GEODEZJA I KARTOGRAFIA
t. XLIX, z. 2, s. 91 – 106 (2000)

Witold Prószyński

Institute of Applied Geodesy
Warsaw University of Technology
(00-661 Warsaw, 1 Politechniki Sq.)

On the space of imperceptible observational errors in linear Gauss-Markov models with examples taken from engineering surveys

This paper presents an analysis of the space, known in the theory of reliability, of observational gross errors or blunders absolutely undetectable in the least-squares estimation process in linear Gauss-Markov models. The analysis is based on a general relationship linking the observational disturbances and a model response. Although the definition of this space is identical with that given by [1] it is arrived at in a slightly different way. Several properties of this space are formulated, one of them showing its connection with the reliability level of a model with uncorrelated observations. Although the linearized models are included in the theory, the approach applied to them, being basically a simple extension of that proposed for linear models, can not be considered as a complete proposal for practical purposes. The theory is illustrated with examples taken from engineering surveys.

1. *Introduction*

It is well known that in Gauss-Markov models (GMMs) with properly distributed observational redundancies, one can detect the outlying observations which carry gross errors or blunders, henceforward called observational disturbances. According to Baarda [2] in a model with uncorrelated observations the minimum detectable disturbance in a single observation depends on the level of measurement accuracy, the reliability measure for the observation in question as well as on the assumed probability levels (α, β) for the type I and type II procedural errors. It is also known [1] that there are gross errors or blunders which do not change the least-squares (LS) residuals and therefore pass undetected through any statistical test. Beside providing an independent proof for the definition of the space of such errors (as given in [1]) the investigation of its properties and finding relationship to the model's reliability characteristics are the main objectives of the present paper. This topic is of special interest to engineering surveys where more and more often the demands for high precision and high reliability of setting out or deformation monitoring are considered essential in the process of the erection of a building structure or in assessing its safety.

To provide a terminological basis for the reliability-oriented analyses concerning this space, the following classification of observational disturbances is proposed, being consistent with the approach presented in [1]:

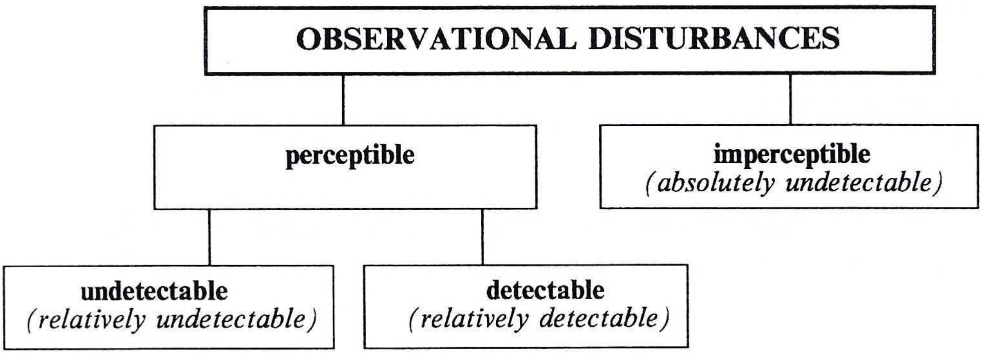


Fig.1 The reliability-oriented classification of observational disturbances

The term "perceptible" covers all those disturbances which cause certain distortions in the LS residuals, the magnitudes of the disturbances being either within the uncertainty area of the measurement method (undetectable) or outside that area (detectable). The term "imperceptible" refers to all those disturbances, which, since they do not induce any distortion in the LS residuals, are absolutely undetectable in the process of the LS estimation, i.e. even with the uncertainty area being reduced to a point (standard deviation of measurement equal to zero). In this theoretical case the minimum detectable disturbance as determined on the basis of the Baarda formula becomes zero, which means that every perceptible disturbance is detectable [2]. The greater the standard deviation of measurement the bigger the minimum detectable disturbance in every observation. To emphasize this dependence (and that of other factors) the adjective "relative" can be added to precisely characterize the "undetectable" and "detectable" disturbances.

This paper concentrates on the space of imperceptible disturbances, verifying its definition and formulating some of its properties. The identification of the space is based here on a general relationship between the observational disturbances and a model response. It should be emphasized that of all possible types of disturbances we take into account those that do not affect any model equation (the model being considered correct) but distort its right hand side terms only. The functioning of this space is investigated also in a random error environment.

2. *The "disturbance/response" relationship for a GMM*

Let us consider the class of linear GMMs, covering the models with minimum constraints and the models with no constraints, defined by:

$$\begin{aligned} \mathbf{A}x + e &= y \qquad e \sim (\mathbf{0}, \mathbf{C}) \\ \mathbf{S}x &= \mathbf{0} \end{aligned} \tag{1}$$

where:

- y — the $n \times 1$ vector of observations (observational increments — in a linearized model);
- $\mathbf{A}$ — the $n \times u$ coefficient matrix; $\operatorname{rank}(\mathbf{A}) \leqslant u$, or introducing d as a defect of $\mathbf{A}$, $\operatorname{rank}(\mathbf{A}) = u - d \ (d \geqslant 0)$;
- x — the unknown $u \times 1$ vector of parameters (parameter increments — in a linearized model);
- e — the unknown $n \times 1$ vector of random errors; $e = y - \mathbf{y}^{\text{truc}}$
- $\mathbf{C}$ — the $n \times n$ covariance matrix (pos. def.);
- $\mathbf{S}$ — the $d \times u$ coefficient matrix, rank $\mathbf{S}^T = d$, such that rank $\begin{bmatrix}\mathbf{A}\\ \mathbf{S}\end{bmatrix} = u$.

From the consistency of the functional model it follows that $y^{\text{truc}} \in M(\mathbf{A})$, where $M(\mathbf{A})$ denotes the space spanned by the columns of $\mathbf{A}$.

Within the models with no constraints in (1) there can also be the over-constrained models ($\mathbf{S}(w \times u)$ where $w > d$, rank $(\mathbf{S}^T) = w$, rank $[\mathbf{A}^T \mathbf{S}^T]^T = u$) reduced to their equivalent forms having full-rank coefficient matrices.

