\frac{\alpha}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right) \times$$
$$\times \left[1-\overline{G}^2(x;\boldsymbol{\xi})\right]^{b-1}\overline{G}(x;\boldsymbol{\xi})g(x;\boldsymbol{\xi}) =$$

$$= \sum_{z=0}^{\infty}\binom{r}{z}(-1)^z 4\alpha b \exp\left(\frac{\alpha(z+1)}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right) \times$$
$$\times \left(1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\right)\right)^{-(\alpha(z+1)+1)} \times$$
$$\times \left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma-1}\left[1-\overline{G}^2(x;\boldsymbol{\xi})\right]^{b-1}\overline{G}(x;\boldsymbol{\xi})g(x;\boldsymbol{\xi}).$$

Now following the same steps leading to Equation (11), we have

$$F(x)^r f(x) = \sum_{u=0}^{\infty} C_{u+1} g_{u+1}(x;\boldsymbol{\xi}),$$

where $g_{u+1}(x;\boldsymbol{\xi}) = (u+1)[G(x;\boldsymbol{\xi})]^u g(x;\boldsymbol{\xi})$ is the exp-G pdf with the power parameter $(u+1)$ and baseline parameter vector $\boldsymbol{\xi}$, and

$$C_{u+1} = \sum_{z,k,i,s,p,q,w=0}^{\infty}\binom{r}{z}\binom{\alpha(z+1)+k}{k}\binom{k}{s}\binom{i}{p}\binom{\gamma(p+1)+q}{q}\binom{b(q+1)-1}{w} \times$$
$$\times \binom{2w+1}{u}\frac{(-1)^{k+s+p+q+w+u}\left(\frac{k+\alpha}{\gamma}\right)^i}{i!}\left(\frac{4\alpha b}{w+1}\right). \quad (13)$$

Thus, the PWMs of TIIEHL-Gom-TL-G family of distributions is given by

$$\eta_{m,r} = E(X^m F(X)^r) = \sum_{u=0}^{\infty} C_{u+1} \int_{-\infty}^{\infty} x^m g_{u+1}(x; \boldsymbol{\xi}) dx.$$

3.2 Moments and generating function

Let $Y_{u+1} \sim Exponentiated\!-\!G(u+1, \boldsymbol{\xi})$, then the η^{th} raw moment, μ'_η of the TIIEHL-Gom-TL-G family of distributions is given by:

$$\mu'_\eta = E(X^\eta) = \int_{-\infty}^{\infty} x^\eta f(x) dx = \sum_{u=0}^{\infty} b_{u+1} E(Y_{u+1}^\eta),$$

where $E(Y_{u+1}^\eta)$ is the η^{th} moment of Y_{u+1} and b_{u+1} is given by Equation (12). The moment generating function (MGF), for $|t| < 1$, is given by:

$$M_X(t) = \sum_{u=0}^{\infty} b_{u+1} M_{u+1}(t),$$

where $M_{u+1}(t)$ is the mgf of Y_{u+1} and b_{u+1} is given by Equation (12).

4 Rényi entropy and order statistics

Order statistics and entropy play important roles in probability and statistics, particularly in reliability, lifetime data analysis and information theory. In this section, we present Rényi entropy and the distribution of the r^{th} order statistic for the TIIEHL-Gom-TL-G family of distributions.

4.1 Rényi entropy

Rényi entropy (Rényi (1960)) is an extension of Shannon entropy. Its importance is seen in ecology and statistics as indices of diversity, uncertainty or randomness of a system. Rényi entropy of the TIIEHL-Gom-TL-G family of distributions is defined to be

$$I_R(v) = \frac{1}{1-v} \log\left(\int_0^{\infty} [f(x; \alpha, \gamma, b, \boldsymbol{\xi})]^v dx \right), v \neq 1, v > 0. \qquad (14)$$

Let the TIIEHL-Gom-TL-G pdf $f(x; \alpha, \gamma, b, \boldsymbol{\xi})$ be written as $f(x)$, then

$$[f(x)]^v = (4\alpha b)^v \left[1 - \overline{G}^2(x; \boldsymbol{\xi})\right]^{vb-v} \left[1 - \left(1 - \overline{G}^2(x; \boldsymbol{\xi})\right)^b\right]^{-v\gamma - v} \overline{G}^v(x; \boldsymbol{\xi}) g^v(x; \boldsymbol{\xi}) \times$$
$$\times \exp\left(\frac{v\alpha}{\gamma} \left(1 - \left[1 - \left(1 - \overline{G}^2(x; \boldsymbol{\xi})\right)^b \right]^{-\gamma} \right) \right) \times$$

$$\times\left(1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^{2}(x;\boldsymbol{\xi})\right)^{b}\right]^{-\gamma}\right)\right)\right)\right)^{-v(\alpha+1)}=$$

$$=\sum_{k,s=0}^{\infty}\binom{v(\alpha+1)+k-1}{k}\binom{k}{s}(-1)^{k+s}(4\alpha b)^{v}\left[1-\left(1-\overline{G}^{2}(x;\boldsymbol{\xi})\right)^{b}\right]^{-v\gamma-v}\times$$
$$\times\exp\left(\frac{s+v\alpha}{\gamma}\left(1-\left[1-\left(1-\overline{G}^{2}(x;\boldsymbol{\xi})\right)^{b}\right]^{-\gamma}\right)\right)\times$$
$$\times\left[1-\overline{G}^{2}(x;\boldsymbol{\xi})\right]^{vb-v}\overline{G}^{v}(x;\boldsymbol{\xi})g^{v}(x;\boldsymbol{\xi})\;=$$

$$=\sum_{k,s,i=0}^{\infty}\binom{v(\alpha+1)+k-1}{k}\binom{k}{s}(-1)^{k+s}(4\alpha b)^{v}\left[1-\left(1-\overline{G}^{2}(x;\boldsymbol{\xi})\right)^{b}\right]^{-v\gamma-v}\times$$
$$\times\frac{\left(\frac{s+v\alpha}{\gamma}\right)^{i}\left(1-\left[1-\left(1-\overline{G}^{2}(x;\boldsymbol{\xi})\right)^{b}\right]^{-\gamma}\right)^{i}}{i!}\left[1-\overline{G}^{2}(x;\boldsymbol{\xi})\right]^{vb-v}\times$$
$$\times\overline{G}^{v}(x;\boldsymbol{\xi})g^{v}(x;\boldsymbol{\xi})=$$

$$=\sum_{k,s,i,p=0}^{\infty}\binom{v(\alpha+1)+k-1}{k}\binom{k}{s}\binom{i}{p}(-1)^{k+s+p}(4\alpha b)^{v}\overline{G}^{v}(x;\boldsymbol{\xi})g^{v}(x;\boldsymbol{\xi})\times$$
$$\times\left[1-\left(1-\overline{G}^{2}(x;\boldsymbol{\xi})\right)^{b}\right]^{-\gamma(v+p)-v}\frac{\left(\frac{s+v\alpha}{\gamma}\right)^{i}}{i!}\left[1-\overline{G}^{2}(x;\boldsymbol{\xi})\right]^{vb-v}=$$

$$=\sum_{k,s,i,p,q,w=0}^{\infty}\binom{v(\alpha+1)+k-1}{k}\binom{k}{s}\binom{i}{p}\binom{\gamma(v+p)+v+q-1}{q}\times$$
$$\times\binom{b(v+q)-v}{w}(-1)^{k+p+q+w}(4\alpha b)^{v}\overline{G}^{2w+v}(x;\boldsymbol{\xi})g^{v}(x;\boldsymbol{\xi})\frac{\left(\frac{s+v\alpha}{\gamma}\right)^{i}}{i!}=$$

$$=\sum_{k,s,i,p,q,w,u=0}^{\infty}\binom{v(\alpha+1)+k-1}{k}\binom{k}{s}\binom{i}{p}\binom{\gamma(v+p)+v+q-1}{q}\times$$
$$\times\binom{b(v+q)-v}{w}\binom{2w+v}{u}(-1)^{k+s+p+q+w+u}(4\alpha b)^{v}\times$$

$$\times G^{u}(x;\boldsymbol{\xi})g^{v}(x;\boldsymbol{\xi})\frac{\left(\frac{s+v\alpha}{\gamma}\right)^{i}}{i!}.$$

Now,

$$\int_{0}^{\infty}(f(x))^{v}dx=\sum_{k,s,i,p,q,w,u=0}^{\infty}\binom{v(\alpha+1)+k-1}{k}\binom{i}{p}\binom{k}{s}\binom{\gamma(v+p)+v+q-1}{q}\times$$
$$\times\binom{b(v+q)-v}{w}\binom{2w+v}{u}(-1)^{k+s+p+q+w+u}(4\alpha b)^{v}\frac{\left(\frac{s+v\alpha}{\gamma}\right)^{i}}{i!}\times$$
$$\times\int_{0}^{\infty}G^{w}(x;\boldsymbol{\xi})g^{v}(x;\boldsymbol{\xi})dx.$$

Consequently, Rényi entropy for the TIIEHL-Gom-TL-G family of distributions is given by

$$I_{R}(v)=\frac{1}{1-v}\log\left[\sum_{k,s,i,p,q,w,u=0}^{\infty}\binom{v(\alpha+1)+k-1}{k}\binom{i}{p}\binom{k}{s}\times\right.$$
$$\times\binom{\gamma(v+p)+v+q-1}{q}\binom{b(v+q)-v}{w}\binom{2w+v}{u}(-1)^{k+s+p+q+w+u}\times$$
$$\left.\times\frac{\left(\frac{s+v\alpha}{\gamma}\right)^{i}}{i!}\frac{(4\alpha b)^{v}}{\left[1+\frac{u}{v}\right]^{v}}\int_{0}^{\infty}\left(\left[1+\frac{u}{v}\right](G(x;\boldsymbol{\xi}))^{\frac{u}{v}}(g(x;\boldsymbol{\xi}))\right)^{v}dx\right]=$$

$$=\frac{1}{1-v}\log\left[\sum_{u=0}^{\infty}\tau_{u}\exp\left((1-v)I_{REG}\right)\right],\tag{15}$$

for $v>0$, $v\neq1$, where $I_{REG}=\frac{1}{1-v}\log\left[\int_{0}^{\infty}\left(\left[1+\frac{u}{v}\right](G(x;\boldsymbol{\xi}))^{\frac{u}{v}}(g(x;\boldsymbol{\xi}))\right)^{v}dx\right]$ is the Rényi entropy of exp-G distribution with power parameter $1+\frac{u}{v}$ and

$$\tau_{u}=\sum_{k,s,i,p,q,w=0}^{\infty}\binom{v(\alpha+1)+k-1}{k}\binom{i}{p}\binom{k}{s}\binom{\gamma(v+p)+v+q-1}{q}\times$$
$$\times\binom{b(v+q)-v}{w}\binom{2w+v}{u}(-1)^{k+s+p+q+w+u}\frac{\left(\frac{s+v\alpha}{\gamma}\right)^{i}}{i!}\frac{(4\alpha b)^{v}}{\left[1+\frac{u}{v}\right]^{v}}.$$

Therefore, Rényi entropy of the TIIEHL-Gom-TL-G family of distributions can be obtained from those of the exp-G family of distributions.

