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On the minimum energy compensation for linear time-varying disturbed systems

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We consider in this work a class of finite dimensional time-varying linear disturbed systems. The main objective of this work is to studied the optimal control which ensures the remediability of a disturbance of time-varying disturbed systems. The remediability concept consist to find a convenient control which bringing back the corresponding observation of disturbed system to the normal one at the final time. We give firstly some characterisations of compensation and in second party we find a control which annul the output of the system and we show also that the Hilbert Uniqueness Method can be used to solve the optimal control which ensure the remediability. A general approach was given to minimize the linear quadratic problem. Examples and numerical simulations are given.

Key words: dynamical systems, remediability, observation, optimal control, disturbance

1. Introduction

Disturbances can cause serious damage to the dynamic system, its disturbances can be caused by infections, radiations or pollutions. Studies of disturbed systems have continued to grow in importance in recent years. Unknown disturbances are detected by observation and several works have been devoted to their detection and reconstruction from the corresponding observation (see [6, 9, 12, 13, 17, 21, 22, 24, 26]).

Though, the detection of a disturbance is generally insufficient, it is however necessary to act by means of controls to attenuate the impact of the disturbances on the system. The notion of remediability consists in studying the existence of an adequate control ensuring the compensation of possible disturbances by attenuating it, and this by bringing back the observation of the disturbed system towards its state without disturbance.

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The concepts of remediability are developed and treated firstly for a class of parabolic systems in the case of a finite time horizon, and hence for discrete systems, hyperbolic systems, regional and asymptotic cases, we can see [1–5, 7].

In [25] the authors was defined the gradient remediability of distributed parabolic systems and the relationship with the gradient controllability. In [23] the problem of regional remediability for a class of nonlinear distributed systems was studied, this problem was solved by the fixed-point theorem and the pseudo inverse techniques. Also the case of multiple input delays for a class of distributed systems was described in [27]. In [29] the authors was studied the remediability problem for a class of discrete delayed systems with application to the discrete version of the wave equation. Also the remediability problem for a category of hyperbolic perturbed systems with the two case constant and time-varying delays is described in [28]. And in [8] the authors studied the possibility of finite time or asymptotic compensation of disturbances for a class of linear lumped systems.

In the case of finite dimensional linear time-varying systems the remediability is not yet discussed. The goal of this paper is to discus the optimal control problem of remidiability for such systems. The quadratic control problem has been the subject of different works for a variety of continuous, discrete, linear, nonlinear systems, see as examples [11, 14, 15, 19, 20]. In this work we investigate a optimal control which make the linear time-varying disturbed system remediably. The minimum energy compensation for discrete delayed systems with disturbances has been studied in [29] and a cheap controls for disturbances compensation in hyperbolic delayed systems is described in [28] and with multi-input delays in [27]. In this paper we give necessary and sufficient conditions for the remediability for time-varing system, and we find a control which annul the output of the system. In second part of this work we show also that the Hilbert Uniqueness Method can be used to solve the optimal control which ensure the remediability. And a general approach was given to minimize the linear quadratic problem. To illustrate our work several examples and numerical simulations are given.

This work is organized as follows: In section 2, we introduce the considered model of perturbed time varying systems and we define the problem statement and we give some results to characterize the remediability of system. In section 3, the minimum energy problem are described and some numerical simulation are given to illustrate the obtained results. Finally, a conclusion is summarized in section 4.

2. Problem statement

In this work, we consider a class of finite dimension time-varying control systems described by a linear state equation as follows:

$$
\begin{aligned}
&\dot{z}(t) = A(t)z(t) + B(t)u(t) + f(t), \quad 0 < t < T, \\
&z(0) = z_0\,,
\end{aligned}
\tag{1}
$$

where $A \in C^\infty([0,T], M_n(\mathbb{R}))$, $B \in C^\infty([0,T], M_{n,p}(\mathbb{R}))$, $u \in L^2(0,T;\mathbb{R}^p)$ and $f \in L^2(0,T;\mathbb{R}^n)$.

The system (1) is augmented by the output equation:

$$y(t) = C(t)z(t), \quad 0 < t < T \tag{2}$$

with $C \in C^\infty([0,T], M_{q,n}(\mathbb{R}))$. We have

$$z(t) = R(t,0)z_0 + H_t u + G_t f \,,$$

where R is the resolvent of the time-varying linear system $\dot{x} = A(t)x$. Then

$$y(t) = C(t)R(t,0)z_0 + C(t)H_t u + C(t)G_t f \,,$$

where H_t and G_t are the operators defined by

$$\begin{aligned} H_t\colon\ L^2(0,t;\mathbb{R}^p) &\longrightarrow \mathbb{R}^n \\ u &\longrightarrow \int_0^t R(t,s)B(s)u(s)\,\mathrm{d}s \end{aligned} \tag{3}$$

and

$$\begin{aligned} G_t\colon\ L^2(0,t;\mathbb{R}^n) &\longrightarrow \mathbb{R}^n \\ f &\longrightarrow \int_0^t R(t,s)f(s)\,\mathrm{d}s. \end{aligned} \tag{4}$$

In the case without disturbance and control, i.e. $f = 0$ and $u = 0$, the observation is given by

$$y_{0,0}(t) = C(t)R(t,0)z_0 \,.$$

But if the system is disturbed by a term f, the observation becomes

$$y_{0,f}(t) = C(t)R(t,0)z_0 + \int_0^t C(t)R(t,s)f(s)\,\mathrm{d}s \neq C(t)R(t,0)z_0.$$

Then we introduce a control term Bu in order to reduce the effect of this disturbance at final time T, i.e. $y_{u,f}(T) = y_{0,0}(T)$.