After standardization the model (1) will take the form:

$$\begin{aligned} \mathbf{A}_* x + e_* &= y_* \qquad e_* \sim (\mathbf{0}, \mathbf{I}) \\ \mathbf{S}x &= \mathbf{0} \end{aligned} \tag{2}$$

where: $y_* = by$; $\mathbf{A}_* = b\mathbf{A}$; $e_* = be$; $b^T b = \mathbf{C}^{-1}$

The least-squares estimator of $(-e_*)$, being the vector of standardized residuals denoted here by v_*, is given by the formula:

$$v_* = -[\mathbf{I} - \mathbf{A}_*(\mathbf{A}_*^T\mathbf{A}_*)_S^- \mathbf{A}_*^T] y_* \tag{3}$$

where $(\mathbf{A}_*^T\mathbf{A}_*)_S^-$ is the reflexive g-inverse of $\mathbf{A}_*^T\mathbf{A}_*$ such that $\mathbf{S}(\mathbf{A}_*^T\mathbf{A}_*)_S^- = \mathbf{0}$, or with

$$\begin{aligned} C_{v*} &= \mathbf{I} - \mathbf{A}_*(\mathbf{A}_*^T\mathbf{A}_*)_S^- \mathbf{A}_*^T \\ v_* &= -\mathbf{C}_{v*} y_* \end{aligned} \tag{4}$$

The orthogonal projector $\mathbf{C}_{v*}$, being the covariance matrix of the standardized residuals, contains full information on internal reliability of a GMM with uncorrelated observations (see [1], [3], [4]).

Replacing $\boldsymbol{y}_*$ and $\boldsymbol{v}_*$ by their increments $\Delta\boldsymbol{y}_*$ and $\Delta\boldsymbol{v}_*$ we get a well known "disturbance/response" relationship for a GMM (in a standardized form), i.e.

$$\Delta\boldsymbol{v}_* = -\mathbf{C}_{v*}\Delta\boldsymbol{y}_* \tag{5}$$

where: $\Delta\boldsymbol{y}_*$ — the vector of disturbances in standardized observations,

$\Delta\boldsymbol{v}_*$ — the vector of resulting distortions in the LS standardized residuals.

Substituting into (5) $\Delta\boldsymbol{y}_* = \boldsymbol{b}\Delta\boldsymbol{y}$, $\Delta\boldsymbol{v}_* = \boldsymbol{b}\Delta\boldsymbol{v}$ with $\boldsymbol{b}$ as in (2) and expressing the result in terms of $\mathbf{A}$ and $\mathbf{C}$ as in (1) we obtain the "disturbance/response" relationship for the original, i.e. non-standardized, GMM (see e.g. [1], [5])

$$\Delta\boldsymbol{v} = -[\mathbf{I} - \mathbf{A}(\mathbf{A}^T\mathbf{C}^{-1}\mathbf{A})_S^- \mathbf{A}^T\mathbf{C}^{-1}]\Delta\boldsymbol{y} \tag{6}$$

or in a form analogous to (5)

$$\Delta\boldsymbol{v} = -\mathbf{C}_{(v)}\Delta\boldsymbol{y} \tag{7}$$

where $\mathbf{C}_{(v)}$ is a projector, which can be considered orthogonal in a vector space with a norm defined as $\|\boldsymbol{y}\| = (\boldsymbol{y}^T\mathbf{C}^{-1}\boldsymbol{y})^{1/2}$ where $\mathbf{C}^{-1}$ is p.d. (see [6]). The brackets used in a subscript are to indicate that $\mathbf{C}_{(v)}$ is not a covariance matrix of the LS residuals $\boldsymbol{v}$. The matrix $\mathbf{C}_{(v)}$ is a basis for computing the reliability measures in GMMs with correlated observations (see [7], [8]).

With reference to linearized models let us denote by $\boldsymbol{x}^0$ the vector of approximate values of parameters and assume that it lies within the specified validity area of linear expansion around $\boldsymbol{x}^{\text{true}}$. Now, with $\mathbf{A}(\boldsymbol{x}^0)$, $\boldsymbol{y}-\boldsymbol{y}(\boldsymbol{x}^0)$, $\mathbf{A}_*(\boldsymbol{x}^0)$, $\boldsymbol{y}_*-\boldsymbol{y}_*(\boldsymbol{x}^0)$ taken instead of, correspondingly $\mathbf{A}$, $\boldsymbol{y}$, $\mathbf{A}_*$, $\boldsymbol{y}_*$, the formulas (1)$\div$(7) become the explicit relationships concerning the linearized models. The requirement for consistency (and hence — for correctness) of their functional models expressed as $[\boldsymbol{y}^{\text{true}} - \boldsymbol{y}(\boldsymbol{x}^0)] \in M[\mathbf{A}(\boldsymbol{x}^0)]$ can be satisfied to a degree resulting from the accepted inaccuracy of linear expansion.

3. *The space of imperceptible observational disturbances*

Taking into consideration the relationship (5) we shall investigate the cases when there is no response of the model to disturbances in the standardized observations. The set of such disturbance vectors, denoted by U_*, i.e.

$$U_* = \{\Delta\boldsymbol{y}_* : \Delta\boldsymbol{y}_* \neq \mathbf{0} \Rightarrow \Delta\boldsymbol{v}_* = \mathbf{0}\} \tag{8}$$

will form the space of imperceptible disturbances in the standardized observations. Consequently, the space of the corresponding imperceptible disturbances in the original observations, denoted by U, will be

$$U=\{\Delta y : \Delta y \neq \mathbf{0} \Rightarrow \Delta v = \mathbf{0}\} \tag{9}$$

where: $\Delta y = b^{-1}\Delta y_*$, $\Delta v = b^{-1}\Delta v_*$

Although we can find immediately from (5) that the model yields no responses in the following two cases:

$$\text{a) } \mathbf{C}_{v*} = \mathbf{0} \tag{10a}$$

$$\text{b) } \mathbf{C}_{v*}\Delta y = \mathbf{0} \quad (\mathbf{C}_{v*} \neq \mathbf{0}) \tag{10b}$$

so that we might analyze them separately, we shall follow a more general approach.