4.2 Order statistics

In this subsection, the pdf of the r^{th} order statistic is presented. Let $X_1, X_2, ..., X_n$ be independent and identically distributed TIIEHL-Gom-TL-G random variables. The pdf of the r^{th} order statistic from the TIIEHL-Gom-TL-G pdf $f(x; \alpha, \gamma, b, \boldsymbol{\xi}) = f(x)$ can be written as

$$\begin{aligned} f_{r:n}(x) &= \frac{n!f(x)}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r} = \\ &= \frac{n!f(x)}{(r-1)!(n-r)!}\sum_{m=0}^{n-r}(-1)^m\binom{n-r}{m}[F(x)]^{m+r-1}. \end{aligned} \tag{16}$$

Using Equations (1) and (2), and letting

$$H = \left(1-\left[\frac{\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)}{1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\right)}\right]^{\alpha}\right)^{m+r-1},$$

we have

$$\begin{aligned} f(x)[F(x)]^{m+r-1} &= \\ &= H4\alpha b\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma-1}\left[1-\overline{G}^2(x;\boldsymbol{\xi})\right]^{b-1}\times \\ &\times\left(1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\right)\right)^{-(\alpha+1)}\times \\ &\times\exp\left(\frac{\alpha}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\overline{G}(x;\boldsymbol{\xi})g(x;\boldsymbol{\xi}) = \\ &= \sum_{z=0}^{\infty}\binom{m+r-1}{z}(-1)^z 4\alpha b\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma-1}\times \\ &\times\exp\left(\frac{\alpha(z+1)}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\overline{G}(x;\boldsymbol{\xi})g(x;\boldsymbol{\xi})\times \\ &\times\left(1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\right)\right)^{-(\alpha(z+1)+1)}\times \\ &\times\left[1-\overline{G}^2(x;\boldsymbol{\xi})\right]^{b-1}. \end{aligned}$$

Now following the same steps leading to Equation (11), we obtain

$$f(x)[F(x)]^{m+r-1} = \sum_{u=0}^{\infty} a_{u+1} g_{u+1}(x;\boldsymbol{\xi}), \tag{17}$$

where $g_{u+1}(x;\boldsymbol{\xi}) = (u+1)[G(x;\boldsymbol{\xi})]^u g(x;\boldsymbol{\xi})$ is the exp-G pdf with the power parameter $(u+1)$ and baseline parameter vector $\boldsymbol{\xi}$, and

$$a_{u+1} = \sum_{z,k,i,s,p,q,w=0}^{\infty} \binom{m+r-1}{z}\binom{\alpha(z+1)+k}{k}\binom{k}{s}\binom{i}{p}\binom{\gamma(p+1)+q}{q} \times \\ \times \binom{b(q+1)-1}{w}\binom{2w+1}{u}\frac{(-1)^{k+s+p+q+w+u}\left(\frac{k+\alpha}{\gamma}\right)^i}{i!}\left(\frac{4\alpha b}{w+1}\right). \tag{18}$$

Thus, by substituting Equation (17) into Equation (16), the pdf of the r^{th} order statistic for the TIIEHL-Gom-TL-G family of distributions can be written as

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}\sum_{u=0}^{\infty}\sum_{m=0}^{n-r}(-1)^m\binom{n-r}{m} a_{u+1} g_{u+1}(x;\boldsymbol{\xi}).$$

5 Maximum likelihood estimation

Let $X \sim TIIEHL-Gom-TL-G(\alpha,\gamma,b,\boldsymbol{\xi})$ and $\boldsymbol{\Delta} = (\alpha,\gamma,b,\boldsymbol{\xi})^T$ be the vector of model parameters. The log-likelihood function $\ell_n = \ell_n(\boldsymbol{\Delta})$ based on a random sample of size n from the TIIEHL-Gom-TL-G family of distributions is given by

$$\ell_n(\boldsymbol{\Delta}) = n\ln(4\alpha b) + (-\gamma-1)\sum_{i=1}^{n}\ln\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right] + \\ -(\alpha+1)\sum_{i=1}^{n}\ln\left(1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\right)\right) + \\ +\frac{\alpha}{\gamma}\sum_{i=1}^{n}\left(1-\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right) + \sum_{i=1}^{n}\ln\overline{G}(x_i;\boldsymbol{\xi}) + \\ + \sum_{i=1}^{n}\ln\left(g(x_i;\boldsymbol{\xi})\right) + (b-1)\sum_{i=1}^{n}\ln\left[1-\overline{G}^2(x_i;\boldsymbol{\xi})\right].$$

By equating the score vector given in the appendix to the zero vector, we obtain a non-linear sytem of equations that are solved using a numerical method such as Newton-Raphson procedure to obtain the MLEs of the parameters. Under standard regularity conditions, we have $(\hat{\mathbf{\Delta}} - \mathbf{\Delta}) \sim N_{p+3}\left(\mathbf{0}, K(\mathbf{\Delta})^{-1}\right)$, where $\sim$ means approximately distributed and $K(\mathbf{\Delta})$ is the expected information matrix. The asymptotic behavior remains valid if $K(\mathbf{\Delta})$ is replaced by the observed information matrix $J(\mathbf{\Delta})$ evaluated at $\hat{\mathbf{\Delta}}$, i.e., $J(\hat{\mathbf{\Delta}})$. The multivariate normal $N_{p+3}\left(\mathbf{0}, J(\hat{\mathbf{\Delta}})^{-1}\right)$ distribution can be used to construct approximate confidence intervals for the model parameters.

5.1 Some special cases

In this section, we consider some special cases of the TIIEHL-Gom-TL-G family of distributions, specifically when the baseline distribution function $G(x;\boldsymbol{\xi})$ are log-logistic, exponential and Weibull distributions, respectively.

TIIEHL-Gom-TL-Log-Logistic (TIIEHL-Gom-TL-LLoG) Distribution
Suppose the cdf and pdf of the baseline distribution are given by $G(x;c) = 1-(1+x^c)^{-1}$ and $g(x;c) = cx^{c-1}(1+x^c)^{-2}$ for $c > 0$ and $x > 0$. The new TIIEHL-Gom-TL-LLoG distribution has cdf and pdf given by

$$F(x;\alpha,\gamma,b,c) = 1 - \left[\frac{\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-(1+x^c)^{-2}\right)^b\right]^{-\gamma}\right)\right)}{1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-(1+x^c)^{-2}\right)^b\right]^{-\gamma}\right)\right)\right)}\right]^{\alpha}$$

and

$$\begin{aligned} f(x;&\alpha,\gamma,b,c) = \\ &= 4\alpha b\left(1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-(1+x^c)^{-2}\right)^b\right]^{-\gamma}\right)\right)\right)\right)^{-(\alpha+1)} \times \\ &\times \left[1-\left(1-(1+x^c)^{-2}\right)^b\right]^{-\gamma-1}\exp\left(\frac{\alpha}{\gamma}\left(1-\left[1-\left(1-(1+x^c)^{-2}\right)^b\right]^{-\gamma}\right)\right) \times \\ &\times \left[1-(1+x^c)^{-2}\right]^{b-1}(1+x^c)^{-1}cx^{c-1}(1+x^c)^{-2}, \end{aligned}$$

respectively, for $\alpha,\gamma,b,c > 0$. The hazard rate function (hrf) is given by

$$\begin{aligned} h_F(x) = 4\alpha b&\left(1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-(1+x^c)^{-2}\right)^b\right]^{-\gamma}\right)\right)\right)\right)^{-(\alpha+1)} \times \\ &\times \left[1-\left(1-(1+x^c)^{-2}\right)^b\right]^{-\gamma-1}\exp\left(\frac{\alpha}{\gamma}\left(1-\left[1-\left(1-(1+x^c)^{-2}\right)^b\right]^{-\gamma}\right)\right) \times \\ &\times \left[1-(1+x^c)^{-2}\right]^{b-1}(1+x^c)^{-1}cx^{c-1}(1+x^c)^{-2} \times \end{aligned}$$

$$\times\left[\frac{\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-(1+x^c)^{-2}\right)^b\right]^{-\gamma}\right)\right)}{1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-(1-(1+x^c)^{-2})^b\right]^{-\gamma}\right)\right)\right)}\right]^{-\alpha}.$$

Figure 1 shows the plots of skewness and kurtosis for the TIIEHL-Gom-TL-LLoG distribution. The plots show the flexibility of the TIIEHL-Gom-TL-LLoG distribution in capturing different levels of skewness and kurtosis by varying the values of the model parameters.
Figure 2 shows the plots of the pdf and hrf of the TIIEHL-Gom-TL-LLoG distribution. The pdf can take various shapes that include almost symmetric, reverse-J, J, left or right-skewed. Furthermore, the graphs of the hrf for the TIIEHL-Gom-TL-LLoG distribution exhibit increasing, decreasing, bathtub, upside-down bathtub and upside-down bathtub followed by bathtub shapes.

TIIEHL-Gom-TL-Weibull (TIIEHL-Gom-TL-W) Distribution
Suppose the cdf and pdf of the Weibull distribution are given by $G(x;\lambda)=1-\exp\left(-x^{\lambda}\right)$, and $g(x;\lambda)=\lambda x^{\lambda-1}\exp\left(-x^{\lambda}\right)$, for $\lambda>0$, and $x>0$, then, the cdf and pdf of the TIIEHL-Gom-TL-W distribution are respectively, given by

$$F(x;\alpha,\gamma,b,\lambda)=1-\left[\frac{\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\exp\left(-2x^{\lambda}\right)\right)^b\right]^{-\gamma}\right)\right)}{1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-(1-\exp\left(-2x^{\lambda}\right))^b\right]^{-\gamma}\right)\right)\right)}\right]^{\alpha}$$

and

$$\begin{aligned}f(x;&\alpha,\gamma,b,\lambda)=\\&=4\alpha b\left(1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\exp\left(-2x^{\lambda}\right)\right)^b\right]^{-\gamma}\right)\right)\right)\right)^{-(\alpha+1)}\times\\&\times\left[1-\left(1-\exp\left(-2x^{\lambda}\right)\right)^b\right]^{-\gamma-1}\exp\left(\frac{\alpha}{\gamma}\left(1-\left[1-\left(1-\exp\left(-2x^{\lambda}\right)\right)^b\right]^{-\gamma}\right)\right)\times\\&\times\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b-1}\exp\left(-x^{\lambda}\right)\lambda x^{\lambda-1}\exp\left(-x^{\lambda}\right),\end{aligned}$$

for $\alpha,\gamma,b,\lambda>0$. The hrf is given by

$$\begin{aligned}h_{F}(x)&=\\&=4\alpha b\left(1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\exp\left(-2x^{\lambda}\right)\right)^b\right]^{-\gamma}\right)\right)\right)\right)^{-(\alpha+1)}\times\\&\times\left[1-\left(1-\exp\left(-2x^{\lambda}\right)\right)^b\right]^{-\gamma-1}\exp\left(\frac{\alpha}{\gamma}\left(1-\left[1-\left(1-\exp\left(-2x^{\lambda}\right)\right)^b\right]^{-\gamma}\right)\right)\times\end{aligned}$$

Figure 1: Plots of the Skewness and Kurtosis for the TIIEHL-Gom-TL-LLoG distribution