Definition 1 *The system (1) augmented with the output (2), or (1) + (2) is said to be remediable on $[0,T]$, if for any $f \in L^2(0,T;\mathbb{R}^n)$, there exists a control $u \in L^2(0,T;\mathbb{R}^p)$ such that*

$$C(T)H_T u + C(T)G_T f = 0.$$

We have the following characterization result.

Proposition 1 *The following properties are equivalent*

i) (1) + (2) *is remediable on* $[0,T]$*;*

ii) $\mathrm{Im}(C(T)G_T) \subset \mathrm{Im}(C(T)H_T)$*;*

iii) $\mathrm{Im}(C(T)H_T) = \mathrm{Im}(C(T))$*;*

iv) $\mathrm{Ker}(H_T^*C(T)^*) = \mathrm{Ker}(G_T^*C(T)^*)$*;*

v) $\mathrm{Ker}(H_T^*C(T)^*) = (\mathrm{Im}(C(T)))^\perp$*;*

vi) $\mathrm{Ker}(B^*G_T^*C(T)^*) = \mathrm{Ker}(G_T^*C(T)^*)$*;*

vii) There exists $\gamma > 0$ *such that for every* $\theta \in \mathbb{R}^q$*, we have*

$$\|R(T,.)^*C(T)^*\theta\|_{L^2(0,T;\mathbb{R}^n)} \leqslant \gamma\, \|B(.)^*R(T,.)^*C(T)^*\theta\|_{L^2(0,T;\mathbb{R}^p)}. \tag{5}$$

Proof. Derive from the definition and the fact that

$$\mathrm{Ker}(H_T^*C(T)^*) = \mathrm{Ker}(B(.)^*R(T,.)^*C(T)^*),$$
$$\mathrm{Ker}(G_T^*C(T)^*) = \mathrm{Ker}(R(T,.)^*C(T)^*)$$

and also the result [10]. □

Let us now define the remediability Gramian of the system (1) + (2).

Definition 2 *Let* $q > 1$*, the remediability Gramian of the system* (1) + (2) *is the symmetric* $q \times q$*-matrix*

$$\Theta(T) = C(T)H_TH_T^*C(T)^* = \int_0^T C(T)R(T,s)B(s)B(s)^*R(T,s)^*C(T)^*\,\mathrm{d}s. \tag{6}$$

Remark 1 *Note that, for every* $\Psi \in \mathbb{R}^q$*, we have*

$$\Psi^*\bar{\Theta}(T)\Psi = \int_0^T \|B(s)^*R(T,s)^*C(T)^*\Psi\|^2\,\mathrm{d}s.$$

Hence the remediability Gramian $\bar{\Theta}(T)$ *is a nonnegative symmetric matrix.*

3. Minimum energy problem

3.1. The optimal control

Let $\bar{z} \in C^0(0,T;\mathbb{R}^n)$ be the solution of the Cauchy problem

$$\begin{aligned} \dot{\bar{z}}(t) &= A(t)\bar{z}(t) + B(t)\bar{u}(t) + f(t), \quad 0 < t < T, \\ \bar{z}(0) &= z_0 \end{aligned} \tag{7}$$

and the system (7) is augmented by the output equation

$$\bar{y}(t) = C(t)\bar{z}(t), \quad 0 < t < T. \tag{8}$$

We assume (7) + (8) is remediable on $[0,T]$, and let $\bar{u} \in L^2(0,T;\mathbb{R}^p)$ be defined by:

$$\bar{u}(s) = B(s)^* R(T,s)^* C(T)^* \bar{\Theta}(T)^{-1}(-C(T)G_T f), \quad s \in [0,T]. \tag{9}$$

Then

$$\begin{aligned} \bar{y}(T) &= C(T)R(T,0)z_0 \\ &+ \int_0^T C(T)R(T,s)B(s)B(s)^* R(T,s)^* C(T)^* \bar{\Theta}(T)^{-1}(-C(T)G_T f)\,\mathrm{d}s \\ &+ C(T)G_T f = C(T)R(T,0)z_0. \end{aligned}$$

We have the following result of uniqueness.

