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General solution of pseudorange equation system for GPS positioning

In this paper the general solution of nonlinear satellite pseudorange equation system has been given. This solution was obtained by the application of the new positional transformation determining relations between the points in three-dimension space.

It has been proved that computation of the position does not require knowledge of the light speed, occuring in pseudorange measurements, or determination of approximate coordinates.

The general solution allows one to obtain all possible solutions including complex conjugate positions.

This work has also stated that there exists some space regions in which it is not possible to determine the positions in the domain of real numbers. This is especially important in navigation of objects moving in the space.

INTRODUCTION

Determination of the position is one of the fundamental problems in geodesy and navigation. Position can be obtained using four pseudoranges measured simultaneously to four GPS satellites of which ephemerides are known.

The pseudorange measurements to four satellites give, known in literature, nonlinear satellite pseudorange equation system. In this system there are four unknowns. Three of them are coordinates of the point and one is a clock error caused by a lack of synchronization between GPS system time and the receiver clock.

The mathematical model is set up of nonlinear equations and the solution of this system is carried out iteratively, Seeber (1993), Hofmann-Wellenhof, Lichtenegger and Collins (1994), Leick (1995). It should be noted that iterative methods require knowledge of approximate coordinates.

Various approaches and their associated mathematical concepts upon which the closed-form and direct solutions of the nonlinear pseudorange equation system have been established and can be found in the literature, Bancroft (1985), Krause (1987), Abel and Chaffee (1991), Kleusberg (1994), Grafarend and Shan (1996). In the approaches based on geocentric and barycentric coordinates, Grafarend and Shan (1996), two solutions are produced and it has been proved there that the solution space is not unique.

In our considerations, all possible solution sets of nonlinear pseudorange equation system together with unknown complex conjugate position have been given. Position computation requires neither knowledge of approximate coordinates nor the speed of light, which must be known for pseudorange measurements and computation of the clock error.

This general solution has been obtained owing to the application of the positional transformation (Martusewicz (2000)). Establishing all possible solution sets for any relations between observation stations and the satellites is especially important in space navigation.

1. *Pseudorange equations*

Let four pseudoranges between four satellites and the observing point *P*, be measured simultaneously. Denoting the satellites by 1, 2, 3 and 4, the components of the geocentric position vectors of these satellites can be written as follows $1(X_1, Y_1, Z_1)$, $2(X_2, Y_2, Z_2)$, $3(X_3, Y_3, Z_3)$ and $4(X_4, Y_4, Z_4)$, respectively. If now we denote four pseudoranges, for simplicity, by the symbols p_1, p_2, p_3 and p_4, three unknown ECEF (Earth-Centered-Earth-Fixed) coordinates of the point *P* by $P(X_P, Y_P, Z_P)$ we have the following system of equations

$$\begin{aligned}
p_1 &= \sqrt{(X_1-X_P)^2+(Y_1-Y_P)^2+(Z_1-Z_P)^2} - cdt \\
p_2 &= \sqrt{(X_2-X_P)^2+(Y_2-Y_P)^2+(Z_2-Z_P)^2} - cdt \\
p_3 &= \sqrt{(X_3-X_P)^2+(Y_3-Y_P)^2+(Z_3-Z_P)^2} - cdt \\
p_4 &= \sqrt{(X_4-X_P)^2+(Y_4-Y_P)^2+(Z_4-Z_P)^2} - cdt
\end{aligned} \tag{1}$$

where *dt* is the receiver clock error, and *c* is the speed of light.

In order to solve this system of nonlinear equations, we put the expression *p* and *cdt* together on the right sides of (1), and having squared both sides, we obtain

$$\begin{aligned}
(X_1-X_P)^2+(Y_1-Y_P)^2+(Z_1-Z_P)^2 &= (p_1+cdt)^2 \\
(X_2-X_P)^2+(Y_2-Y_P)^2+(Z_2-Z_P)^2 &= (p_2+cdt)^2 \\
(X_3-X_P)^2+(Y_3-Y_P)^2+(Z_3-Z_P)^2 &= (p_3+cdt)^2 \\
(X_4-X_P)^2+(Y_4-Y_P)^2+(Z_4-Z_P)^2 &= (p_4+cdt)^2
\end{aligned} \tag{2}$$

The above equations have definite geometrical interpretation. The left sides of equations are the squares of the distances between the satellites and the point *P*, and the right sides are the squares of sum of two distances being the differences between the pseudoranges and the receiver clock offset. Because the distances are independent of coordinate systems, or more precise, are invariants of isometric mapping, the system of equations does not change when we write these equations in another coordinate system. To do this, we first introduce a transformation of the pseudorange equations based on the positional transformation, Martusewicz (2000).