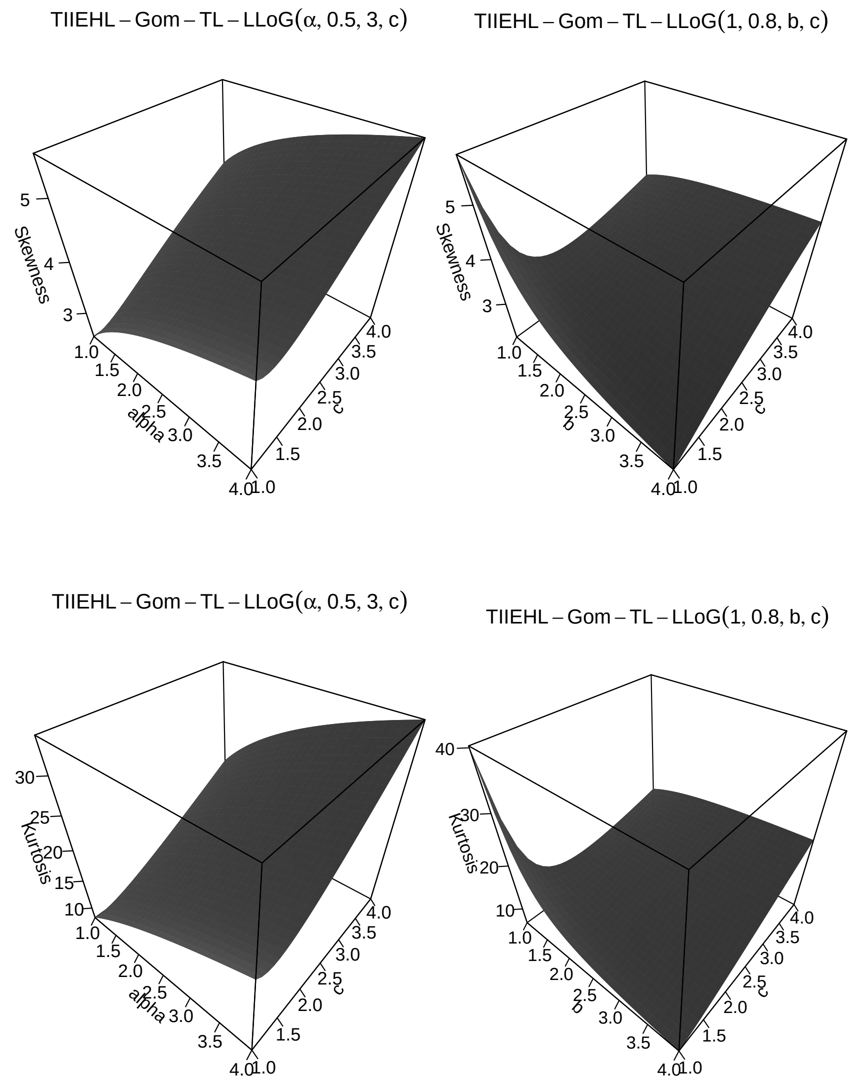


Figure 2: Plots of the pdf and hrf for the TIIEHL-Gom-TL-LLoG distribution

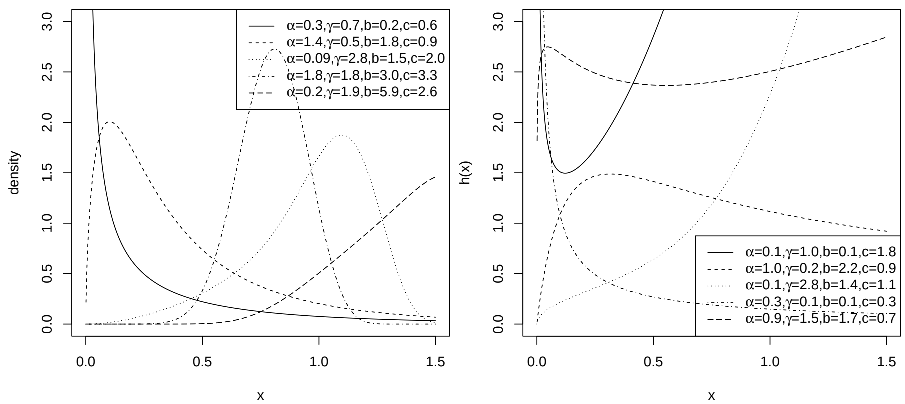
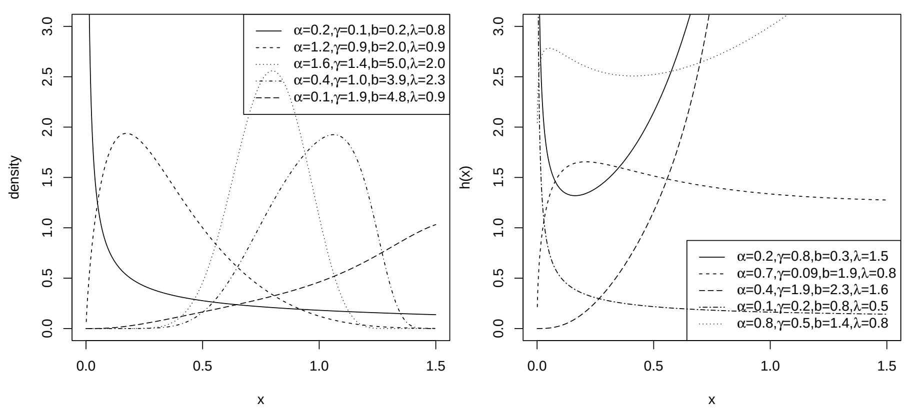


Figure 3: Plots of the pdf and hrf for the TIIEHL-Gom-TL-W distribution



$$\times \left[1-\exp\left(-2x^{\lambda}\right)\right]^{b-1}\exp\left(-x^{\lambda}\right)\lambda x^{\lambda-1}\exp\left(-x^{\lambda}\right)\times$$
$$\times\left[\frac{\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\exp\left(-2x^{\lambda}\right)\right)^{b}\right]^{-\gamma}\right)\right)}{1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\exp\left(-2x^{\lambda}\right)\right)^{b}\right]^{-\gamma}\right)\right)\right)}\right]^{-\alpha}.$$

Figure 3 shows the plots of the pdf and hrf of the TIIEHL-Gom-TL-W distribution. The pdf can take various shapes that include almost symmetric, reverse-J, J, left or right-skewed. Furthermore, plots of the hrf for the TIIEHL-Gom-TL-W distribution exhibit increasing, decreasing, bathtub, upside-down bathtub and upside-down bathtub followed by bathtub shapes.

Figure 4 shows the plots of skewness and kurtosis for the TIIEHL-Gom-TL-W distribution. We can see that the TIIEHL-Gom-TL-W distribution can exhibit various levels of skewness and kurtosis by varying the values of the model parameters. This illustrates the flexibility and ability of the distribution to model data sets with varying degrees of skewness and kurtosis.

TIIEHL-Gom-TL-Exponential (TIIEHL-Gom-TL-E) Distribution

Suppose the cdf and pdf of the baseline distribution are given by $G(x;\lambda)=1-e^{-\lambda x}$, $x\geq 0$ and $g(x;\lambda)=\lambda e^{-\lambda x}$, $x>0,\lambda>0$. Then, the new TIIEHL-Gom-TL-E distribution has cdf and pdf given by

$$F(x;\alpha,\gamma,b,\lambda)=1-\left[\frac{\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-e^{-2\lambda x}\right)^{b}\right]^{-\gamma}\right)\right)}{1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-e^{-2\lambda x}\right)^{b}\right]^{-\gamma}\right)\right)\right)}\right]^{\alpha}$$

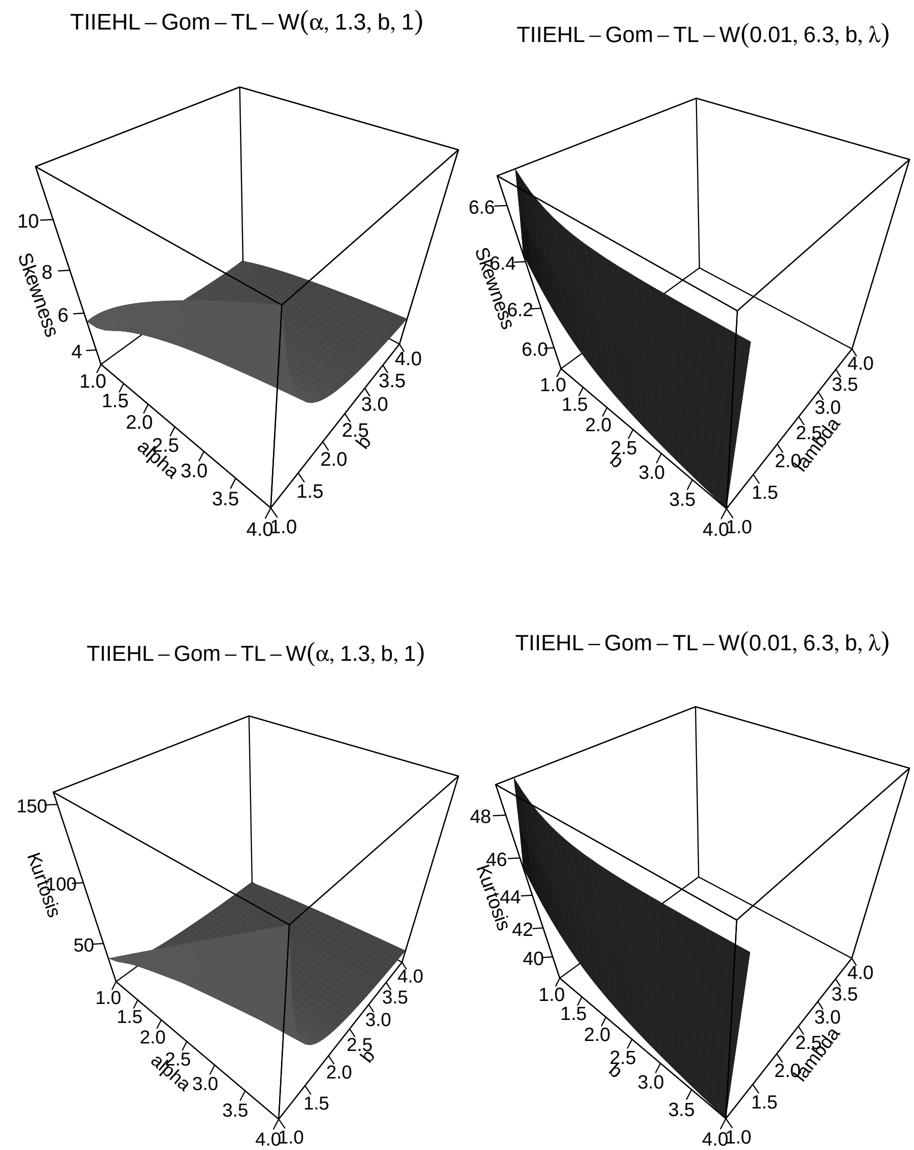
and

$$\begin{aligned} f(x;&\alpha,\gamma,b,\lambda)=\\ &=4\alpha b\left(1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-e^{-2\lambda x}\right)^{b}\right]^{-\gamma}\right)\right)\right)\right)^{-(\alpha+1)}\times\\ &\times\left[1-\left(1-e^{-2\lambda x}\right)^{b}\right]^{-\gamma-1}\exp\left(\frac{\alpha}{\gamma}\left(1-\left[1-\left(1-e^{-2\lambda x}\right)^{b}\right]^{-\gamma}\right)\right)\times\\ &\times\left[1-e^{-2\lambda x}\right]^{b-1}e^{-\lambda x}\lambda e^{-\lambda x}, \end{aligned}$$

respectively, for $\alpha,\gamma,b,\lambda>0$. The hrf is given by

$$h_{F}(x)=4\alpha b\left(1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-e^{-2\lambda x}\right)^{b}\right]^{-\gamma}\right)\right)\right)\right)^{-(\alpha+1)}\times$$

Figure 4: Plots of Skewness and Kurtosis for the TIIEHL-Gom-TL-W distribution



$$\times \left[1-\left(1-e^{-2\lambda x}\right)^{b}\right]^{-\gamma-1} \exp\left(\frac{\alpha}{\gamma}\left(1-\left[1-\left(1-e^{-2\lambda x}\right)^{b}\right]^{-\gamma}\right)\right) \times$$
$$\times \left[1-e^{-2\lambda x}\right]^{b-1} e^{-\lambda x}\lambda e^{-\lambda x} \times$$
$$\times \left[\frac{\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-e^{-2\lambda x}\right)^{b}\right]^{-\gamma}\right)\right)}{1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-e^{-2\lambda x}\right)^{b}\right]^{-\gamma}\right)\right)\right)}\right]^{-\alpha}.$$

Figure 5 shows the plots of skewness and kurtosis for the TIIEHL-Gom-TL-E distribution. We can see that the skewness becomes right-skewed and kurtosis becomes leptokurtic with increasing values of α and b, and also with increasing values of γ and b.
Figure 6 shows the plots of the pdf and hrf of the TIIEHL-Gom-TL-E distribution. The pdf can take various shapes that include almost symmetric, reverse-J, J, left or right-skewed. Furthermore, the plots of the hrf for the TIIEHL-Gom-TL-E distribution exhibit increasing, decreasing, bathtub, upside-down bathtub and upside-down bathtub followed by bathtub shapes.

6 Simulation study

In this section, we present some simulation results for the TIIEHL-Gom-TL-W distribution to assess the reliability of the maximum likelihood estimates (MLEs). For different values of α, γ, b and λ, samples of sizes n = 25, 50,100, 200, 400, 800 and 1600 were generated from the TIIEHL-Gom-TL-W disttribution via the R package. We repeated the simulation N= 3000 times and calculated the mean MLEs, average bias (ABias) and the root mean square errors (RMSEs). The average bias and RMSE for the estimated parameter, say, $\hat{\theta}$, are given by:

$$ABias(\hat{\theta}) = \frac{\sum_{i=1}^{N}\hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\theta}_i-\theta)^2}{N}},$$

respectively.
It can be observed in Tables 1 and 2 that the mean MLEs get close to the true values of the parameters and the RMSEs decay towards zero as the sample size increases. This indicates that the maximum likelihood method works well for obtaining estimate of the parameters of the TIIEHL-Gom-TL-W distribution.