Proposition 2 *Let $(z, z_0) \in \mathbb{R}^n \times \mathbb{R}^n$ and let $u \in L^2(0,T;\mathbb{R}^p)$ be such that the solution of the Cauchy problem*

$$\begin{aligned} \dot{z}(t) &= A(t)z(t) + B(t)u(t) + f(t), \quad 0 < t < T, \\ z(0) &= z_0. \end{aligned} \tag{10}$$

The system (10) is augmented by the output equation

$$y(t) = C(t)z(t), \quad 0 < t < T. \tag{11}$$

We assume that (10) + (11) is remediable on $[0,T]$, and satisfies

$$y(T) = C(T)R(T,0)z_0.$$

Then

$$\int_0^T \|\bar{u}(s)\|^2 \mathrm{d}s \leqslant \int_0^T \|u(s)\|^2 \mathrm{d}s$$

with equality if and only if

$$u(s) = \bar{u}(s) \quad \textit{for almost every } s \in [0,T].$$

Proof. Let $v = u - \bar{u}$. Then, $\bar{z}$ and z being the solutions of the Cauchy problems (7) and (10), respectively, one has

$$\begin{aligned} C(T)\int_0^T R(T,s)B(s)v(s)\,\mathrm{d}s &= C(T)\int_0^T R(T,s)B(s)u(s)\,\mathrm{d}s \\ &\quad - C(T)\int_0^T R(T,s)B(s)\bar{u}(s)\,\mathrm{d}s \\ &= (C(T)H_T u) - (C(T)H_T\bar{u})\,. \end{aligned}$$

Hence

$$\begin{aligned} C(T)\int_0^T R(T,s)B(s)v(s)\,\mathrm{d}s &= (y(T) - C(T)R(T,0)z_0 - C(T)G_T f) \\ &\quad - (\bar{y}(T) - C(T)R(T,0)z_0 - C(T)G_T f) \\ &= 0. \end{aligned} \tag{12}$$

We have

$$\int_0^T \|u(s)\|^2\mathrm{d}s = \int_0^T \|\bar{u}(s)\|^2\mathrm{d}s + \int_0^T \|v(s)\|^2\mathrm{d}s + 2\int_0^T \bar{u}^*(s)v(s)\,\mathrm{d}s. \tag{13}$$

From (9) (note also that $\Theta(T)^* = \Theta(T)$),

$$\int_0^T \bar{u}^*(s)v(s)\,\mathrm{d}s = (-C(T)G_T f)^*\bar{\Theta}(T)^{-1}\int_0^T C(T)R(T,s)B(s)v(s)\,\mathrm{d}s,$$

which, together with (12), gives

$$\int_0^T \bar{u}^*(s)v(s)\,\mathrm{d}s = 0. \tag{14}$$

Then Proposition 2 follows from (13) and (14). □

3.1.1. Numerical simulations

Let us define A where $n = 2$ by

$$A(t) = \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix},$$

compute the resolvent of $\dot{z}(t) = A(t)z(t)$.

One has

$$R(T,t) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{T^2-t^2}{2}} \end{pmatrix}.$$

We consider the case where $p = q = 2$ and

$$B(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The disturbance term is given as follows

$$f(t) = \begin{pmatrix} 0 \\ e^{\frac{t^2}{2}} \end{pmatrix}.$$

Using proposition 2, one gets

$$u(t) = \begin{pmatrix} 0 \\ \dfrac{-20e^{\frac{-t^2}{2}}}{\sqrt{\pi}\,\mathrm{erf}(10)} \end{pmatrix},$$

with the error function (also called the Gauss error function), often denoted by erf such that

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int\limits_0^x e^{-t^2}\,\mathrm{d}t.$$

The initial state is considered null $z_0 = 0$, then $y_{(0,0)} = 0$. Then

$$y_{(u,f)}(t) = \begin{pmatrix} 0 \\ \dfrac{-10e^{\frac{t^2}{2}} erf(t)}{\mathrm{erf}(10)} + te^{\frac{t^2}{2}} \end{pmatrix},$$

$$y_{(0,f)}(t) = \begin{pmatrix} 0 \\ te^{\frac{t^2}{2}} \end{pmatrix}.$$

We obtain the following numerical results which illustrate the previous developments.

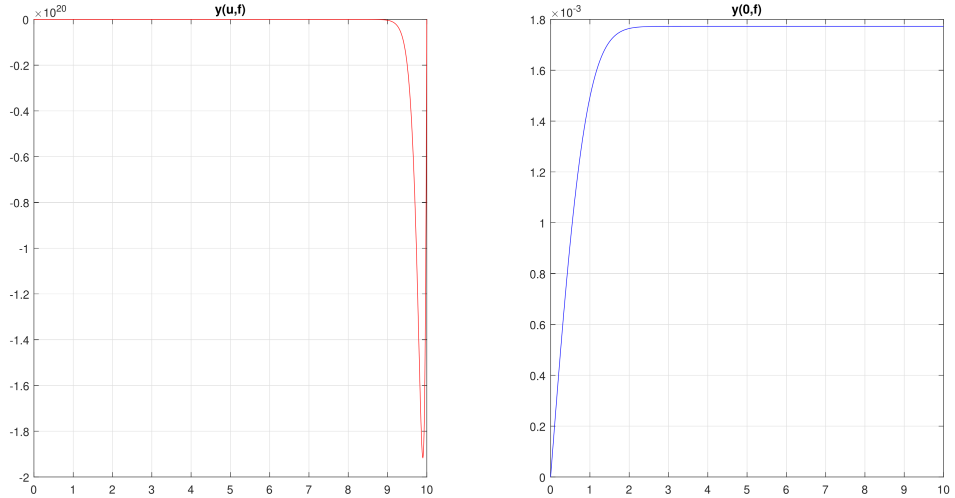

Figure 1: Representation of $y_{(u,f)}$ and $y_{(0,f)}$

Hence, in Figure 1, we give the representation of the observations $y_{(u,f)}$ and $y_{(0,f)}$. This figure show the effect of our control, which bringing back the output the normal one at time $T = 10$. i.e., $y_{(u,f)}(T) = y_{(0,0)}(T) = 0$.