2. *Transformation of pseudorange equations*

In order to transform pseudorange equations let us introduce a new a rectangular Cartesian coordinate system, *xyz,* based on satellites, which we consider as points. Transformation of the coordinates of the points to the new coordinate system, *xyz,* is carried out using the positional transformation.

Let us assume that the three points, 1, 2, and 3, are not lying on the same straight line. Let the point 1 be the origin of the new Cartesian *xyz* coordinate system, the *x*-axis passes through the points 1 and 2, and the *xy*-plane contains the point 3, Fig. 1.

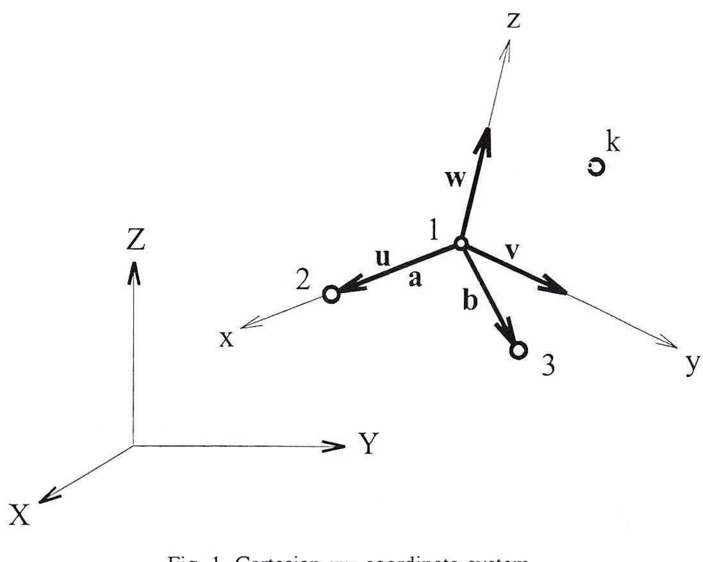

Fig. 1. Cartesian *xyz* coordinate system

Denoting vectors starting at point 1 and ending at point 2 and 3 by $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, and establishing the axis vectors of new coordinate system, for which the axes *x, y, z* are denoted by $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, we write

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{a} \\ \boldsymbol{v} &= (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{a} \\ \boldsymbol{w} &= \boldsymbol{a} \times \boldsymbol{b} \end{aligned} \tag{3}$$

From these vectors and their modules

$$u = \sqrt{u_X^2 + u_Y^2 + u_Z^2}$$
$$v = \sqrt{v_X^2 + v_Y^2 + v_Z^2} \tag{4}$$
$$w = \sqrt{w_X^2 + w_Y^2 + w_Z^2}$$

and also unit vectors of axes XYZ system $\boldsymbol{e}_X = [1, 0, 0]$; $\boldsymbol{e}_Y = [0, 1, 0]$; $\boldsymbol{e}_Z = [0, 0, 1]$, we obtain an orthogonal transformation matrix

$$\mathbf{M} = \begin{bmatrix} u_x^0 & u_y^0 & u_z^0 \\ v_x^0 & v_y^0 & v_z^0 \\ w_x^0 & w_y^0 & w_z^0 \end{bmatrix} \tag{5}$$

which elements are as follows

$$u_x^0 = \frac{u_x}{u} \quad u_y^0 = \frac{u_y}{u} \quad u_z^0 = \frac{u_z}{u}$$
$$v_x^0 = \frac{v_x}{v} \quad v_y^0 = \frac{v_y}{v} \quad v_z^0 = \frac{v_z}{v} \tag{6}$$
$$w_x^0 = \frac{w_x}{w} \quad w_y^0 = \frac{w_y}{w} \quad w_z^0 = \frac{w_z}{w}$$

Finding the coordinates for any point k in xyz coordinate system, we finally have

$$\begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} = \begin{bmatrix} u_x^0 & u_y^0 & u_z^0 \\ v_x^0 & v_y^0 & v_z^0 \\ w_x^0 & w_y^0 & w_z^0 \end{bmatrix} \begin{bmatrix} \Delta X_{1k} \\ \Delta Y_{1k} \\ \Delta Z_{1k} \end{bmatrix} \tag{7}$$

where ΔX_{1k}, ΔY_{1k}, ΔZ_{1k} are coordinate differences between any point k and the point 1 being the origin of xyz coordinate system.