According to [9], for any type of g-inverse in (4) we have $\mathbf{C}_{v*}\mathbf{A}_* = \mathbf{0}$, which means that $\mathbf{C}_{v*}$ carries out the projection onto the space orthogonal to that spanned by the columns of $\mathbf{A}_*$, i.e. $M(\mathbf{A}_*)$. Consequently, $M(\mathbf{A}_*)$ is the null space of $\mathbf{C}_{v*}$, i.e. $M(\mathbf{A}_*) = N(\mathbf{C}_{v*})$, so $M(\mathbf{A}_*) \perp M(\mathbf{C}_{v*})$ and hence, dim $M(C_{v*}) + \dim M(\mathbf{A}_*) = n$, where $\dim M(\mathbf{C}_{v*}) = n - u + d$, $\dim M(\mathbf{A}_*) = u - d$.

By distinguishing between dim $M(\mathbf{A}_*) = n (\Rightarrow \mathbf{C}_{v*} = \mathbf{0})$ and $\dim M(\mathbf{A}_*) < n$ $(\Rightarrow \mathbf{C}_{v*} \neq \mathbf{0})$ we cover the cases a) and b) listed above. Unlike the latter case, the former one excludes the occurrence of any dependence between the rows of $\mathbf{A}$, which applies to the model with no observational redundancies.

Thus, the space of imperceptible observational disturbances in the standardized GMM can be defined as

$$U_* = \{\Delta y_* : \Delta y_* \in M(\mathbf{A}_*)\} \tag{11}$$

For Δy_* such that $\Delta y_* \in M(\mathbf{A}_*)$ the corresponding Δy (see (9)) satisfies $\Delta y \in M(\mathbf{A})$ and hence, the space of imperceptible observational disturbances in the non-standardized GMM can be defined as

$$U = \{\Delta y : \Delta y \in M(\mathbf{A})\} \tag{12}$$

Multiplying $\mathbf{A}$ (or $\mathbf{A}_*$) by different nonsingular matrices $\mathbf{K} \in R^{u\mathrm{x}u}$ (excluding the identity matrix and the permutation matrix) we can generate different sets of Δy (or Δy_*) belonging to the space U (or U_*). There can be such column vectors in $\mathbf{K}$ that with $\mathbf{A}$ being of incomplete rank (i.e. rank $(\mathbf{A}) < u$) the multiplication will yield $\Delta y = \mathbf{0}$ (or $\Delta y_* = \mathbf{0}$).

The definition of the space U as in (12), which is independent of the given covariance matrix $\mathbf{C}$, could also be obtained directly on the basis of (6) by analogous derivations.

Still another method of arriving at the same definition of U (or U_*) would be to resort to the algebraic properties of the original model (1) (or the standardized model (2)) and apply the n.s. condition for the model consistency (see [6]) to the incremental form of (1) (or (2)), i.e. $\mathbf{A} \cdot \Delta x = \Delta y$ (or $\mathbf{A}_*\Delta x = \Delta y_*$). Using the duality properties of the least squares method (see [10]) one can easily prove a full consistency in the definition of the space of imperceptible observational disturbances (the SID for short) between the parametric method and the method of conditions.

Maintaining the assumption of correctness of a model, in the case of linearized models we have to modify the definitions of the SID (see (11) and (12)) as shown below for the definition (12)

$$U=\{\Delta y:\Delta y\in[M(\mathbf{A}(\boldsymbol{x}^0))\cap(R_1\cup R_2)]\} \tag{13}$$

where: R_1 — the space of those Δy, which, when contained in the observations used in computing $\boldsymbol{x}^0$, do not affect the model correctness;

R_2 — the space of those Δy, which, when contained in the observations not used in finding $\boldsymbol{x}^0$, can not be detected by the individual "free-term" checks. The assumption is here made that $\boldsymbol{x}^0$, which is found on other basis, ensures the model correctness.

By introducing these two spaces without defining them precisely we would like to emphasize only that the actual SID for linearized models constitutes a certain subspace of $M[\mathbf{A}(\boldsymbol{x}^0)]$.

4. *Disturbances in a random error environment*

Let us consider the following structure of the observation vector:

$$y=y^{\text{true}}+e+\Delta y \tag{14}$$

where: y^{true} — the vector of true values of measured quantities;

e — the vector of random errors, as in (1);

Δy — the vector of disturbances (gross errors, blunders).

In a standardized model the formula (14) will take the form

$$y_*=y_*^{\text{true}}+e_*+\Delta y_* \tag{15}$$

Let each of the vectors e_* and Δy_* be a superimposition of the component belonging to U_* and the component not belonging to U_*, i.e.

$$y_*=y_*^{\text{true}}+e_*^{(-)}+e_*^{(+)}+\Delta y_*^{(-)}+\Delta y_*^{(+)} \tag{16}$$

where: $e_*^{(-)}\notin U_*$, $e_*^{(+)}\in U_*$ and $\Delta y_*^{(-)}\notin U_*$, $\Delta y_*^{(+)}\in U_*$. The consistency of the model equations implies that $y_*^{\text{true}}\in M(\mathbf{A}_*)$ and hence $y_*^{\text{true}}\in U_*$. Either directly from (14) or by multiplying both sides of (16) by $\boldsymbol{b}^{-1}$ ($\boldsymbol{b}$ as in (2)) we may get the analogous form of the observation vector in the initial model.

Substituting the relationship (16) into the formula (4) and taking into account that $\mathbf{C}_{v*}y^{\text{true}}=\mathbf{0}$ and $\mathbf{C}_{v*}e_*^{(+)}=\mathbf{0}$; $\mathbf{C}_{v*}\Delta y_*^{(+)}=\mathbf{0}$, we get

$$v_*=-\mathbf{C}_{v*}e_*^{(-)}-\mathbf{C}_v{}^*\Delta y_*^{(-)} \tag{17}$$

or in a condensed form

$$v_* = v_{*(r)} + \Delta v_* \tag{18}$$

where: $v_{*(r)}$ — the vector of standardized residuals corresponding to perceptible standardized random errors;

Δv_* — the vector of standardized model responses to perceptible standardized observational disturbances.