In Figure 2, we give the representation of the optimal control u.

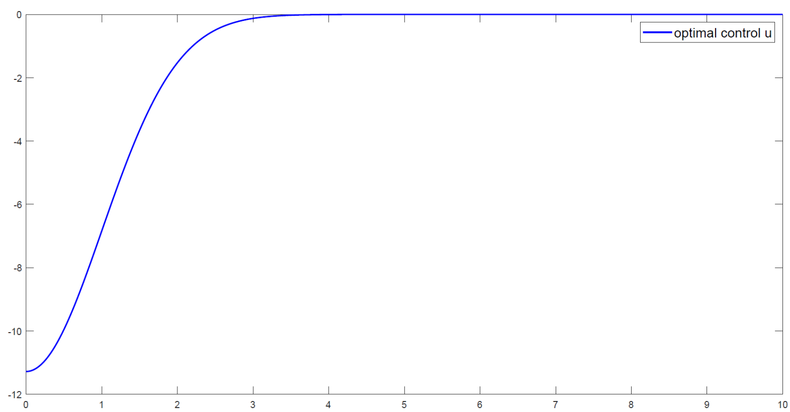

Figure 2: Representation of the optimal control u

3.2. HUM method

In this section we give an other method for the optimal control ensuring the compensation by the Hilbert uniqueness method. The HUM method introduced by J.L. Lions [20] was generalized for different systems [16] and [18].

We consider the following minimum energy problem:

For $z_0 \in \mathbb{R}^n$ and $f \in L^2(0,T;\mathbb{R}^n)$, does there exist an optimal control $u \in L^2(0,T;\mathbb{R}^p)$ such that

$$C(T)H_T u + C(T)G_T f = 0,$$

i.e. minimizing the function $J(v) = \|v\|^2$ on the set $\{v \in L^2(0,T;\mathbb{R}^p) \mid C(T)H_T v + C(T)G_T f = 0\}$?

For this, we use an extension of the Hilbert Uniqueness Method. Indeed, for $\theta \in \mathbb{R}^q$, let us note:

$$\|\theta\|_* = \left(\int_0^T \|(H_T)^* C(T)^*\theta\|^2_{\mathbb{R}^p} \,\mathrm{d}s\right)^{\frac{1}{2}} = \left(\int_0^T \|B(T)^* R(T,s)^* C(T)^*\theta\|^2_{\mathbb{R}^p} \,\mathrm{d}s\right)^{\frac{1}{2}}$$

$\|\theta\|_*$ is a semi-norm on $\mathbb{R}^q$.

We assume that $\|.\|_*$ is a norm on $\mathbb{R}^q$. If $\mathrm{Ker}\,[(H_T)^* C(T)^*] = \{0\}$, this is equivalent to the remediability of the system (1) + (2) on $[0,T]$. The corresponding inner product is given by:

$$< \theta, \sigma >_* = \int_0^T < B(T)^* R(T,s)^* C(T)^*\theta, B(T)^* R(T,s)^* C(T)^*\sigma > \mathrm{d}s$$

and the operator $\Lambda : \mathbb{R}^q \longrightarrow \mathbb{R}^q$ defined by

$$\Lambda\theta = C(T)H_T(H_T)^* C(T)^*\theta = \int_0^T C(T)R(T,s)B(T)B(T)^* R(T,s)^* C(T)^*\theta \,\mathrm{d}s$$

is symmetric and positive definite and then invertible. We give hereafter the expression of the optimal control ensuring the compensation of a disturbance f at the final time T.

Proposition 3 *For $f \in L^2(0,T;\mathbb{R}^n)$, there exists a unique $\theta_f \in Y^q$ such that*

$$\Lambda\theta_f = -C(T)G_T f$$

and the control

$$u_{\theta_f}(.) = B(T)^* R(T,.)^* C(T)^*\theta_f$$

verify

$$C(T)H_T u_{\theta_f} + C(T)G_T f = 0.$$

Moreover, it is optimal and

$$\|u_{\theta_f}\|_{L^2(0,T;\mathbb{R}^p)} = \|\theta_f\|_*.$$

Proof. Λ extends uniquely into an isomorphism such that:

1. $< \Lambda\theta, \sigma >_{\mathbb{R}^q} = < \theta, \sigma >_*$.

2. $\|\Lambda\theta\|_{\mathbb{R}^q} = \|\theta\|_*$.