From (7) we can determine the coordinates of the points 2, 3 and 4 in xyz system

$$\begin{bmatrix} x_2 & x_3 & x_4 \\ 0 & y_3 & y_4 \\ 0 & 0 & z_4 \end{bmatrix} = \begin{bmatrix} u_x^0 & u_y^0 & u_z^0 \\ v_x^0 & v_y^0 & v_z^0 \\ w_x^0 & w_y^0 & w_z^0 \end{bmatrix} \begin{bmatrix} \Delta X_{12} & \Delta X_{13} & \Delta X_{14} \\ \Delta Y_{12} & \Delta Y_{13} & \Delta Y_{14} \\ \Delta Z_{12} & \Delta Z_{13} & \Delta Z_{14} \end{bmatrix} \tag{8}$$

where the coordinates $y_2 = z_2 = z_3 = 0$ is the result of introducing the new coordinate system.

Now it is possible, taking into account (8), to express the left sides of equations (2), which are the squares of the distances independent of coordinate system, by x_i, y_i, z_i, $i =$ $= 1, 2, 3, 4$, and x_P, y_P, z_P.

Then finally we can write the equations (2) in the most suitable form

$$
\begin{aligned}
x_P^2 \quad &+ \quad y_P^2 \quad + \quad z_P^2 &= (p_1+cdt)^2 \\
(x_2-x_P)^2 &+ \quad y_P^1 \quad + \quad z_P^2 &= (p_2+cdt)^2 \\
(x_3-x_P)^2 &+ (y_3-y_P)^2 + \quad z_P^2 &= (p_3+cdt)^2 \\
(x_4-x_P)^2 &+ (y_4-y_P)^2 + (z_4-z_P)^2 &= (p_4+cdt)^2
\end{aligned}
\tag{9}
$$

where the coordinates, in *xyz* system, are obtained from (8).

In this way we have a new set of four equations which are equivalent to the starting equation (1).

3. *Determination of the clock error*

Now, let us consider the determination of the unknown *dt*. Squaring the expression in brackets in (9), and subtracting the first equation from the three remaining equations, having transformed them and taking the first equation in its original form, we finally write

$$
\begin{aligned}
x_P^2 + y_P^1 + z_P^2 &= (p_1+cdt)^2 \\
x_2 x_P &= a_1 cdt + c_1 \\
x_3 x_P + y_3 y_P &= a_2 cdt + c_2 \\
x_4 x_P + y_4 y_P + z_4 z_P &= a_3 cdt + c_3
\end{aligned}
\tag{10}
$$

where coefficients are

$$
\begin{aligned}
a_1 &= p_1 - p_2 \\
a_2 &= p_1 - p_3 \\
a_3 &= p_1 - p_4
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
c_1 &= (p_1^2 - p_2^2 + x_2^2)/2 \\
c_2 &= (p_1^2 - p_3^2 + x_3^2 + y_3^2)/2 \\
c_3 &= (p_1^2 - p_4^2 + x_4^2 + y_4^2 + z_4^2)/2
\end{aligned}
\tag{12}
$$

The last three equations, in which *dt* is taken as a parameter, can be written in matrix form

$$
\begin{bmatrix} x_2 & 0 & 0 \\ x_3 & y_3 & 0 \\ x_4 & y_4 & z_4 \end{bmatrix}
\begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix}
=
\begin{bmatrix} a_1 cdt + c_1 \\ a_2 cdt + c_2 \\ a_3 cdt + c_3 \end{bmatrix}
\tag{13}
$$

solving these equations, we have

$$\begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix} = \begin{bmatrix} x_2 & 0 & 0 \\ x_3 & y_3 & 0 \\ x_4 & y_4 & z_4 \end{bmatrix}^{-1} \begin{bmatrix} a_1cdt + c_1 \\ a_2cdt + c_2 \\ a_3cdt + c_3 \end{bmatrix} \tag{14}$$

where $x_2y_3z_4 \neq 0$.

The obtained results can be expressed as the function of two systems of equations having the same matrix of coefficients.