The analysis presented above leads to the following statements which hold true also for the initial (i.e. non-standardized) model:

i) both $\Delta y_*^{(+)}$ and $e_*^{(+)}$ — the components of the vector of standardized observational results — are not perceptible in the LS estimation process;

ii) the standardized disturbances $\Delta y_*^{(-)}$ being within the magnitude of the standardized random errors may not be detectable in a particular GMM. Hence, potentially, the space of disturbances which can be overlooked in the LS estimation process will be the space of imperceptible disturbances extended by the space of perceptible but undetectable disturbances;

$$\Delta y_* \text{ (overlooked)} = \Delta y_*^{(+)} \text{ (imperceptible)} + \Delta y_*^{(-)} \text{ (undetectable)} \tag{19}$$

iii) for a particular GMM we can define the LS equivalent sets of observational results, being those vectors y_* (see (16)) for which we get the same vectors v_*, i.e.

$$\{y_{*\,\mathrm{eqv}}\} = \{y_* : y_* = y_*^{\mathrm{true}} + e_*^{(-)} + e_{*i}^{(+)} + \Delta y_*^{(-)} + \Delta y_{*i}^{(+)}\} \tag{20}$$

where the subscript i denotes an arbitrary non-zero vector belonging to U_*

iv) the imperceptible disturbances reside in the LS estimated values of the observed quantities

As $\hat{y}_* = y_* + v_*$ and v_* does not contain any compensation for the disturbances $\Delta y_*^{(+)}$ and $e_*^{(+)}$, they are transferred onto the LS estimator $\hat{y}_*$. We can verify it immediately using the relationship $\hat{y} = (I - C_{v*})\{y_{*\,\mathrm{eqv}}\}_i = \mathbf{A}_*(\mathbf{A}_*^T\mathbf{A}_*)_S^- \mathbf{A}_*^T\{y_{*\,\mathrm{eqv}}\}_i$, where $\{y_{*\,\mathrm{eqv}}\}_i$ is an arbitrary element of $\{y_{*\,\mathrm{eqv}}\}$ as in (20). The matrix $\mathbf{I} - \mathbf{C}_{v*}$ carries out the projection upon $M(\mathbf{A}_*)$, so we obtain finally

$$\hat{y}_* = y_*^{\mathrm{true}} + (\mathbf{I} - \mathbf{C}_{v*})[e_*^{(-)} + \Delta y_*^{(-)}] + e_*^{(+)} + \Delta y_*^{(+)}$$

v) the distortion of the LS estimated parameter values due to disturbances overlooked in the estimation process is

$$\begin{aligned} \Delta\hat{x} &= (\mathbf{A}^T\mathbf{C}^{-1}\mathbf{A})_S^- \mathbf{A}^T\mathbf{C}^{-1}(\Delta y^{(+)}\Delta y^{(-)}) = (\mathbf{A}^T\mathbf{C}^{-1})_S^- \mathbf{A}^T\mathbf{C}^{-1}(\mathbf{A}k + \Delta y^{(-)}) = \\ &= (\mathbf{I} - \mathbf{S}^-\mathbf{S})k + (\mathbf{A}^T\mathbf{C}^{-1}\mathbf{A})_S^- \mathbf{A}^T\mathbf{C}^{-1}\Delta y^{(-)} \end{aligned} \tag{21}$$

where k is a non-zero column vector, such that $\mathbf{A}k \neq \mathbf{0}$ (i.e. $k \notin N(\mathbf{A}^T\mathbf{A})$) and $(\mathbf{I}-\mathbf{S}^-\mathbf{S})$ is a projector onto the space $\perp$to $M(\mathbf{S}^-)$;

The above-mentioned requirement for k excludes the column vectors of the transposed coefficient matrix $\mathbf{S}_0$ of free-net conditions (i.e. such that $\mathbf{A}\mathbf{S}_0^T=\mathbf{0}$) as they span the null-space of $\mathbf{A}^T\mathbf{A}$.

For rank$(\mathbf{A})=u$ we have instead of $(\mathbf{A}^T\mathbf{C}^{-1}\mathbf{A})_S^-$ a regular inverse $(\mathbf{A}^T\mathbf{C}^{-1}\mathbf{A})^{-1}$, $N(\mathbf{A}^T\mathbf{A})=\{\mathbf{0}\}$ and hence

$$\Delta\hat{x}=k+(\mathbf{A}^T\mathbf{C}^{-1}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{C}^{-1}\Delta y^{(-)} \tag{22}$$

where k is an arbitrary non-zero column vector;

vi) it seems reasonable to recommend that the space of imperceptible disturbances $(\dim(U_*)=\dim M(\mathbf{A}_*)=u-d)$ should constitute a small part of the whole n - dimensional observational space, hence $(u-d)/n$ should be as small as possible, or equivalently, that the space of perceptible disturbances $(\dim M(\mathbf{C}_{v*})=n-u+d)$ should constitute a dominating part of the whole n — dimensional observational space, and hence $(n-u+d)/n$ should be as big as possible. We recognize at once that the latter is a requirement for a high value of the global measure of the standarized model's internal reliability The higher the value of the global measure of the standarized model's internal reliability the smaller is the space of imperceptible disturbances. Assuming the uniform distribution of redundancies throughout the model the above global measure is identical with each of the local measures (i.e. concerning each observation).

The models with a low level of internal reliability ($\{\mathbf{C}_{v*}\}_{ii}<0.5$ $(i=1, \ldots, n)$, as specified in [3] for the models with uncorrelated observations), do not satisfy the requirement stated above.