In particular, if $-C(T)G_T f \in \mathbb{R}^q$, then $\exists! \theta_f \in \mathbb{R}^q$ such that

$$\Lambda\theta_f = -C(T)G_T f$$

one gets

$$\int_0^T C(T)R(T,s)B(T)B(T)^* R(T,s)^* C(T)^* \theta_f \, \mathrm{d}s = -C(T)G_T f$$

from which we have

$$\int_0^T C(T)R(T,s)B(T)u_{\theta_f}(s) \, \mathrm{d}s = -C(T)G_T f$$

$$\text{with } u_{\theta_f}(s) = B(T)^* R(T,s)^* C(T)^* \theta_f$$

that implies

$$C(T)H_T u_{\theta_f} = -C(T)G_T f.$$

We have

$$C(T)H_T u_{\theta_f} + C(T)G_T f = 0$$

then $u_{\theta_f} \in \{v \in L^2(0,T;\mathbb{R}^p) \mid C(T)H_T v + C(T)G_T f = 0\}$.

By assumptions of Lax-Milgram theorem, $J(v) = a(v,v) - 2L(v)$ has a unique minimum u^* such that

$$< u^*, v - u^* > \geqslant L(v - u^*), \ \forall v \in \{v \in L^2(0,T;\mathbb{R}^p) \mid C(T)H_T v + C(T)G_T f = 0\}$$

with $a(u,v) =< u, v >_{L^2(0,T;\mathbb{R}^p)}$, and $L \equiv 0$ since $a(u,u) = \|u\|^2$.

Let us prove $u_{\theta_f} = u^*$.

Let $v \in \{v \in L^2(0,T;\mathbb{R}^p) \mid C(T)H_T v + C(T)G_T f = 0\}$, we have

$$
\begin{aligned}
< u_{\theta_f}, v - u_{\theta_f} >_{L^2(0,T;\mathbb{R}^p)} &= \int_0^T < u_{\theta_f}(s), v(s) - u_{\theta_f}(s) > \mathrm{d}s \\
&= \int_0^T < B(T)^* R(T,s)^* C(T)^* \theta_f, v(s) - u_{\theta_f}(s) > \mathrm{d}s \\
&= \int_0^T < \theta_f, C(T)R(T,s)B(T)(v(s) - u_{\theta_f}(s)) > \mathrm{d}s \\
&= < \theta_f, \int_0^T C(T)R(T,s)B(T)v(s)\mathrm{d}s - \int_0^T C(T)R(T,s)B(T)u_{\theta_f}(s)\mathrm{d}s > \\
&= < \theta_f, Hv - Hu_{\theta_f} > \\
&= 0
\end{aligned}
$$

then $u_{\theta_f} = u^*$. $\square$

3.3. Linear Quadratic problem

In this section, we present a more general approach which consists to consider the compensation problem as minimization one of a cost function defined on $L^2(0,T;\mathbb{R}^p)$ as follows

$$
\begin{aligned}
J(u) = &< Q(C(T)H_T u + C(T)G_T f), C(T)H_T u + C(T)G_T f > \\
&+ \int_0^T < W(t)(C(t)H_t u + C(t)G_t f), C(t)H_t u + C(t)G_t f > \mathrm{d}t \\
&+ \int_0^T < U(t)u(t), u(t) > \mathrm{d}t,
\end{aligned}
$$

where $Q \in M_n(\mathbb{R})$ and $W \in L^\infty([0,T], M_n(\mathbb{R}))$ are positive symmetric matrixes, and $U \in L^\infty([0,T], M_p(\mathbb{R}))$ is a positive definite symmetric matrix.

Let's remember that the map J is strictly convex and

$$
\forall u \in L^2(0,T;\mathbb{R}^p), \quad \int_0^T < U(t)u(t), u(t) > \mathrm{d}t \geqslant \alpha \int_0^T u(t)^* u(t)\mathrm{d}t \tag{15}
$$

with $\alpha > 0$ is the minimal eigenvalue of U.

We have the following result.

Theorem 1 *Under the hypothese (15), there exists a unique control* $u \in L^2(0,T;\mathbb{R}^p)$ *such that*

$$J(u) = \inf_{v \in L^2(0,T;\mathbb{R}^p)} J(v).$$

Proof. Let us prove initially the existence of such control. Let us consider a sequence minimizing of controls $(u_n)_{n\in\mathbb{N}}$ on $[0,T]$ such that

$$C(T)H_T u_n + C(T)G_T f = 0, \tag{16}$$

i.e. the sequence $J(u_n)$ converges to the lower bound of the costs, in particular this sequence is bounded. By assumption,

$$\exists\,\alpha > 0,\ \forall u \in L^2(0,T;\mathbb{R}^p),\ J(u) \geqslant \alpha \|u\|^2_{L^2(0,T;\mathbb{R}^p)}$$

then, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded on $L^2(0,T;\mathbb{R}^p)$, which implies, the subsequence of $(u_n)_{n\in\mathbb{N}}$ converges weakly to $u \in L^2(0,T;\mathbb{R}^p)$, using (16), one gets

$$C(T)H_T u + C(T)G_T f = 0.$$

Hence $u_n \longrightarrow u$ on $L^2(0,T;\mathbb{R}^p)$ implies that

$$\int_0^T < U(t)u(t), u(t) > \mathrm{d}t \leqslant \liminf \int_0^T < U(t)u_n(t), u_n(t) > \mathrm{d}t$$

from which we get

$$J(u) \leqslant \liminf J(u_n).$$

In particular, $(u_n)_{n\in\mathbb{N}}$ is a sequence minimizing of controls on $[0,T]$, then $J(u)$ is equal to the lower bound of the costs, we then have the existence of such a control optimal u.