Figure 5: Plots of Skewness and Kurtosis for the TIIEHL-Gom-TL-E distribution

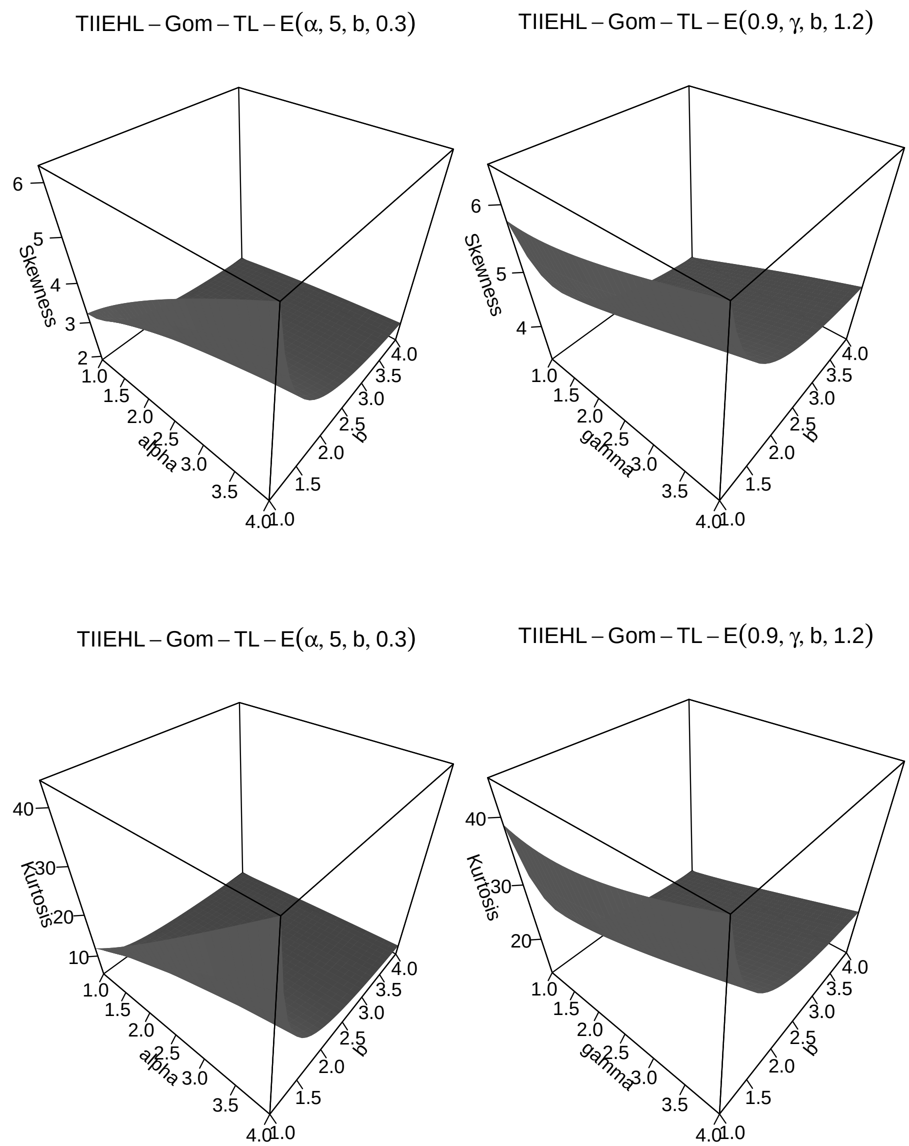
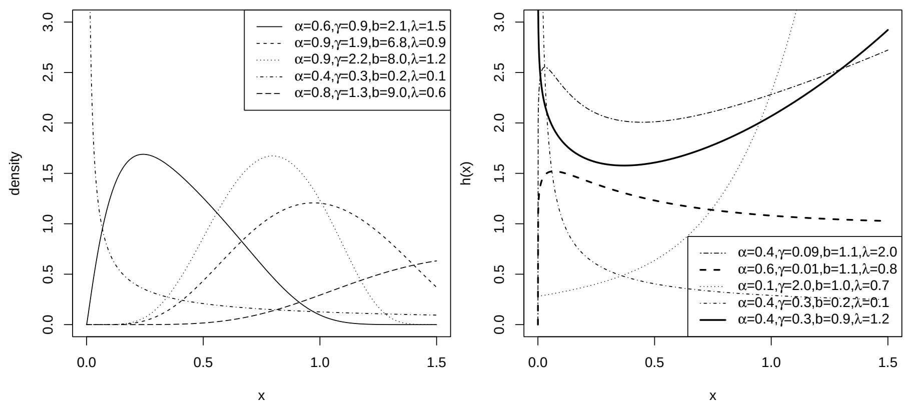


Figure 6: Plots of the pdf and hrf for the TIIEHL-Gom-TL-E distribution



7 Data analysis

In this section, the importance of the TIIEHL-Gom-TL-G family of distributions is illustrated by fitting its special case, namely, TIIEHL-Gom-TL-W distribution to four real data sets. The TIIEHL-Gom-TL-W model is compared to its nested models and competitive distributions: namely, type II general inverse exponential Burr III (TIIGIE-BIII) distribution by Jamal et al. (2020) with the pdf

$$f_{TIIGIE-BIII}(x;\lambda,\theta,c,k)=\frac{\lambda\theta ckx^{-c-1}\left(1+x^{-c}\right)^{-k-1}\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\theta-1}}{\left(1-\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\theta}\right)^{2}}\times$$
$$\times\exp\left(-\lambda\frac{\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\theta}}{1-\left[1-\left(1+x^{-c}\right)^{-k}\right]^{\theta}}\right),$$

for $\lambda,\theta,c,k>0$ and $x>0$, type II exponentiated half-logistic-Topp-Leone-Weibull logarithmic (TIIEHL-TL-WL) distribution by Moakofi et al. (2021a) with the pdf

$$f_{TIIEHL-TL-WL}(x;a,b,\theta,\lambda)=\frac{4ab\theta\lambda x^{\lambda-1}\exp\left(-x^{\lambda}\right)\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b-1}\exp\left(-x^{\lambda}\right)}{\left(1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}\right)^{a+1}}\times$$
$$\times\left(1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}\right)^{a-1}\times$$

Table 1: Monte Carlo Simulation Results (part 1)

		(0.2, 1.6, 1.6, 1.6)			(0.2, 0.9, 1.6, 0.9)			(1.6, 1.2, 0.2, 0.9)		
parameter	Sample Size	Mean	RMSE	Bias	Mean	RMSE	Bias	Mean	RMSE	Bias
α	25	0.4175	0.5526	0.2175	0.3006	0.3794	0.1006	2.4813	3.5132	0.8813
	50	0.3486	0.4707	0.1486	0.2767	0.3206	0.0767	2.4548	3.4280	0.8548
	100	0.2738	0.2016	0.0738	0.2551	0.2308	0.0551	2.3633	3.0459	0.7633
	200	0.2406	0.1053	0.0406	0.2299	0.0990	0.0299	2.2259	2.4937	0.6259
	400	0.2224	0.0625	0.0224	0.2200	0.0508	0.0200	2.0625	1.9169	0.4625
	800	0.2100	0.0396	0.0100	0.2117	0.0309	0.0117	1.8538	1.2574	0.2538
	1600	0.2038	0.0248	0.0038	0.2066	0.0191	0.0066	1.7083	0.7552	0.1083
γ	25	3.4213	11.0508	1.8213	1.7852	1.6985	0.8852	2.3948	2.3373	1.1948
	50	3.1388	6.9203	1.5388	1.6361	1.4543	0.7361	2.2204	1.9942	1.0204
	100	2.9510	5.8947	1.3510	1.4430	1.2342	0.5430	1.9054	1.6285	0.7054
	200	2.5486	4.0898	0.9486	1.2614	0.9926	0.3614	1.666	1.3074	0.4667
	400	2.1108	2.4077	0.5108	1.1141	0.7578	0.2141	1.4915	1.0191	0.2915
	800	1.8842	1.5186	0.2842	0.9862	0.4626	0.0862	1.3029	0.6070	0.1029
	1600	1.6920	0.7107	0.0920	0.9357	0.2641	0.0357	1.2479	0.3592	0.0479
b	25	2.6331	3.9481	1.0331	2.5298	3.6353	0.9298	0.3422	0.4780	0.1422
	50	2.4397	3.6217	0.8397	2.4278	3.3434	0.8278	0.3097	0.4732	0.1097
	100	2.3136	2.9971	0.7136	2.4059	3.3229	0.8059	0.2591	0.2087	0.0591
	200	2.1607	2.3799	0.5607	2.2632	2.5273	0.6632	0.2345	0.0932	0.0345
	400	1.9599	1.6495	0.3599	2.1548	2.0385	0.5548	0.2218	0.0520	0.0218
	800	1.8236	1.1200	0.2236	1.9504	1.4100	0.3504	0.2124	0.0351	0.0124
	1600	1.6804	0.5932	0.0804	1.7547	0.9162	0.1547	0.2062	0.0235	0.0062
λ	25	3.1036	2.9479	1.5036	1.2168	0.7183	0.3168	1.4228	1.0537	0.5228
	50	2.8937	2.5628	1.2937	1.1596	0.6561	0.2596	1.3555	0.9537	0.4555
	100	2.5209	2.1252	0.9209	1.1000	0.6283	0.2000	1.2578	0.8570	0.3578
	200	2.2223	1.7229	0.6223	1.0771	0.5946	0.1771	1.1807	0.7788	0.2807
	400	1.9789	1.3389	0.3789	1.0367	0.5503	0.1367	1.1169	0.7025	0.2169
	800	1.7403	0.8011	0.1403	0.9887	0.4675	0.0887	1.0239	0.5370	0.1239
	1600	1.6622	0.4735	0.0622	0.9758	0.3910	0.0758	0.9681	0.3832	0.0681

$$\times \frac{\left(1-\theta\left[\frac{1-\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}{1+\left[1-\exp\left(-2x^{\lambda}\right)\right]^{b}}\right]^{a}\right)^{-1}}{-\log(1-\theta)},$$

for a, b, θ, λ and $x > 0$, type II general inverse exponential Lomax (TIIGIE-Lx) distribution by Hamedani et al. (2019) with the pdf

$$f_{TIIGIE-Lx}(x;\lambda,\alpha,a,b) = \lambda\alpha\frac{a}{b}\left(1+\frac{x}{b}\right)^{-(a+1)}\left(1+\frac{x}{b}\right)^{a(\alpha+1)} \times \\ \times \exp\left(\lambda\left(1-\left(1+\frac{x}{b}\right)^{a\alpha}\right)\right),$$

Table 2: Monte Carlo Simulation Results (part 2)

		(0.8, 2.0, 2.0, 2.0)			(0.8, 0.3, 2.0 2.0)			(1.6, 1.0, 0.8, 0.2)		
parameter	Sample Size	Mean	RMSE	Bias	Mean	RMSE	Bias	Mean	RMSE	Bias
α	25	1.3990	1.6385	0.5990	1.1242	0.8711	0.3242	2.5927	4.2553	0.9927
	50	1.3274	1.4677	0.5274	1.0706	0.6886	0.2706	2.5725	3.8841	0.9725
	100	1.1914	1.1317	0.3914	1.0227	0.5692	0.2227	2.4242	3.6283	0.8242
	200	1.0661	0.7815	0.2661	0.9656	0.4032	0.1656	2.2637	2.8957	0.6637
	400	0.9538	0.3954	0.1538	0.9061	0.2619	0.1061	2.1687	2.4090	0.5687
	800	0.8884	0.1647	0.0884	0.8796	0.2035	0.0796	1.9858	1.7200	0.3858
	1600	0.8553	0.1135	0.0553	0.8657	0.1664	0.0657	1.6835	0.6189	0.0835
γ	125	4.7738	10.0610	2.7738	1.0998	2.1064	0.7998	2.6340	5.7228	1.6340
	50	4.4584	8.0375	2.4584	0.8412	1.2205	0.5412	2.3847	4.4453	1.3847
	100	3.7249	6.0870	1.7249	0.7345	0.9534	0.4345	2.0326	3.2177	1.0326
	200	3.1704	4.3931	1.1704	0.6216	0.7064	0.3216	1.7641	2.4845	0.7641
	400	2.6298	2.6344	0.6298	0.5168	0.5381	0.2168	1.4484	1.6150	0.4484
	800	2.3921	1.6026	0.3921	0.4707	0.4475	0.1707	1.2910	1.0267	0.2910
	1600	2.1835	1.1046	0.1835	0.4476	0.3798	0.1476	1.1875	0.7093	0.1875
b	25	3.5067	4.2735	1.5067	3.7457	3.2952	1.7457	1.2108	1.2958	0.4108
	50	3.4328	4.1344	1.4328	3.4799	2.7800	1.4799	1.2033	1.1335	0.4033
	100	3.0867	3.5315	1.0867	3.2566	2.4791	1.2566	1.1027	0.9643	0.3027
	200	2.8415	2.8652	0.8415	2.9932	2.0990	0.9932	1.0236	0.7460	0.2230
	400	2.5169	2.0491	0.5169	2.6766	1.6330	0.6766	0.9306	0.4420	0.1306
	800	2.3610	1.4796	0.3610	2.5363	1.3930	0.5363	0.8701	0.1706	0.0701
	1600	2.1782	1.0664	0.1782	2.4692	1.1994	0.4692	0.8423	0.0942	0.0423
λ	25	3.8007	5.2500	1.8007	3.4604	3.8367	1.4604	0.3170	0.4081	0.1170
	50	3.3475	4.1697	1.3475	3.1970	3.2854	1.1970	0.2873	0.3553	0.0873
	100	3.1011	3.4993	1.1011	2.9744	2.7835	0.9744	0.2480	0.1918	0.0480
	200	3.0681	3.1840	1.0681	2.6255	2.0748	0.6255	0.2291	0.1019	0.0291
	400	2.8062	2.5545	0.8060	2.4355	1.4881	0.4355	0.2188	0.0509	0.0188
	800	2.6831	2.2006	0.6831	2.2946	1.0865	0.2946	0.2114	0.0306	0.0114
	1600	2.4630	1.7268	0.4630	2.1373	0.9658	0.1373	0.2058	0.0185	0.0058