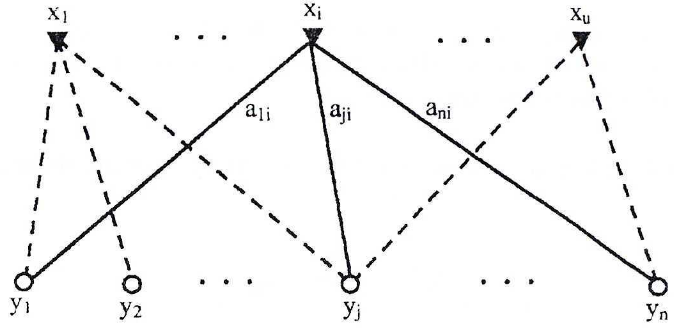
5. *Further properties of the space of imperceptible disturbances*

Here are some other properties of the SID together with their proofs:

— in a GMM with redundancies such that $\{\mathbf{C}_{v*}\}_{ii}>0$ $(i=1, \ldots, n)$ the space of imperceptible disturbances does not contain any vector with a single observational disturbance. Assuming the standardized disturbance vector $\Delta y_{*,i}=[0\ldots0\ \Delta y_{*i}\ 0\ldots0]^T$ $(i=1, \ldots, n)$ and substituting it into (10b) we obtain the requirement for the i-th column vector of $\mathbf{C}_{v*}$ being $\{\mathbf{C}_{v*}\}_{\bullet i}=0$ $(i=1, \ldots, n)$, which contradicts the assumption, and thus $\Delta y_{*,i} \notin U_*$ $(i=1, \ldots, n)$. The single disturbance being perceptible may be a detectable or an undetectable quantity.

— the disturbance vector $\Delta y_i=\boldsymbol{a}_i=[a_{1,i},\ a_{2,i},\ \ldots,\ a_{n,i}]^T$, where $\boldsymbol{a}_i$ is the i-th column-vector of $\mathbf{A}$ $(\Delta y_i \in U)$, contains the disturbances in the observations which approach the i-th parameter node as shown on a structural network of $\mathbf{A}$ in Fig.2. The connecting lines are drawn only for the non-zero componenets of the vector Δy_i (in the figure $a_{2,i}=0$). Hence, each of the column vectors of $\mathbf{A}$ contains the single-node imperceptible disturbances.

parameters



observations

Fig.2 Structural network of the matrix **A**

— the disturbance vector $\Delta y = \mathbf{A}k = k_1 a_1 + k_2 a_2 + \ldots + k_u a_u$, with k (as in (20)) having at least two non-zero components, contains the disturbances in the observations belonging to different parameter nodes;

— neither reordering of the parameters in a vector x (see (1)) nor rescaling of each of them affects the space of imperceptible disturbances.

The first modification corresponds to using as **K** (see section 3) the permutation matrix, and the second — the diagonal matrix, i.e. $\mathbf{A}x = \mathbf{AKK}^{-1}x = \mathbf{A}_M x_M$, and hence $M(\mathbf{A}_m) = M(\mathbf{A})$;

— the disturbance vector $\Delta y = \mathbf{A}k = k_1 a_1 + k_2 a_2 + \ldots + k_u a_u$, with k as in (22), corresponds to the vector of parameter increments $\Delta x = [k_1 k_2 \ldots k_u]^T$.

On the basis of (1) we have $\mathbf{A} \cdot \Delta x = \Delta y$ and thus $\mathbf{A} \cdot \Delta x = \mathbf{A} \cdot [k_1 k_2 \ldots k_u]^T = k_1 a_1 + \ldots + k_u a_u$. In linearized models the property holds for parameter shifts which are within the area of validity of linear expansion.

Special cases, which are characteristic from the practical point of view, are as follows:

a) $\Delta x = [0 \ldots 0 \; 1_{(i)} \; 0 \ldots 0]^T \Rightarrow \Delta y = a_i$;
The vector of the i-th node disturbances corresponds to the change in the i-th parameter;

b) $\Delta x = [0 \ldots 0 \; k_i \; k_j \; 0 \ldots 0]^T \Rightarrow \Delta y = k_i a_i + k_j a_j$;
When the i-th and the j-th parameters are X, Y coordinates of a point, a linear combination of the i-th and the j-th column-vectors of **A** corresponds to the change in the (X, Y) position of this point.

— the disturbance vector Δy corresponding to Δx as above, distorts the LS estimation of the parameter vector x by $\Delta\hat{x}$ according to the relationship (for notation see property v., sect. 4)

$$\Delta\hat{x} = (\mathbf{A}^T\mathbf{C}^{-1}\mathbf{A})_S^- \mathbf{A}^T\mathbf{C}^{-1}\Delta y = (\mathbf{A}^T\mathbf{C}^{-1}\mathbf{A})_S^- \mathbf{A}^T\mathbf{C}^{-1}\mathbf{A} \cdot \Delta x = (\mathbf{I} - \mathbf{S}^-\mathbf{S})\Delta x \,,$$

and with **A** being of full rank we have $\Delta\hat{x} = \Delta x$.

6. *Examples*

For simple measuring schemes, as can be met in the practice of engineering surveys, we shall find some of the SID vectors and demonstrate the way of generating the equivalent observation vectors.

E x a m p l e 1. Let us consider a local levelling scheme shown in Figure 3.

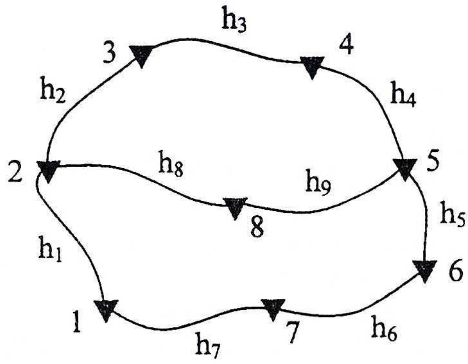


Fig.3 The levelling scheme used in the example

Instead of presenting the whole matrix $\mathbf{A}(9 \times 8)$ ($\mathrm{rank}(\mathbf{A})=7$) we shall show only two of its columns, representing the characteristic types of the SID vectors:

— for the node H_1 $\quad \Delta\boldsymbol{h}^{(+)}=[-1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0]^T$
— for the node H_5 $\quad \Delta\boldsymbol{h}^{(+)}=[0\ 0\ 0\ 1\ -1\ 0\ 0\ 0\ 1]^T$

The components of these vectors are single-node disturbances. By applying the transformation $\mathbf{A}\boldsymbol{k}$, where $\boldsymbol{k}$ is as in (21), we may get other forms of the SID vectors, e.g. for $\boldsymbol{k}=[0\ 0\ 5\ 5\ 0\ 0\ 0\ 0\ 0]^T$ and $\boldsymbol{k}=[0\ 0\ 0\ 0\ 4\ 1\ 0\ 0\ 0]^T$ we get

$$\Delta\boldsymbol{h}^{(+)}=[0\ 5\ 0\ -5\ 0\ \ 0\ 0\ 0\ 0]^T$$

$$\Delta\boldsymbol{h}^{(+)}=[0\ 0\ 0\ \ 4-3\ -1\ 0\ 0\ 0]^T$$

The examples of the SID vectors demonstrate the well known effects of gross error compensation in levelling networks.