Since J is strictly convex we deduce uniqueness of such a control optimal u.

In the following result, we give a necessary and sufficient condition for our optimal control problem.

Theorem 2 *The control u is optimal such that*

$$J(u) = \inf_{v \in L^2(0,T;\mathbb{R}^p)} J(v),$$

if and only if there exists a certain row vector $p(t) \in \mathbb{R}^n \backslash \{0\}$, *called adjoint vector satisfying*

$$\dot{p}(t) = -p(t)A(t) + (C(t)H_t u + C(t)G_t f)^* W(t)C(t),\ \forall t \in [0,T] \tag{17}$$

such that

$$p(T) = -(C(T)H_T u + C(T)G_T f)^* Q \tag{18}$$

or

$$u(t) = U^{-1}(t)B(t)^* p(t)^*. \tag{19}$$

Proof. Let u the optimal control such that

$$J(u) = \inf_{v \in L^2(0,T;\mathbb{R}^p)} J(v).$$

Let $u \in L^2(0,T;\mathbb{R}^p)$ be such that

$$u_{\text{pert}}(t) = u(t) + \delta u(t)$$

and

$$z_{\text{pert}}(t) = z(t) + \delta z(t) + \circ(\|\delta u\|_{L^2}),$$

with

$$\delta z(0) = 0.$$

z_{pert} is the solution of the system $\dot{z}(t)_{\text{pert}} = A(t)z_{\text{pert}} + B(t)u_{\text{pert}}$, and it is augmented by the output equation

$$y_{\text{pert}}(t) = C(t)z_{\text{pert}}(t), \quad 0 < t < T$$

then

$$\begin{aligned} \delta\dot{z}(t) &= A(t)\delta z(t) + B(t)\delta u(t), \quad 0 < t < T, \\ \delta y(t) &= C(t)\delta z(t) \end{aligned}$$

one has

$$C(t)\delta z(t) = C(t)H_t\delta u = \int_0^t M(t)M^{-1}(s)B(s)\delta u(s)\,\mathrm{d}s. \tag{20}$$

with

$$M(t) = R(t,0).$$

We have the cost function J is Frechet differentiable, and u is optimal control, then

$$dJ(u) = 0.$$

In particular,

$$
\begin{aligned}
J(u_{\text{pert}}) = &< Q(C(T)H_T u_{\text{pert}} + C(T)G_T f), C(T)H_T u_{\text{pert}} + C(T)G_T f > \\
&+ \int_0^T < W(t)(C(t)H_t u_{\text{pert}} + C(t)G_t f), C(t)H_t u_{\text{pert}} + C(t)G_t f > \mathrm{d}t \\
&+ \int_0^T < U(t)u_{\text{pert}}(t), u_{\text{pert}}(t) > \mathrm{d}t
\end{aligned}
$$

and for every $\delta u \in L^2(0,T;\mathbb{R}^p)$, we have

$$
\begin{aligned}
\frac{1}{2} dJ(u)\delta u = &\big(C(T)H_T u + C(T)G_T f\big)^* Q C(T) H_T \delta u \\
&+ \int_0^T \big(C(t)H_t u + C(t)G_t f\big)^* W(t) C(t) H_t \delta u \, \mathrm{d}t + \int_0^T u(t)^* U(t) \delta u(t) \, \mathrm{d}t \\
= &\, 0. \qquad (21)
\end{aligned}
$$

Let the adjoint vector $p(t)$ is the solution of the system

$$
\begin{aligned}
&\dot{p}(t) = -p(t)A(t) + \big(C(t)H_t u + C(t)G_t f\big)^* W(t)C(t), \ 0 < t < T, \\
&p(T) = -\big(C(T)H_T u + C(T)G_T f\big)^* Q
\end{aligned}
$$

then

$$
\begin{aligned}
p(t) = &\,\Delta M^{-1}(t) \\
&+ \int_0^t (C(s)H_s u + C(s)G_s f)^* W(s)C(s)M(s)\,\mathrm{d}s M^{-1}(t), \ \forall t \in [0,T]
\end{aligned}
$$

with

$$
\begin{aligned}
\Delta = &-(C(T)H_T u + C(T)G_T f)^* Q M(T) \\
&- \int_0^T (C(s)H_s u + C(s)G_s f)^* W(s)C(s)M(s)\,\mathrm{d}s.
\end{aligned}
$$