for $\lambda, \alpha, a, b > 0$ and $x > 0$, exponentiated half logistic-power generalized Weibull-log-logistic (EHL-PGW-LLoG) by Oluyede et al. (2020) with the pdf

$$f_{EHL-PGW-LLoG}(x;\alpha,\beta,\delta,c) =$$
$$= 2\alpha\beta\delta\left[1+\left(\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)^{\alpha}\right]^{\beta-1} e^{\left(1-\left[1+\left(\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)^{\alpha}\right]^{\beta}\right)} \times$$
$$\times\left((1+x^c)^{-1}\right)^{-(\alpha+3)}\left(1+e^{\left(1-\left[1+\left(\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)^{\alpha}\right]^{\beta}\right)}\right)^{-2} \times$$

$$\times\left[\frac{1-e^{\left(1-\left[1+\left(\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)^{\alpha}\right]^{\beta}\right)}}{1+e^{\left(1-\left[1+\left(\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right)^{\alpha}\right]^{\beta}\right)}}\right]^{\delta-1} cx^{c-1}\left(1-(1+x^c)^{-1}\right)^{\alpha-1},$$

for $\alpha, \beta, \delta, c > 0$, and odd exponentiated half logistic- Burr XII (OEHL-BXII) by Aldahlan and Afify (2018) with the pdf

$$f_{OEHLBXII}(x;\alpha,\lambda,a,b) = \\ = \frac{2\alpha\lambda abx^{a-1}\exp(\lambda[1-(1+x^a)^b])(1-\exp(\lambda[1-(1+x^a)^b]))^{\alpha-1}}{(1+x^a)^{-b-1}(1+\exp(\lambda[1-(1+x^a)^b]))^{\alpha+1}},$$

for $\alpha, \lambda, a, b > 0$.

Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. (1983)) are given in Figure 7, Figure 9, Figure 11 and Figure 13. For the probability plot, we plotted $F(x_{(j)};\hat{\alpha},\hat{\gamma},\hat{b},\hat{\boldsymbol{\xi}})$ against $\frac{j-0.375}{n+0.25}$, $j=1,2,\dots,n$, where $x_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares

$$SS=\sum_{j=1}^{n}\left[F(x_{(j)})-\left(\frac{j-0.375}{n+0.25}\right)\right]^2.$$

The goodness-of-fit statistics: -2log-likelihood statistic $(-2\ln(L))$, Akaike Information Criterion $(AIC = 2p - 2\ln(L))$, Bayesian Information Criterion $(BIC = p\ln(n) - 2\ln(L))$, and Consistent Akaike Information Criterion $(CAIC = AIC + 2\frac{p(p+1)}{n-p-1})$, where $L = L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are used to assess the performance of the models. All the results were obtained using R programming language.

Also, the goodness-of-fit statistics W^* and A^*, described by Chen and Balakrishnan (1995) and the Kolmogorov-Smirnov (KS) defined below are considered.

Let $\hat{\theta}$ be an estimate of θ. For a given random sample $x_1, x_2, \dots, x_n$ from a continuous distribution with cdf $F(x;\theta)$, where the form of F is known but θ is unknown parameter vector. Define $u_i = F(X_i;\hat{\theta}), i = 1,2,\dots,n$. Without loss of generality, suppose $x_i's$ and $u_i's$ have been arranged in order, then the Cramér-von Mises statistic and the Anderson-Darling statistic are given by

$$W^2=\sum_{i=1}^{n}\left[u_i-\left(\frac{(2i-1)}{(2n)}\right)\right]^2+\frac{1}{(12n)}$$

and

$$A^2=-n-n^{-1}\sum_{i=1}^{n}\Big[(2i-1)\ln(u_i)+(2n+1-2i)\ln(1-u_i)\Big].$$

We can obtain goodness-of-fit statistics W^* and A^* by modifying W^2 into W^* and A^2 into A^* as follows

$$W^* = W^2\left(1+\frac{0.5}{n}\right), \quad A^* = A^2\left(1+\frac{0.75}{n}+\frac{2.25}{n^2}\right). \tag{19}$$

The Kolmogorov-Smirnov (K-S) statistic denoted D_n and scaled with $\sqrt{n}$ is given by

$$D_n = Sup_x \sqrt{n} \mid F_n(x) - F_0(x) \mid,$$

where F_0 is what may be taken as the null hypothesis or called the evidence and $F_n(x)$ is the empirical distribution function given as $F_n(x) = (\#X_i's \leq x)/n$. In general, the smaller the values of all the goodness-of-fit statistics, the better the fit.

7.1 Turbocharger failure times data

The data represents failure times (10^3h) of turbocharger of a type of engine. The data set can be found in Xu et al. (2003) and in Afify et al. (2021). The data is given below:
1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0.
The estimated variance-covariance matrix for TIIEHL-Gom-TL-W model on turbocharger failure times data is given by

$$\begin{bmatrix} 1.4666\times10^{-05} & 4.8125\times10^{-05} & -1.7259\times10^{-07} & -2.7790\times10^{-04} \\ 4.8125\times10^{-05} & 1.2458\times10^{-02} & -1.0267\times10^{-05} & -2.0533\times10^{-02} \\ -1.7259\times10^{-07} & -1.0267\times10^{-05} & 9.8106\times10^{-09} & 1.8945\times10^{-05} \\ -2.7790\times10^{-04} & -2.0533\times10^{-02} & 1.8945\times10^{-05} & 3.6876\times10^{-02} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by
$\alpha \in [7.6797\times10^{-03} \pm 0.0075]$, $\gamma \in [2.0480\times10^{-01} \pm 0.2187]$, $b \in [6.1986 \pm 0.0001]$ and $\lambda \in [1.1218 \pm 0.3763]$, respectively.
Maximum likelihood estimates (MLEs) of the parameters of TIIEHL-Gom-TL-W distribution (standard errors in parenthesis), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC) are given in Tables 3 and 4 for the turbocharger failure times data. The goodness−of−fit statistics W^* and A^* are also given. Plots of histogram and fitted densities, and observed probability versus predicted probabilty for the turbocharger failure times data are given in Figure 7.
From Table 4, we find that the TIIEHL-Gom-TL-W distribution has the lowest value of the $K-S$ statistic and the highest p-value. Furthermore, the goodness-of-fit measures $-2Log(L)$, W^* and A^* are smallest for the TIIEHL-Gom-TL-W distribution. This means that the TIIEHL-Gom-TL-W distribution is statistically superior to the fits by the nested and competitive lifetime models considered.

Table 3: Input and output indicators for performance benchmarking

	Estimates			
Model	α	γ	b	λ
TIIEHL-Gom-TL-W	7.6797×10^{-03}	2.0480×10^{-01}	6.1986	1.1218
	(3.8297×10^{-03})	(1.1162×10^{-01})	(9.9049×10^{-05})	(1.9203×10^{-01})
TIIEHL-Gom-TL-W(1, γ, b, λ)	1	2.3846	90.3154	0.4115
	-	(1.0494)	(20.5659)	(0.0358)
TIIEHL-Gom-TL-W(α, 1, b, λ)	5.9933×10^{04}	1	1.3796×10^{02}	1.1808×10^{-01}
	(9.5465×10^{-04})	-	(9.0293)	(1.2282×10^{-02})
TIIEHL-Gom-TL-W(1, 1, b, λ)	1	1	86.3189	0.4329
	-	-	(17.8188)	(0.0264)
	λ	θ	c	k
TIIGIE-BIII	7.7081	8.4126×10^{04}	1.1417×10^{-01}	1.6152×10^{01}
	(5.54640)	(2.2992×10^{-05})	(1.9800×10^{-02})	(5.9742×10^{-01})
	a	b	θ	λ
TII-EHL-TL-WL	1.0950	9.0645	1.1350×10^{-08}	2.7996×10^{-01}
	(4.0099×10^{-01})	(2.3093)	(1.1409×10^{-02})	(4.6888×10^{-02})
	λ	α	a	b
TIIGIE-Lx	2.9529×10^{02}	3.6376×10^{04}	1.2838×10^{-04}	3.6203×10^{03}
	(5.5597×10^{-12})	(4.5155×10^{-14})	(1.2813×10^{-05})	(4.5352×10^{-13})
	α	β	δ	c
EHL-PGW-LLoG	2.8303	0.5994	7.2104	0.4452
	(0.0335)	(0.2950)	(3.3381)	(0.2123)
	α	λ	a	b
OEHL-BXII	1.4554	0.0005	0.5826	5.6606
	(0.0065)	(0.0002)	(0.0549)	(0.0048)

Table 4: Goodness-of-Fit Statistics for Turbocharger Failure Times Data

	Statistics							
Model	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TIIEHL-Gom-TL-W	159.4886	167.4888	168.6316	174.2443	0.0305	0.2279	0.0881	0.9149
TIIEHL-Gom-TL-W(1, γ, b, λ)	171.5633	177.5633	178.2300	182.6299	0.1611	1.1032	0.1378	0.4329
TIIEHL-Gom-TL-W(α, 1, b, λ)	167.3201	173.3201	173.9867	178.3867	0.1052	0.7577	0.1165	0.6494
TIIEHL-Gom-TL-W(1, 1, b, λ)	179.0737	183.0737	183.3981	186.4515	0.2721	1.7396	0.1762	0.1664
TIIGIE-BIII	186.9018	194.9020	196.0449	201.6575	11.71232	76.9800	0.9995	2.2×10^{-16}
TII-EHL-TL-WL	243.2460	251.2460	252.3888	258.0015	0.1593	1.0906	0.3637	5.057×10^{-05}
TIIGIE-Lx	161.2419	169.2418	170.3847	175.9974	0.0373	0.2830	0.0915	0.8910
EHL-PGW-LLoG	181.9284	189.9284	191.0713	196.6839	0.2998	1.8928	0.1543	0.2961
OEHL-BXII	207.6965	215.6965	216.8393	222.4520	0.0763	0.5671	0.1339	0.4692

Figure 7: Fitted Densities and Probability Plots for the Turbocharger Failure Times Data

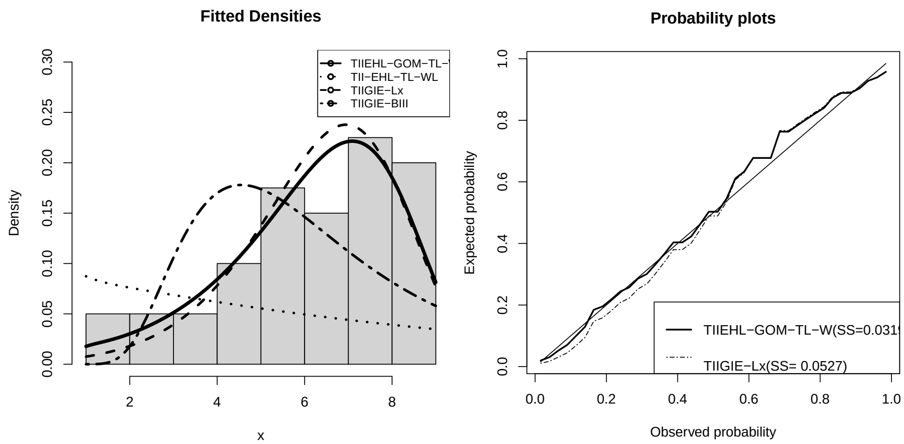


Figure 8: Fitted TTT and Kaplan-Meier Survival Plots for Turbocharger Failure Times Data

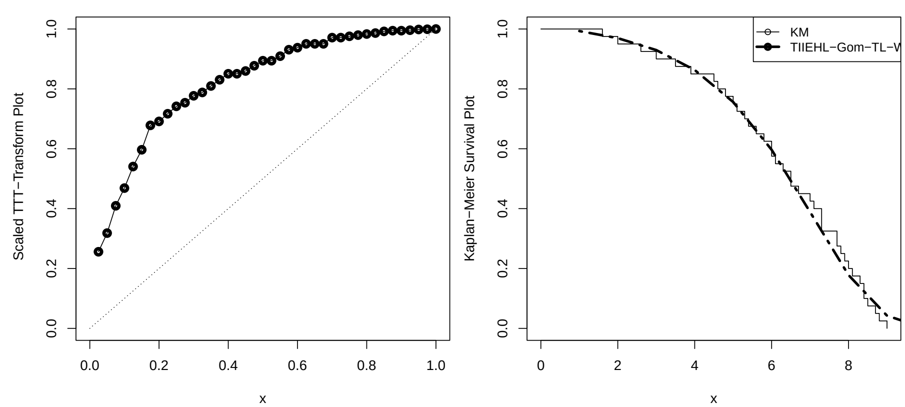


Figure 8 shows the TTT scaled plots, observed and the fitted Kaplan-Meier survival plots. We conclude that the TIIEHL-Gom-TL-W distribution performs better since the observed and fitted Kaplan-Meier survival curves are close to each other.