In analogy to (20) the LS equivalent observation vectors can be generated by the formula:

$$\boldsymbol{h}_{eqv,i}=\boldsymbol{h}+\Delta\boldsymbol{h}_i^{(+)}=\boldsymbol{h}+\mathbf{A}\boldsymbol{k}_i$$

where

$\Delta\boldsymbol{h}_i^{(+)}$ — an arbitrary non-zero element of U;

k_i — an arbitrary non-zero column vector except for $k \in M(S_0^T)$, where $S_0 = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]$.

Here are the examples of the equivalent observation vectors h_{eqv} for a given observation vector h (without showing the corresponding vectors k)

$$h = [-4.99\quad 14.86\quad -2.94\quad -20.12\quad 11.09\quad 19.11\quad -16.93\quad 0.57\quad -9.12]^T$$

$$h_{eqv} = [-4.99\quad 14.86\quad \mathbf{2.06}\quad \mathbf{-25.12}\quad 11.09\quad \mathbf{29.11}\quad \mathbf{-26.93}\quad 0.57\quad -9.12]^T$$

$$h_{eqv} = [\ 0\quad 0\quad -8.20\quad 0\quad 0\quad 0\quad 8.28\quad 0\quad -8.55]^T$$

$$h_{eqv} = [\ 0\quad 0\quad 0\quad 0.35\quad -0.27\quad 0\quad 0\quad 0\quad 0]^T$$

The bold-typed numbers in the first example of h_{eqv} indicate the observations carrying the disturbance. The last two examples have been deliberately constructed to show the possibility of aggregating the observational results along the individual lines or within the individual loops.

Assuming that the observations are equally accurate and uncorrelated we get the vector of residuals

$$v = [0.025\quad -0.060\quad -0.060\quad -0.060\quad 0.025\quad 0.025\quad 0.025\quad 0.085\quad 0.085]^T,$$

being identical for all the four observation vectors presented above.

It is obvious that the parameter estimates corresponding to the observation vectors in question will differ considerably between themselves.

E x a m p l e 2. Figure 4 presents a horizontal measuring scheme with the observations being: the directions $k_1, k_2, \ldots, k_{15}$ and distances L_1, L_2. The parameters

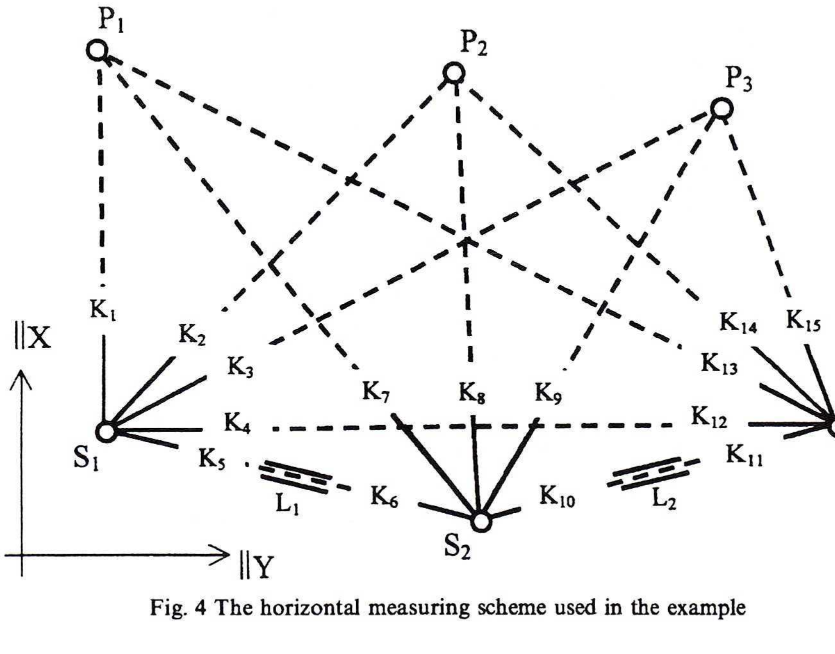


Fig. 4 The horizontal measuring scheme used in the example

to be determined in a local reference system are the coordinate increments dX, dY for each point and the orientation constants Z for each theodolite station. To construct the matrix $\mathbf{A}$ we assume the point coordinates as listed in Table 1.

Table 1

Point	X^0[m]	Y^0[m]
S1	120.030	16.950
S2	85.132	221.193
S3	93.735	384.216
P1	280.145	16.950
P2	255.688	200.007
P3	240.346	360.771

For the purpose of this analysis they represent the true coordinates, so we have $\boldsymbol{x}^0=\boldsymbol{x}^{\text{true}}$. Maintaining the notation $\Delta \boldsymbol{y}^{(+)}$ used for the SID vectors (see section 4) we introduce an auxiliary symbol $\Delta \boldsymbol{y}^+$ to denote those vectors $\Delta \boldsymbol{y}$ which satisfy the condition

$$\Delta \boldsymbol{y} \in M[\mathbf{A}(\boldsymbol{x}^0)] .$$

but were not checked against the requirement $\Delta \boldsymbol{y} \in (R_1 \cup R_2)$, which constitutes the second component in formula (13).