Using (20), (21) and integrations by parts, one gets

$$\int_0^T (C(t)H_t u + C(t)G_t f)^* W(t)C(t)H_t \delta u \,\mathrm{d}t =$$

$$\int_0^T (C(t)H_t u + C(t)G_t f)^* W(t)C(t) \int_0^t M(t)M^{-1}(s)B(s)\delta u(s)\,\mathrm{d}s\,\mathrm{d}t$$

$$\int_0^T (C(s)H_s u + C(s)G_s f)^* W(s)C(s)M(s)\,\mathrm{d}s \int_0^T M^{-1}(s)B(s)\delta u(s)\,\mathrm{d}s$$

$$-\int_0^T \int_0^t (C(s)H_s u + C(s)G_s f)^* W(s)C(s)M(s)\,\mathrm{d}s$$

$$M^{-1}(t)B(t)\delta u(t)\,\mathrm{d}t.$$

We have

$$p(t) - \Delta M^{-1}(t) = \int_0^t (C(s)H_s u + C(s)G_s f)^* W(s)C(s)M(s)\,\mathrm{d}s M^{-1}(t), \ \forall t \in [0,T]$$

then

$$\int_0^T (C(t)H_t u + C(t)G_t f)^* W(t)C(t)H_t \delta u \,\mathrm{d}t =$$

$$-(C(T)H_T u + C(T)G_T f)^* Q M(T) \int_0^T M^{-1}(t)B(t)\delta u(t)\,\mathrm{d}t - \int_0^T p(t)B(t)\delta u(t)\,\mathrm{d}t.$$

From (21), one gets

$$(C(T)H_T u + C(T)G_T f)^* Q C(T)H_T \delta u =$$

$$(C(T)H_T u + C(T)G_T f)^* Q M(T) \int_0^T M^{-1}(t)B(t)\delta u(t)\,\mathrm{d}t,$$

which implies that

$$\frac{1}{2} dJ(u)\delta u = \int_0^T (u(t)^* U(t) - p(t)B(t))\,\delta u(t)\,\mathrm{d}t = 0, \quad \forall \delta u \in L^2(0,T;\mathbb{R}^p)$$

and

$$u(t)^* U(t) - p(t)B(t) = 0, \quad \forall t \in [0, T].$$

Conversely, if there exists a adjoint vector $p(t)$ satisfying (17), (18) and (19), then

$$dJ(u) = 0.$$

Moreover, J is strictly convex, which implies the control u is optimal such that

$$J(u) = \inf_{v \in L^2(0,T;\mathbb{R}^p)} J(v).$$

Remark 2 *Let* $H\colon [0,T] \times \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}^p \longrightarrow \mathbb{R}$ *is the Hamiltonian function defined by*

$$\begin{aligned} H(t, z(t), p(t), u(t)) = p(t)\big(A(t)z(t) + B(t)u(t) + f(t)\big) \\ - \frac{1}{2}\big[(C(t)z(t) - C(t)M(t)z_0)^* W(t)(C(t)z(t) \\ - C(t)M(t)z_0) + u(t)^* U(t)u(t)\big], \end{aligned}$$

then

$$\dot{z}(t) = \frac{\partial H}{\partial p}(t, z(t), p(t), u(t)) = A(t)z(t) + B(t)u(t) + f(t),$$

$$\dot{p}(t) = -\frac{\partial H}{\partial z}(t, z(t), p(t), u(t)) = -p(t)A(t) + (C(t)H_t u + C(t)G_t f)^* W(t)C(t),$$

and

$$\frac{\partial H}{\partial u}(t, z(t), p(t), u(t)) = p(t)B(t) - u(t)^* U(t) = 0.$$

This the general maximal principle.

Example 1

i) We consider the case where $n = 1$, $p = q = 1$ and

$$A(t) = t\,, \qquad B(t) = t\,, \qquad C(t), = -t,$$

and the cost function

$$J(u) = \int_0^T u(t)^2 \mathrm{d}t.$$

Using the maximal principle, one gets

$$\begin{aligned} \dot{p}(t) &= -p(t)t, \\ p(T) &= 0, \end{aligned}$$

and

$$u(t) = tp(t).$$

Then

$$p(t) = p(0)\exp\left(\frac{-t^2}{2}\right),$$

from which we get the optimal control minimizing the cost function J

$$u(t) = tp(0)\exp\left(\frac{-t^2}{2}\right).$$

ii) Let us define A where $n = 2$ by

$$A(t) = \begin{pmatrix} t & -1 \\ 1 & t \end{pmatrix},$$

compute the resolvent of $\dot{z}(t) = A(t)z(t)$.

One has

$$R(T,t) = \begin{pmatrix} \cos(T-t)e^{\frac{T^2-t^2}{2}} & -\sin(T-t)e^{\frac{T^2-t^2}{2}} \\ \sin(T-t)e^{\frac{T^2-t^2}{2}} & \cos(T-t)e^{\frac{T^2-t^2}{2}} \end{pmatrix}.$$

We consider the case where $p = q = 1$ and

$$B(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}, \quad C(t) = \begin{pmatrix} t & 0 \end{pmatrix},$$

and the cost function

$$J(u) = -(C(T)H_T u + C(T)G_T f)^* + \int_0^T u(t)^2 \mathrm{d}t.$$

Using the maximal principle, one gets

$$\dot{p}(t) = \begin{pmatrix} -tp_{z_1}(t) - p_{z_2}(t) & p_{z_1}(t) - tp_{z_2}(t) \end{pmatrix},$$

$$p(T) = \begin{pmatrix} p_{z_1}(T) & p_{z_2}(T) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix},$$

and

$$u(t) = tp_{z_1}(t).$$

Then

$$p(t) = \begin{pmatrix} \frac{1}{2}\cos(T-t)e^{\frac{T^2-t^2}{2}} & -\frac{1}{2}\sin(T-t)e^{\frac{T^2-t^2}{2}} \end{pmatrix},$$

from which we get the optimal control minimizing the cost function J

$$u(t) = \frac{t}{2}\cos(T-t)e^{\frac{T^2-t^2}{2}}. \qquad \square$$

3.3.1. Numerical simulations

Let us define A where $n = 2$ by

$$A(t) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix},$$

compute the resolvent of $\dot{z}(t) = A(t)z(t)$.