7.2 Carbon fibres data

The data set consists of 66 observations on breaking stress of carbon fibres (Gba). The data set was reported by Nichols and Padgett (2006) and also analyzed by Al-Babtain et al. (2021). The observations are:
$3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19,$
$3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67,$
$2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55,$
$2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68$
$1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08,$
$2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70,$
$2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.$
The estimated variance-covariance matrix for TIIEHL-Gom-TL-W model on carbon fibres data is given by

$$\begin{bmatrix} 0.0251 & -0.0107 & -0.3450 & -0.0015 \\ -0.0107 & 0.0045 & 0.1474 & 0.0006 \\ -0.3450 & 0.1474 & 4.7339 & 0.0213 \\ -0.0015 & 0.0006 & 0.0213 & 0.0001 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by
$\alpha \in [103.5515 \pm 0.3107]$, $\gamma \in [34.4358 \pm 0.1328]$, $b \in [58.6658 \pm 4.2645]$ and $\lambda \in [0.1814 \pm 0.0216]$, respectively.
Tables 5 and 6 give the MLEs of the TIIEHL-Gom-TL-W distribution (standard errors in parenthesis), and the goodness-of-fit measures for the TIIEHL-Gom-TL-W distribution, its nested and non-nested models. The TIIEHL-Gom-TL-W distribution has the smallest $AIC, AICC, BIC, K-S$, W^*, A^*, and the largest K-S p-value. These indicate that the TIIEHL-Gom-TL-W distribution is the best lifetime model to represent the carbon fibres data compared to the other competitive models.
Figure 10 shows the TTT scaled plots, observed and the fitted Kaplan-Meier survival plots. We conclude that the TIIEHL-Gom-TL-W distribution performs better since the observed and fitted Kaplan-Meier survival curves are very close to each other.

7.3 Insurance data

This data set from the insurance field represents monthly metrics on unemployment insurance from July 2008 to April 2013 from the Department of Labor, Licensing and Regulation. It consists of 58 observations and 21 variables, we studied the variable

Table 5: Estimates of Models for Carbon Fibres Data

Model	Estimates			
	α	γ	b	λ
TIIEHL-Gom-TL-W	103.5515	34.4358	58.6658	0.1814
	(0.1585)	(0.0677)	(2.1757)	(0.0110)
TIIEHL-Gom-TL-W(1, γ, b, λ)	1	0.5631	30.7580	0.5630
	-	(0.2866)	(2.9851)	(0.0374)
TIIEHL-Gom-TL-W(α, 1, b, λ)	7.6340×10^{-05}	1	4.8578×10^{-03}	6.5686×10^{-01}
	(1.5598×10^{-05})	-	(4.2709×10^{-05})	(2.7930×10^{-02})
TIIEHL-Gom-TL-W(1, 1, b, λ)	1	1	28.8388	0.5188
	-	-	(2.6588)	(0.0232)
	λ	θ	c	k
TIIGIE-BIII	3.1749×10^{01}	$2.6754\times10^{-01}1$	7.9665	4.0638×10^{-04}
	(9.3495×10^{-06})	(1.8422×10^{-02})	(6.0839×10^{-04})	(3.3017×10^{-04})
	a	b	θ	λ
TII-EHL-TL-WL	1.9102×10^{-01}	1.9104×10^{-01}	4.0382×10^{-09}	9.8158×10^{-01}
	(3.3242×10^{-02})	(5.4900×10^{-02})	(8.9914×10^{-03})	(1.0597×10^{-01})
	λ	α	a	b
TIIGIE-Lx	9.6850×10^{-02}	2.9253×10^{04}	4.2828×10^{-02}	1.5798×10^{03}
	(3.0797×10^{-02})	(6.1048×10^{-09})	(4.2018×10^{-03})	(1.0491×10^{-07})
	α	β	δ	c
EHL-PGW-LLoG	1.8986	0.6360	3.2914	0.9252
	(0.1152)	(0.1994)	(1.0464)	(0.2365)
	α	λ	a	b
OEHL-BXII	0.3078	0.0019	11.9671	0.4005
	(0.0616)	(0.0024)	(0.0016)	(0.0666)

Table 6: Goodness-of-Fit Statistics for Carbon Fibres Data

	Statistics							
Model	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TIIEHL-Gom-TL-W	283.1216	291.1216	291.5426	301.5422	0.0784	0.4563	0.0654	0.7848
TIIEHL-Gom-TL-W(1, γ, b, λ)	294.2130	177.5633	178.2300	182.6299	0.1611	1.1032	0.1378	0.4329
TIIEHL-Gom-TL-W(α, 1, b, λ)	296.2110	300.2130	300.4630	308.0285	0.2527	1.3317	0.1187	0.1194
TIIEHL-Gom-TL-W(1, 1, b, λ)	289.1973	293.1973	293.3210	298.4076	0.1800	0.9409	0.1107	0.1717
TIIGIE-BIII	323.2151	331.2151	331.6361	341.6357	30.9238	194.9732	0.9998	2.2×10^{-16}
TII-EHL-TL-WL	482.1752	490.1755	490.5966	500.5962	0.0977	0.5078	0.5321	2.2×10^{-16}
TIIGIE-Lx	298.2180	306.2180	306.6391	316.6387	0.1594	1.2587	0.0961	0.3131
EHL-PGW-LLoG	286.7524	294.7524	295.1734	305.1730	0.1568	0.7964	0.1003	0.2664
OEHL-BXII	318.6305	326.6305	327.0515	337.0512	0.2041	1.4189	0.1301	0.0679

Figure 9: Fitted Densities and Probability Plots for the Carbon Fibres Data

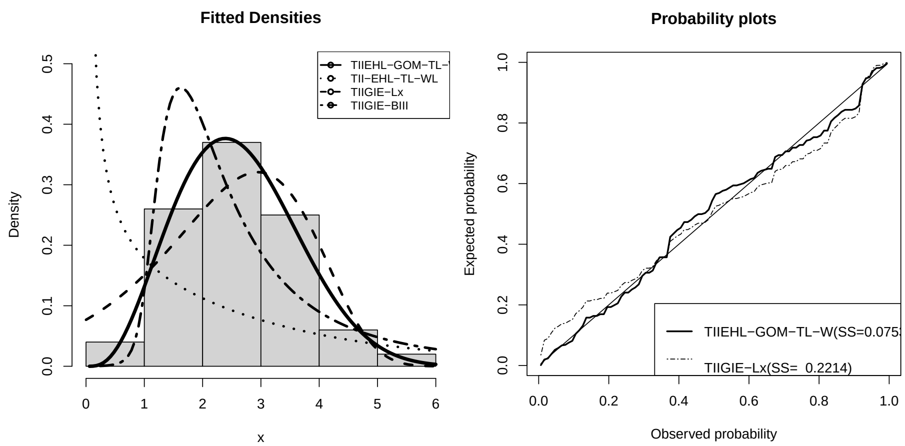
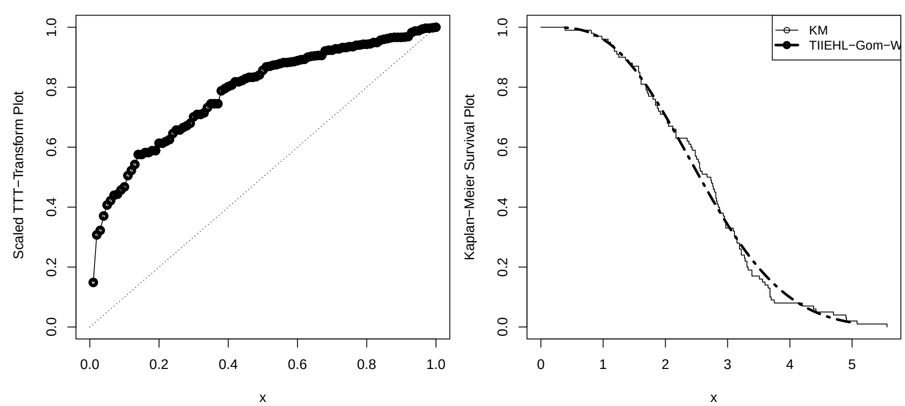


Figure 10: Fitted TTT and Kaplan-Meier Survival Plots for Carbon Fibres Data



number 6. It is available at: https://catalog.data.gov/dataset/ unemployment-insurance-data-july-2008-to-april-2013. The data are:
8.0, 18.2, 22.7, 16.3, 11.1, 26.9, 26.6, 18.9, 24.3, 20.6, 40.7, 56.9, 49.3, 55.0, 49.4, 50.7, 46.6, 87.9, 79.2, 78.9, 82.1, 63.7, 61.7, 68.8, 60.8, 69.2, 55.9, 54.8, 65.4, 53.7, 65.5, 52.7, 52.9, 50.7, 61.3, 49.6, 47.5, 58.9, 47.4, 56.0, 46.9, 46.5, 57.9, 45.7, 44.5, 53.1, 44.1, 41.8, 48.2, 37.1, 32.7, 37.6, 42.8, 47.4, 35.6, 32.2, 30.1, 31.2.
The estimated variance-covariance matrix for TIIEHL-Gom-TL-W model on insurance data is given by

$$\begin{bmatrix} 3.1456\times10^{-05} & 3.6150\times10^{-04} & 1.4184\times10^{-07} & -1.8823\times10^{-04} \\ 3.6150\times10^{-04} & 1.9416\times10^{-02} & 7.1434\times10^{-06} & -6.0484\times10^{-03} \\ 1.4184\times10^{-07} & 7.1434\times10^{-06} & 2.6315\times10^{-09} & -2.2548\times10^{-06} \\ -1.8823\times10^{-04} & -6.0484\times10^{-03} & -2.2548\times10^{-06} & 2.1369\times10^{-03} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by
$\alpha \in [0.0110 \pm 0.0109]$, $\gamma \in [0.4609 \pm 0.2731]$, $b \in [233.1400 \pm 0.0001]$ and $\lambda \in [0.4796 \pm 0.0906]$, respectively.
The MLEs of the parameters of the distributions (standard errors in parenthesis) and the goodness-of-fit statistics for the insurance data are given in Tables 7 and 8, respectively. The TIIEHL-Gom-TL-W distribution gives the smallest $AIC, AICC, BIC, K-S$, W^*, A^*, and the largest K-S p-value from the results in Table 8. These goodness-of-fit statistics indicate that the TIIEHL-Gom-TL-W distribution fits the data better than other competitive distributions for the insurance data.
Figure 12 shows the TTT scaled plots, observed and the fitted Kaplan-Meier plots. From the plots, it is clear that the TIIEHL-Gom-TL-W distribution best describes the insurance data. The TTT scaled plot demonstrates that the shape of the hazard rate function of the data set is increasing.

7.4 Actual taxes data

The data consists of the monthly actual taxes revenue in Egypt from January 2006 to November 2010. The data was analyzed by Mead (2016). The actual taxes revenue data (in 1000 million Egyptian pounds) are:
5.90, 20.4, 14.9, 16.2, 17.2, 7.80, 6.10, 9.20, 10.2, 9.60, 13.3, 8.50, 21.6, 18.5, 5.10, 6.70, 17.0, 8.60, 9.70, 39.2, 35.7, 15.7, 9.70, 10.0, 4.10, 36.0, 8.50, 8.00, 9.20, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.50, 10.6, 19.1, 20.5, 7.10, 7.70, 18.1, 16.5, 11.9, 7.0, 8.60, 12.5, 10.3, 11.2, 6.10, 8.40, 11.0, 11.6, 11.9, 5.20, 6.80, 8.90, 7.10, 10.8.