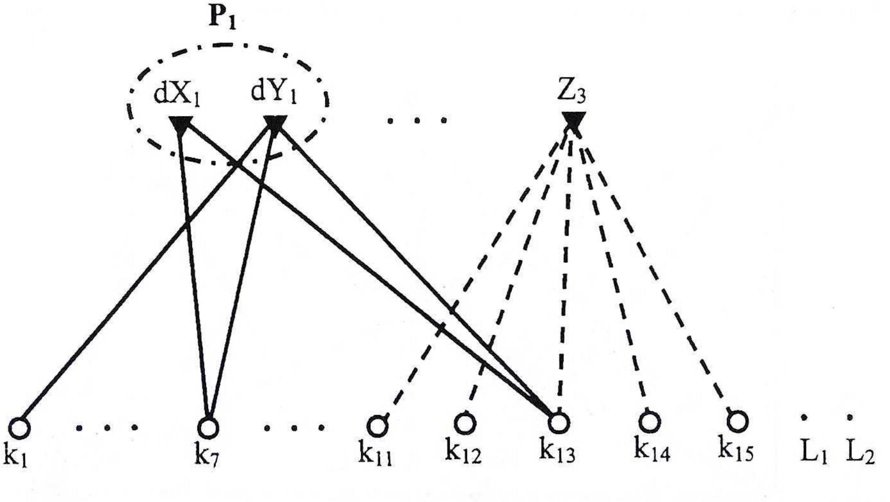


Fig. 5. A fragment of the structural network of the matrix $\mathbf{A}$ for horizontal measuring scheme

Here are the vectors $\Delta \boldsymbol{y}^+$, being the columns of the matrix $\mathbf{A}(\boldsymbol{x}^0)$, corresponding to the parameter nodes dX_1, dY_1, Z_3 (see Figure 5):

— for the node dX_1 $\quad \Delta \boldsymbol{y}^+ = [0\ldots0 \quad 1.63_{(7)} \quad 0\ldots0 \quad 1.38_{(13)} \quad 0\ldots0]^T$

— for the node dY_1 $\Delta y^+=[3.98_{(1)} \quad 0\ldots0 \quad 1.56_{(7)} \quad 0\ldots0 \quad 0.70_{(13)} \quad 0\ldots0]^T$
— for the node Z_3 $\Delta y^+=[0\ldots0 \quad -1 \quad -1 \quad -1 \quad -1 \quad -1 \quad 0 \quad 0]^T$

When we center the signal over the point P1 making the eccentricity error $\Delta r=[2 \quad -1]^T$ and keep the position of the signal fixed in all the measurements in a scheme we automatically generate the vector Δy^+ having the form

$$\Delta y^+=[-3.98_{(1)} \quad 0\ldots0 \quad 1.70_{(7)} \quad 0\ldots0 \quad -2.06_{(13)} \quad 0\ldots0]^T$$

With y being a given observation vector the equivalent observation vectors can be gene-rated by the same formula as in Example 1, i.e.

$$y_{\mathrm{eqv},i}=y+\Delta y_i^{(+)}=y+\mathbf{A}(x^0)\cdot k_i$$

where $k_i \notin N\{[\mathbf{A}(x^0)]^T\mathbf{A}(x^0)\}$ (see formula (21)).

Following the formula (19) we shall construct the vector $\Delta y_{(\mathrm{overlooked})}$ on the basis of Δy^+ for the node dX_1, recognized after an appropriate check as $\Delta y^{(+)}$:

$$\Delta y^{(+)} \quad =[0\ldots0 \quad 16.3_{(7)} \quad 0\ldots0 \quad 13.8_{(13)} \quad 0\ldots0]^T$$

$$\Delta y^{(-)} \quad =[0\ldots0 \quad -1.3_{(7)} \quad 0\ldots0 \quad 1.2_{(13)} \quad 0\ldots0]^T$$

$$\Delta y_{(\mathrm{ovrlooked})} \quad =[0\ldots0 \quad 15.0_{(7)} \quad 0\ldots0 \quad 15.0_{(13)} \quad 0\ldots0]^T$$

In checking whether Δy^+ qualifies to be $\Delta y^{(+)}$ the following condition for the model correctness was assumed:

$$[\mathbf{A}(x_d^0)](x_d^0-x^0)-[y(x_d^0)-y(x^0)]=\varepsilon \qquad |\,\varepsilon_i\,| \leqslant 0.01\sigma_{y,i} \qquad i=1, 2, \ldots, n$$

where x_d^0 — is the vector x^0 with the use of the values of k_7 and k_{13} containing the disturbances, $\sigma_{y,i}$ — the a priori standard deviations being here $\sigma_k=3^{cc}$; $\sigma_L=1$ mm. Using the simulated vector y, the LS residual vectors v were computed for the following three options of the observation vector:

$$y_1=y; \quad y_2=y+\Delta y_{(\mathrm{overlooked})}; \quad y_3=y+\Delta y^{(-)};$$

The results of the computation were as follows

$$v(y_2)=v(y_3)\cong v(y_1)$$

In spite of discrepancies (within $\pm 0.3\sigma_y$) in the second equality, the results of the adjustment for the options 2 and 3 are, in terms of the tests based on residuals, equivalent with the results for the option 1.

7. *Concluding remarks*

The analysis of the space of imperceptible disturbances, carried out in this paper, allows one to give a more complete description of the advantages of designing a GMM to a suitably high reliability level:

— better detection of perceptible disturbances thanks to the reduction of the space of undetectable disturbances;

— reduction of the space of imperceptible disturbances.

Obviously, both these advantages are closely interrelated by their reference to the same model relationships.

For the models being the result of linearization of the initial non-linear models the analysis of SID should have a specific approach adjusted to the complexity and peculiarity of the problem being modelled. Note that the non-linear models with no observational redundancies (which are in fact more theoretical than practical structures) may, in general, have more than one solution. The approach to linearized models presented in this paper is only a straightforward extension of that proposed for the linear models and hence, requires further investigation. The spaces R_1 and R_2 cannot easily be defined precisely as they depend on the procedures for finding the approximate coordinates and on the degree of precision with which one can estimate their accuracy level. There are procedures which, prior to the main adjustment process, make it possible to eliminate a certain number of the evidently outlying observations.