One has

$$R(T,t) = \begin{pmatrix} 1 & 0 \\ \frac{T^2-t^2}{2} & 1 \end{pmatrix}.$$

We consider the case where $p = q = 1$ and

$$B(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \; C(t) = \begin{pmatrix} e^{\frac{-t^2}{2}} & 0 \end{pmatrix},$$

and the cost function

$$J(u) = -(C(T)H_T u + C(T)G_T f)^* + \int_0^T u(t)^2 dt.$$

Using the maximal principle, one gets

$$\dot{p}(t) = \begin{pmatrix} -t p_{z_2}(t) & 0 \end{pmatrix},$$

$$p(T) = \begin{pmatrix} p_{z_1}(T) & p_{z_2}(T) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix},$$

and

$$u(t) = p_{z_1}(t),$$

then

$$p(t) = \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix}.$$

From which we get the optimal control minimizing the cost function J

$$u(t) = \frac{1}{2}.$$

The initial state is considered null $z_0 = 0$, then $y_{(0,0)} = 0$. The disturbance term is given as follows

$$f(t) = \begin{pmatrix} \dfrac{e^{\frac{T^2}{2}-t^2}}{500} \\ 0 \end{pmatrix}.$$

To simplify the notations, let us note $y_{(u,f)}$ the observation corresponding to the control u and the disturbance f. Then

$$y_{(u,f)}(t) = \frac{t}{2}e^{\frac{-t^2}{2}} + \frac{\sqrt{\pi}}{1000}\mathrm{erf}(t),$$
$$y_{(0,f)}(t) = \frac{\sqrt{\pi}}{1000}\mathrm{erf}(t).$$

We obtain the following numerical results which illustrate the previous developments.

Hence, in Figure 3, we give the representation of the observations $y_{(u,f)}$ and $y_{(0,f)}$. This figure show that for $T = 10$, we have $y_{(u,f)}(T) = y_{(0,0)}(T) = 0$.

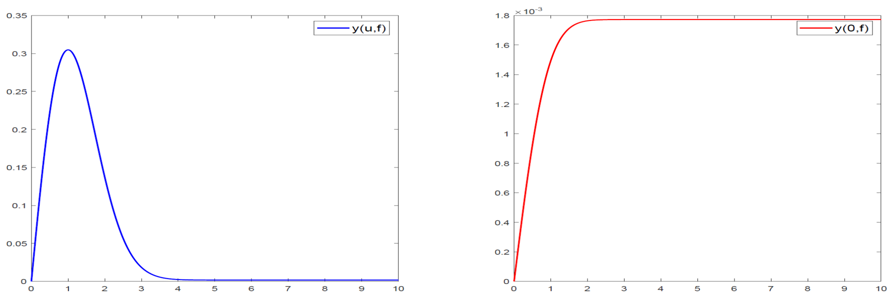

Figure 3: Representation of $y_{(u,f)}$ and $y_{(0,f)}$ for $T = 10$

In Figure 4, we give the representation of the optimal control u.

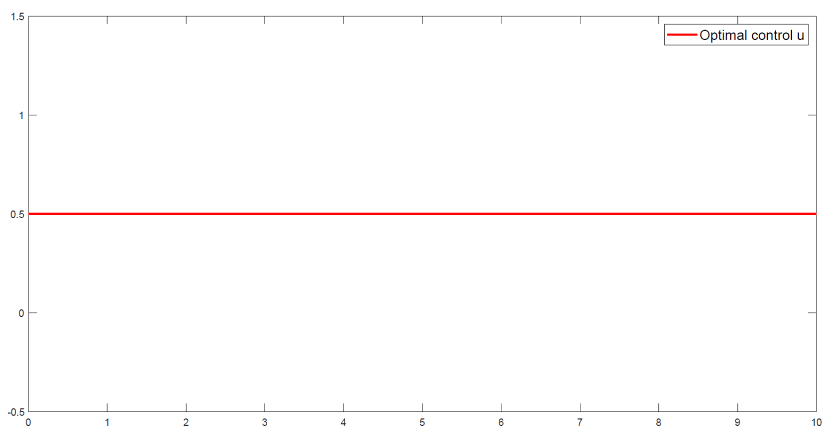

Figure 4: Representation of the optimal control u

4. Conclusion

This paper is about a class of time-varying linear dynamical systems. The concept of remediability is an important technique in perturbation theory. It consists of studying the possibility of attenuating the effect of any disturbance, through observation. We show in this work how to find a practical input operator ensuring the compensation of the disturbance. We find a control that cancels the output of the system and we also show that Hilbert uniqueness method can be used to solve the optimal control that ensures remediability. And finally a general approach has been given to minimize the linear quadratic problem. To illustrate our work, some examples and numerical simulations are given.

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