Table 7: Estimates of Models for Insurance Data

	Estimates			
Model	α	γ	b	λ
TIIEHL-Gom-TL-W	0.0110	0.4609	233.1400	0.4796
	(0.0056)	(0.1393)	(5.1299×10^{-05})	(0.0462)
TIIEHL-Gom-TL-W(1, γ, b, λ)	1	1.7316×10^{-09}	0.2268	0.0975
	-	(0.0142)	(0.0315)	(0.0116)
TIIEHL-Gom-TL-W(α, 1, b, λ)	12.2143	1	199.1155	0.1848
	(8.0295)	-	(0.8648)	(0.0124)
TIIEHL-Gom-TL-W(1, 1, b, λ)	1	1	301.1000	0.2713
	-	-	(6.8380×10^{-07})	(4.5902×10^{-03})
	λ	θ	c	k
TIIGIE-BIII	8.5208	2.1061×10^{03}	0.1436	14.5490
	(9.3495×10^{-06})	(1.8422×10^{-02})	(6.0839×10^{-04})	(3.3017×10^{-04})
	a	b	θ	λ
TII-EHL-TL-WL	1208.8000	89.2660	3.2663×10^{-08}	0.0893
	(0.4055)	(7.1292)	(0.0250)	(0.0069)
	λ	α	a	b
TIIGIE-Lx	0.0028	1.5944×10^{03}	0.0024	15.7860
	(0.0019)	(4.1258×10^{-10})	(2.6355×10^{-04})	(5.4894×10^{-07})
	α	β	δ	c
EHL-PGW-LLoG	2.8681	0.6125	24.2925	0.2458
	(0.0191)	(0.3272)	(12.6625)	(0.1286)
	α	λ	a	b
OEHL-BXII	0.6856	6.2764e-05	3.1619	0.7772
	(3.8214×10^{-10})	(8.7130×10^{-06})	(1.7428×10^{-09})	(7.0904×10^{-09})

Table 8: Goodness-of-Fit Statistics for Insurance Data

Model	Statistics							
	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TIIEHL-Gom-TL-W	494.9707	502.9704	503.7252	511.2122	0.0743	0.4149	0.0759	0.8919
TIIEHL-Gom-TL-W(1, γ, b, λ)	1163.6270	1169.6270	1170.0710	1175.8080	0.3754	1.9905	0.98976	2.2×10^{-16}
TIIEHL-Gom-TL-W(α, 1, b, λ)	505.4172	511.4172	511.8616	517.5985	0.2697	1.4074	0.1527	0.1335
TIIEHL-Gom-TL-W(1, 1, b, λ)	513.7312	517.7312	517.9494	521.8521	0.4000	2.1172	0.2058	0.0146
TIIGIE-BIII	521.5000	529.4999	530.2546	537.7416	17.1631	112.0893	0.9999	2.2×10^{-16}
TII-EHL-TL-WL	531.1409	539.1414	539.8961	547.3831	0.2695	1.4057	0.2668	0.0005
TIIGIE-Lx	495.6523	506.133	506.8877	514.3748	0.0803	0.5259	0.0814	0.8369
EHL-PGW-LLoG	521.8651	529.8651	530.6198	538.1069	0.5104	2.7334	0.1882	0.0327
OEHL-BXII	533.4179	541.4188	542.1736	549.6606	0.0896	0.4622	0.1246	0.3287

Figure 11: Fitted Densities and Probability Plots for Insurance Data

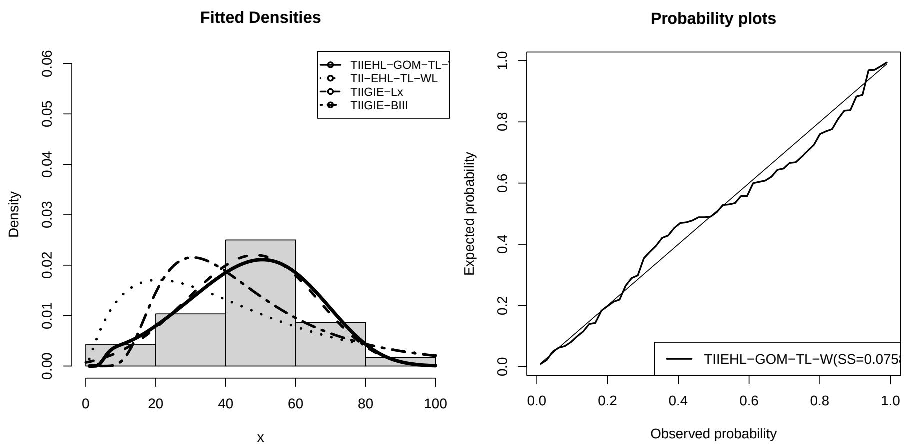
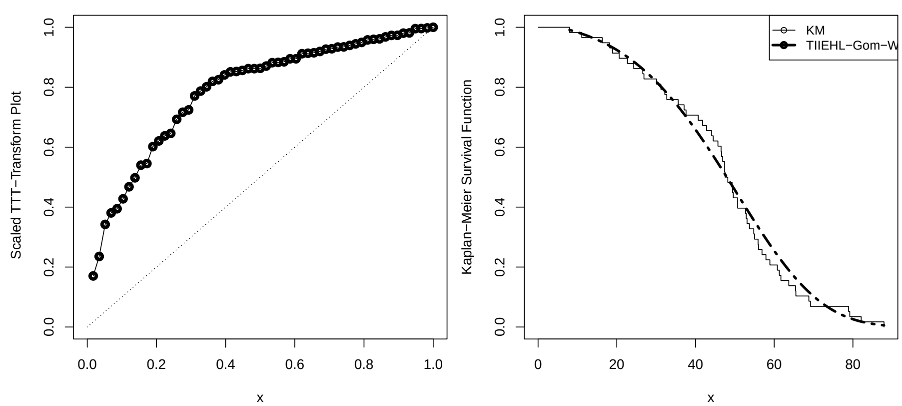


Figure 12: Fitted TTT and Kaplan-Meier Survival Plots for Insurance Data



The estimated variance-covariance matrix for TIIEHL-Gom-TL-W model on actual taxes data is given by

$$\begin{bmatrix} 3.8760\times 10^{-02} & -0.0118 & 6.5452\times 10^{-05} & -5.6791\times 10^{-03} \\ -1.1895\times 10^{-02} & 0.0107 & -1.7238\times 10^{-05} & 1.0625\times 10^{-03} \\ 6.5452\times 10^{-05} & -0.00001 & 1.1173\times 10^{-07} & -9.9543\times 10^{-06} \\ -5.6791\times 10^{-03} & 0.0010 & -9.9543\times 10^{-06} & 1.0474\times 10^{-03} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\alpha \in [0.4894 \pm 0.3858]$, $\gamma \in [0.1092 \pm 0.2028]$, $b \in [259.0900 \pm 0.0006]$ and $\lambda \in [0.4700 \pm 0.0634]$, respectively.

The MLEs of the parameters of the TIIEHL-Gom-TL-W distribution, its nested models, and the competing models (standard errors in parenthesis) and the goodness-of-fit statistics for the actual taxes data are given in Tables 9 and 10. These results show that the TIIEHL-Gom-TL-W distribution provides a significantly better fit than the nested and non-nested models. This can be confirmed by the smallest values of goodness-of-fit statistics: $AIC, AICC, BIC, K-S$, W^*, A^* and largest p-value for the TIIEHL-Gom-TL-W distribution as compared to other fitted distributions.

Figure 14 shows the TTT scaled plots, observed and the fitted Kaplan-Meier curves. We can see that the TIIEHL-Gom-TL-W distribution follows the Kaplan-Meier survival curves very closely. The TTT scaled plot shows an increasing hazard rate function for the actual taxes data.

Figure 13: Fitted Densities and Probability Plots for Actual Taxes Data

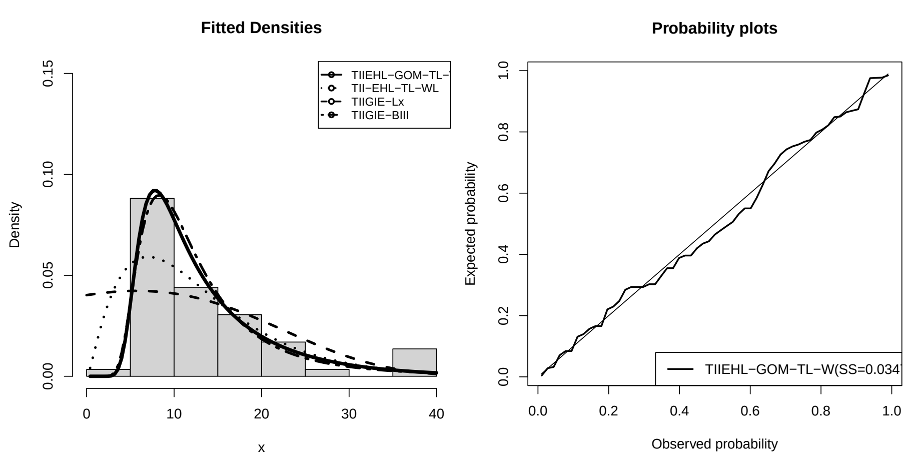


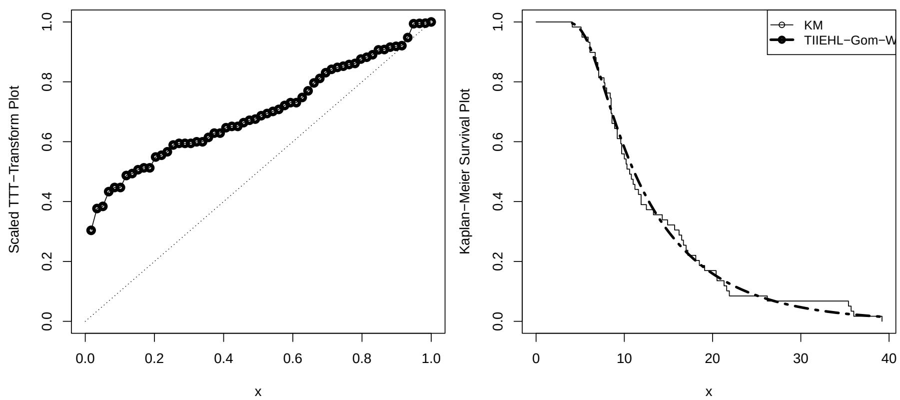
Table 9: Estimates of Models for Actual Taxes Data

	Estimates			
Model	α	γ	b	λ
TIIEHL-Gom-TL-W	0.4894	0.1092	259.0900	0.4700
	(0.1968)	(0.1034)	(3.3426×10^{-04})	(0.0323)
TIIEHL-Gom-TL-W(1, γ, b, λ)	1	1.1902×10^{-09}	0.2236	0.1137
	-	(0.0077)	(0.0296)	(0.1366)
TIIEHL-Gom-TL-W(α, 1, b, λ)	0.0033	1	0.5510	0.3373
	(0.0010)	-	(2.7525×10^{-06})	(0.0158)
TIIEHL-Gom-TL-W(1, 1, b, λ)	1	1	91.8397	0.30569
	-	-	(15.8987)	(0.0136)
	λ	θ	c	k
TIIGIE-BIII	127.5000	42.7280	0.2375	4.8342
	(9.3235×10^{-04})	(5.7923×10^{-03})	(2.1019×10^{-02})	(1.8493×10^{-01})
	a	b	θ	λ
TII-EHL-TL-WL	130.1300	73.3480	2.2988×10^{-08}	0.1025
	(0.2011)	(4.5136)	(0.0219)	(0.0079)
	λ	α	a	b
TIIGIE-Lx	0.6590	2.4030×10^{03}	0.0064	254.9100
	(0.2783)	(1.3936×10^{-06})	(1.4510×10^{-03})	(1.2842×10^{-04})
	α	β	δ	c
EHL-PGW-LLoG	5.9463	0.0807	7.1952	1.1892
	(0.8745)	(0.5230)	(1.6057)	(7.5270)
	α	λ	a	b
OEHL-BXII	0.4385	0.0005	3.4796	0.7169
	(0.1324)	(0.0011)	(0.0277)	(0.1441)

Table 10: Goodness-of-Fit Statistics for Actual Taxes Data

	Statistics							
Model	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
TIIEHL-Gom-TL-W	375.4370	383.4370	384.1778	391.7472	0.0354	0.2371	0.0599	0.9838
TIIEHL-Gom-TL-W(1, γ, b, λ)	994.4397	1000.4390	1000.8750	1006.6720	0.1192	0.7083	0.9887	2.2×10^{-16}
TIIEHL-Gom-TL-W(α, 1, b, λ)	407.9135	413.9135	414.3499	420.1462	2.4042	12.8609	0.1596	0.0989
TIIEHL-Gom-TL-W(1, 1, b, λ)	381.8731	385.8731	386.0874	390.0282	0.1385	0.8905	0.1344	0.2368
TIIGIE-BIII	376.5826	384.5826	385.3234	392.8928	20.0576	118.3375	0.9935	2.2×10^{-16}
TII-EHL-TL-WL	393.2713	401.2717	402.0124	409.5818	0.1909	1.1810	0.1643	0.0827
TIIGIE-Lx	409.5538	417.5538	418.2945	425.8639	0.4130	2.6630	0.1945	0.0230
EHL-PGW-LLoG	377.9272	385.9272	386.6680	394.2374	0.0786	0.4541	0.0829	0.8121
OEHL-BXII	428.8780	436.8780	437.6187	445.1881	0.4909	3.1437	0.1862	0.0334

Figure 14: Fitted TTT and Kaplan-Meier Survival Plots for Actual Taxes Data



7.5 Likelihood ratio test

The likelihood ratio test results for comparing the full and nested models are given in this section.