It should be emphasized that, either being correct or incorrect, every GMM possesses SID. However, in incorrect models the imperceptible disturbances can be a superimposition of the observational disturbances and the disturbances being the effect of model imperfections or of the model's inadequacy. The complexity of such situations in linearized models speaks in favour of giving such models a separate treatment

There are numerous potential sources of disturbances (and hence — also of the imperceptible disturbances) in geodetic observations so we will not attempt to list them here. However, in the light of the analyses carried out so far we shall only comment that a permanent eccentric signalling of a network point will always generate an SID vector, whereas the disturbances induced in the measurement process (e.g. from the influence of external conditions) may only incidentally form a vector falling into that space.

Here are some practical conclusions which emerge from the fact of the SID existence in every GMM. By containing this space such a model incorporates some potential risks for the technology which uses this model as a data-processing tool. The identification of such risks, which can be done on the basis of the matrix **A**, is specially important in the models with a low level of internal reliability. Such models can be encountered in the engineering survey practice. Having in mind a proper quality of the survey products it would be well worth while to investigate the measurement procedure and the measurement process itself with respect to possible disturbing factors which may result in the occurrence of the SID component in the observation vector. A surveyor may substantially reduce the risk of the occurrence of imperceptible disturbances if he designs the measurement procedure in correlation with the shaping of the structure of a data-processing model. The analysis and technological identification of the SID can also be helpful in planning the post-measurement checks of the established framework.

A knowledge, albeit incomplete, of the probabilistic nature of gross observational errors or blunders would make it possible to compare (or at least – to rank) the chances of occurence of the specified disturbance vectors, and thus also those belonging to the SID. The results of research aimed at formulating, on an empirical basis, a probability law for gross errors in geodetic networks (see [12]) are promising and further research in this direction would be well worth while. It seems that the transfer of this approach to the sphere of engineering surveying, if successful, might result in improvements in the methodology of planning measuring schemes and procedures.

The findings of this paper can also be applied to linear regression models.

REFERENCES

[1] Caspary W.F., *Concepts of network and deformation analysis*, Monograph 11, School of Surveying, The University of New South Wales, Kensington, N.S.W., Australia, 1988.

[2] Baarda W., *A testing procedure for use in geodetic networks*, Neth. Geod. Comm., Publ. on Geodesy, New Series 2, No.5, Delft 1968.

[3] Prószyński W., *Criteria for internal reliability of linear least-squares models*. Bull. Geodesique, **68**, 162–167 (1994).

[4] Prószyński W., *Measuring the robustness potential of the least-squares estimation: geodetic illustration*, Journal of Geodesy, **71**, 652–659 (1997).

[5] Ding X., Coleman R., *Multiple outlier detection by evaluating redundancy contributions of observations*, Bull. Geodesique (1995).

[6] Rao C.R., Mitra S.K., *Generalized inverse of matrices and its applications*, John Wiley & Sons, INC., New York 1971.

[7] Wang J., Chen Y. *On the reliability measure of observations*, Acta Geodaet. et Cartograph. Sinica, English Edition, pp. 42–51 (1994).

[8] Schaffrin B., *Reliability measures for correlated observations*, J. of Surv. Engin. (1997).

[9] Rao C.R., *Linear statistical inference and its applications*. Wiley, New York, 1973.

[10] Perelmuter A., Nowak E., Prószyński W., *Practical implications of duality of the least squares method*, Proceedings of the Perelmuter Workshop on Dynamic Deformation Models, Technion City, Haifa, Israel, 1994.

[11] Chatterjee S., Hadi A., S., *Sensitivity analysis in linear regression*, John Wiley, New York 1988.

[12] Adamczewski Z. *Prawo błędów grubych (The law of gross errors)*, Research report in Polish, Warsaw University of Technology (1998) (to be published in Przegląd Geodezyjny in 2000).

Received April 21. 2000
Accepted July 3. 2000

Witold Prószyński

O przestrzeni niedostrzegalnych błędów obserwacyjnych w liniowych modelach Gaussa-Markowa z przykładami zaczerpniętymi z pomiarów inżynieryjnych

S t r e s z c z e n i e

Przedstawiono analizę znanej w teorii niezawodności przestrzeni błędów grubych bądź omyłek niewykrywalnych w procesie estymacji metodą najmniejszych kwadratów w liniowych modelach Gaussa-Markova. Podstawę analizy stanowi ogólna zależność wiążąca zaburzenia obserwacyjne i odpowiedzi modelu. Chociaż definicja tej przestrzeni jest identyczna z podaną w pracy [1] dochodzi się do niej w nieco odmienny sposób. Sformułowano kilka własności tej przestrzeni. Jedna z nich pokazuje związek tej przestrzeni z poziomem niezawodności modelu z obserwacjami nieskorelowanymi. Wprawdzie w rozważaniach teoretycznych uwzględnione są modele zlinearyzowane, podejście do nich jest zasadniczo zwykłym rozszerzeniem podejścia zaproponowanego dla modeli liniowych i nie można go traktować jako gotowej propozycji dla potrzeb praktycznych. Teoria ilustrowana jest przykładami zaczerpniętymi z pomiarów inżynieryjnych.

Витольд Прушыньски

О пространстве незаметных ошибок наблюдений в линейных моделях Гаусса-Маркова с примерами из инженерных измерений

Р е з ю м е

Представлены анализы пространства, известного в теории надёжности, грубых ошибок наблюдений или грубых ошибок абсолютно необнаруживаемых в линейных моделях Гаусса-Маркова в процессе определения методом наименьших квадратов. Анализы основаны на общих зависимостях, связывающих помехи наблюдений и реакцию модели. Хотя определение этого пространства тождественно с данным Каспарым [1], то оно получено немного другим способом. Определены некоторые свойства этого пространства, одно из них указывает его связи с уровнем надёжности модели с некоррелированными наблюдениями. Хотя слинеаризованные модели включены в теорию, применяемый подход к ним, являющийся в основном простым расширением способа предлагаемого для линейных моделей, не может рассматриваться как полное предложение для практических применений. Теория пояснена примерами из инженерских измерений.