Table 11: Likelihood ratio test results (part 1)

		Turbocharger Data	Carbon Fibres Data
Model	df	$\chi^2(p-value)$	$\chi^2(p-value)$
TIIEHL-Gom-TL-W$(1, \gamma, b, \lambda)$	1	12.0744(0.0005)	11.0914(0.0008)
TIIEHL-Gom-TL-W$(\alpha, 1, b, \lambda)$	1	7.8312(0.0051)	13.0894(0.0003)
TIIEHL-Gom-TL-W$(1, 1, b, \lambda)$	2	19.5848(0.0001)	6.0757(0.0479)

Table 12: Likelihood ratio test results (part 2)

		Insurance Data	Actual Taxes Data
Model	df	$\chi^2(p-value)$	$\chi^2(p-value)$
TIIEHL-Gom-TL-W$(1, \gamma, b, \lambda)$	1	668.6563($<$0.00001)	619.0027($<$0.00001)
TIIEHL-Gom-TL-W$(\alpha, 1, b, \lambda)$	1	10.4465(0.0012)	32.4765($<$0.00001)
TIIEHL-Gom-TL-W$(1, 1, b, \lambda)$	2	18.7605(0.00002)	6.4361(0.0112)

The likelihood ratio test results in Tables 11 and 12 indicate that the TIIEHL-Gom-TL-W performs better than its nested models at 5% level of significance, since all p-values are less than 0.05 for all the data sets considered.

8 Concluding remarks

We have proposed and studied a new type II exponentiated half-logistic Gompertz-Topp-Leone-G (TIIEHL-Gom-TL-G) family of distributions. Some properties of our proposed TIIEHL-Gom-TL-G family of distributions were derived. The estimation of parameters was obtained by maximum likelihood method and evaluated via a simulation study. Four applications are provided to assess the flexibility of the TIIEHL-Gom-TL-G family of distributions, and reveal better fits to real data than several other well-known models.

Acknowledgements

The authors are grateful to the editor and referees for their useful comments on an earlier version of this manuscript which led to this improved version.

References

[1] Afify A. Z., Al-Mofleh H., Dey S., (2021), Topp-Leone Odd Log-Logistic Exponential Distribution: Its Improved Estimators and Applications, *Anais da Academia Brasileira de Ciências* 93(4), 1–14.

[2] Ahmad Z., (2020), The Zubair-G Family of Distributions: Properties and Applications, *Annals of Data Science* 7(2), 195–208.

[3] Alizadeh M., Cordeiro G. M., Pinho L. G. B., Ghosh I., (2017), The Gompertz-G Family of Distributions, *Journal of Statistical Theory and Practice* 11(1), 179–207.

[4] Al-Babtain A. A., Elbatal I., Chesneau C., Elgarhy M., (2020), Sine Topp-Leone-G Family of Distributions: Theory and applications, *Open Physics* 18(1), 574–593.

[5] Al-Babtain A. A., Elbatal I., Al-Mofleh H., Gemeay A. M., Afify A. Z., Sarg A. M., (2021), The Flexible Burr X-G Family: Properties, Inference, and Applications in the Engineering Science, *Symmetry* 13, 474.

[6] Aldahlan M., Afify A. Z., (2018), The Odd Exponentiated Half-Logistic Burr XII Distribution, *Pakistan Journal of Statistics and Operation Research* 14(2), 305–317.

[7] Al-Marzouki S., Jamal F., Chesneau C., Elgarhy M., (2020), Topp-Leone Odd Fréchet Generated Family of Distributions with Applications to Covid-19 Datasets, *Computational Modelling in Engineering and Sciences* 125, 437–458.

[8] Al-Mofleh H., Elgarhy M., Afify A., Zannon M., (2020), Type II Exponentiated Half Logistic Generated Family of Distributions with Applications, *Electronic Journal of Applied Statistical Analysis* 13(2), 536–561.

[9] Al-Shomrani A., Arif O., Shawky A., Hanif S., Shahbaz M.Q., (2016), Topp Leone Family of Distributions: Some Properties and Application, *Pakistan Journal of Statistics and Operation Research* 12(3), 443–451.

[10] Balogun O. S., Arshad M. Z., Iqbal M. Z., Ghamkhar M., (2021), A New Modified Lehmann Type II G Class of Distributions: Exponential Distribution With Theory, Simulation, and Applications to Engineering Sector, *F1000Research* 10(483), 483.

[11] Chambers J., Cleveland W., Kleiner B., Tukey J., (1983), Graphical Methods for Data Analysis, Chapman and Hall, London.

[12] Chen G., Balakrishnan N., (1995), A General Purpose Approximate Goodness-of-fit Test, *Journal of Quality Technology* 27, 154–161.

[13] El-Sherpieny E. S. A., Elsehetry M. M., (2019), Type II Kumaraswamy Half-Logistic Family of Distributions with Applications to Exponential Model, *Annals of Data Science* 6(1), 1–20.

[14] Elgarhy M., Nasir A., Jamal F., Kadilar G. O., (2018), Type II Topp-Leone Generated Family of Distributions: Properties and Applications, *Journal of Statistics and Management Systems* 21(8), 1529–1551.

[15] Eghwerido J. T., Oguntunde P. E., Agu F. I., (2021), The Alpha Power Marshall-Olkin-G Distribution: Properties, and Applications, *Sankhya A*, 1-26.

[16] Hamedani G. G., Rasekhi M., Najibi S., Yousof H. M., Alizadeh M., (2019), Type II General Exponential Class of Distributions, *Pakistan Journal of Statistics and Operation Research*, 503–523.

[17] Jamal F., Chesneau C., Elgarhy M., (2020), Type II General Inverse Exponential Family of Distributions, *Journal of Statistics and Management Systems* 23(3), 617–641.

[18] Khaleel M. A., Oguntunde P. E., Al Abbasi J. N., Ibrahim N. A., AbuJarad M. H., (2020), The Marshall-Olkin Topp Leone-G Family of Distributions: A Family for Generalizing Probability Models, *Scientific African* 8, e00470.

[19] Mead M. E., (2016), On Five-parameter Lomax Distribution: Properties and Applications, *Pakistan Journal of Statistics and Operation Research* 12, 185-199.

[20] Moakofi T., Oluyede B., Chipepa F., (2021), Type II Exponentiated Half-Logistic-Topp-Leone-G Power Series Class of Distributions with Applications, *Pakistan Journal of Statistics and Operation Research* 17(4), 885–909.

[21] Moakofi T., Oluyede B., Chipepa F., Makubate B., (2021), Odd Power Generalized Weibull-G Family of Distributions: Properties and Applications, *Journal of Statistical Modelling: Theory and Applications* 2(1), 121–142.

[22] Nichols, M. D. and Padgett, W. J. A., (2006), A Bootstrap Control Chart for Weibull Percentiles, *Quality and Reliability Engineering International* 22(2), 141–151.

[23] Oluyede B., Chipepa F., Wanduku D., (2020), The Exponentiated Half Logistic-Power Generalized Weibull-G Family of Distributions: Model, Properties and Applications, *Eurasian Bulletin of Mathematics* 3(3), 134–161.

[24] Rényi A., (1960), On Measures of Entropy and Information, *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability* 1, 547–561.

[25] Soliman A. H., Elgarhy M. A. E., Shakil M., (2017), Type II Half Logistic Family of Distributions with Applications, *Pakistan Journal of Statistics and Operation Research* 13, 245–264.

[26] Tahir M. H., Hussain M. A., Cordeiro G. M., El-Morshedy M., Eliwa M. S., (2020), A New Kumaraswamy Generalized Family of Distributions with Properties, Applications, and Bivariate Extension, *Mathematics* 8(11), 1989.

[27] Xu K., Xie M., Tang L.C., Ho S. L., (2003), Application of Neural Networks in Forecasting Engine Systems Reliability, *Applied Software Computing* 2(4), 255–268.

Appendix

The first derivative of the log-likelihood function ($\ell_n(\boldsymbol{\Delta})$) is given by

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \ln\left(1 + \left(1 - \exp\left(\frac{1}{\gamma}\left(1 - \left[1 - \left(1 - \overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\right)\right) + \\ + \frac{1}{\gamma}\sum_{i=1}^{n}\left(1 - \left[1 - \left(1 - \overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right),$$

$$
\begin{aligned}
\frac{\partial \ell}{\partial \gamma} &= \frac{n}{\gamma} - (\alpha+1)\sum_{i=1}^{n} \frac{\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)}{\left(1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\right)\right)} \times \\
&\times \left[\gamma^{-2}\left(1-\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right) + \right. \\
&\left. + \frac{1}{\gamma}\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right]^{-\gamma} \ln\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right]\right] + \\
&- \sum_{i=1}^{n} \ln\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right] + \\
&+ \sum_{i=1}^{n} \alpha\gamma^{-2}\left(1-\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right) + \\
&+ \frac{\alpha}{\gamma}\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right]^{-\gamma} \ln\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \ell}{\partial b} &= \sum_{i=1}^{n} \frac{\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right]^{-\gamma-1}}{\left(1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\right)\right)} \times \\
&\times (\alpha+1)\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b \ln\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right) + \frac{n}{b} + \\
&- (-\gamma-1)\sum_{i=1}^{n} \frac{\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b \ln\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)}{\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right]} + \sum_{i=1}^{n} \ln\left[1-\overline{G}^2(x_i;\boldsymbol{\xi})\right] + \\
&- \frac{\alpha}{\gamma}\sum_{i=1}^{n}\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right]^{-\gamma-1}\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b \ln\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right),
\end{aligned}
$$

$$
\frac{\partial \ell}{\partial \xi_k} = \sum_{i=1}^{n} \frac{\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right]^{-\gamma-1}}{\left(1+\left(1-\exp\left(\frac{1}{\gamma}\left(1-\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^b\right]^{-\gamma}\right)\right)\right)\right)} \times
$$

$$\times 2b(\alpha+1)\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^{b-1}\overline{G}(x_i;\boldsymbol{\xi})\frac{\partial\overline{G}(x_i;\xi)}{\partial\xi_k}+$$

$$+(-\gamma-1)\sum_{i=1}^{n}\frac{2b\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^{b-1}\overline{G}(x_i;\boldsymbol{\xi})\frac{\partial\overline{G}(x_i;\xi)}{\partial\xi_k}}{\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^{b}\right]}+$$

$$+(b-1)\sum_{i=1}^{n}\frac{2\overline{G}(x_i;\boldsymbol{\xi})\frac{\partial\overline{G}(x_i;\xi)}{\partial\xi_k}}{\left[1-\overline{G}^2(x_i;\boldsymbol{\xi})\right]}+$$

$$+\alpha\sum_{i=1}^{n}\left[1-\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^{b}\right]^{-\gamma-1}2b\left(1-\overline{G}^2(x_i;\boldsymbol{\xi})\right)^{b-1}\overline{G}(x_i;\boldsymbol{\xi})\frac{\partial\overline{G}(x_i;\xi)}{\partial\xi_k}+$$

$$+\sum_{i=1}^{n}\frac{\frac{\partial\overline{G}(x_i;\xi)}{\partial\xi_k}}{\overline{G}(x_i;\boldsymbol{\xi})}+\sum_{i=1}^{n}\frac{\frac{\partial g(x_i;\xi)}{\partial\xi_k}}{(g(x_i;\boldsymbol{\xi}))}.$$