Passing to the first system of equations, which unknowns are denoted by x_a, y_a, z_a, we write

$$\begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} = \begin{bmatrix} x_2 & 0 & 0 \\ x_3 & y_3 & 0 \\ x_4 & y_4 & z_4 \end{bmatrix}^{-1} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{15}$$

and to the second, unknowns of which are denoted by x_c, y_c, z_c, we have

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} x_2 & 0 & 0 \\ x_3 & y_3 & 0 \\ x_4 & y_4 & z_4 \end{bmatrix}^{-1} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \tag{16}$$

where a_i, c_i, $i = 1, 2, 3$, are given by (11) and (12), respectively.

Then, taking into account (15) and (16), equation (14) can be written in the form

$$\begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix} = cdt \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} + \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} \tag{17}$$

Finding the sum of squares of the coordinates, expressed by (17), we obtain

$$x_P^2 + y_P^2 + z_P^2 = Ac^2dt^2 + 2Bcdt + C \tag{18}$$

where the following expressions are introduced

$$\begin{aligned} A &= x_a^2 + y_a^2 + z_a^2 \\ B &= x_ax_c + y_ay_c + z_az_c \\ C &= x_c^2 + y_c^2 + z_c^2 \end{aligned} \tag{19}$$

and x_i, y_i, z_i, $i = a, c$, we get from (15) and (16), respectively.

If we now substitute (18) into the first equation of the system (10), we get

$$Ac^2dt^2 + 2Bcdt + C = (p_1 + cdt)^2 \tag{20}$$

Before solving the above equation, (20), let us express the coefficients A, B and C as a function of the XYZ coordinates. In order to do that we transpose both sides of the equation (8)

$$\begin{bmatrix} x_2 & 0 & 0 \\ x_3 & y_3 & 0 \\ x_4 & y_4 & z_4 \end{bmatrix} = \begin{bmatrix} \Delta X_{12} & \Delta Y_{12} & \Delta Z_{12} \\ \Delta X_{13} & \Delta Y_{13} & \Delta Z_{13} \\ \Delta X_{14} & \Delta Y_{14} & \Delta Z_{14} \end{bmatrix} \begin{bmatrix} u_x^0 & v_x^0 & w_x^0 \\ u_y^0 & v_y^0 & w_y^0 \\ u_z^0 & v_z^0 & w_z^0 \end{bmatrix} \tag{21}$$

and having found the inverse of this equation, we write

$$\begin{bmatrix} x_2 & 0 & 0 \\ x_3 & y_3 & 0 \\ x_4 & y_4 & z_4 \end{bmatrix}^{-1} = \begin{bmatrix} u_x^0 & u_y^0 & u_z^0 \\ v_x^0 & v_y^0 & v_z^0 \\ w_x^0 & w_y^0 & w_z^0 \end{bmatrix} \begin{bmatrix} \Delta X_{12} & \Delta Y_{12} & \Delta Z_{12} \\ \Delta X_{13} & \Delta Y_{13} & \Delta Z_{13} \\ \Delta X_{14} & \Delta Y_{14} & \Delta Z_{14} \end{bmatrix}^{-1} \tag{22}$$

where it is taken into account that the inverse of the orthogonal matrix is equal to the transpose. Substituting (22) into (15), we have

$$\begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} = \begin{bmatrix} u_x^0 & u_y^0 & u_z^0 \\ v_x^0 & v_y^0 & v_z^0 \\ w_x^0 & w_y^0 & w_z^0 \end{bmatrix} \begin{bmatrix} \Delta X_{12} & \Delta Y_{12} & \Delta Z_{12} \\ \Delta X_{13} & \Delta Y_{13} & \Delta Z_{13} \\ \Delta X_{14} & \Delta Y_{14} & \Delta Z_{14} \end{bmatrix}^{-1} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{23}$$

Premultiplying both sides of equation (23) by the inverse of the transformation matrix and introducing new unknowns, which we denote by X_a, Y_a, Z_a, we write

$$\begin{bmatrix} X_a \\ Y_a \\ Z_a \end{bmatrix} = \begin{bmatrix} \Delta X_{12} & \Delta Y_{12} & \Delta Z_{12} \\ \Delta X_{13} & \Delta Y_{13} & \Delta Z_{13} \\ \Delta X_{14} & \Delta Y_{14} & \Delta Z_{14} \end{bmatrix}^{-1} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{24}$$

where the expressions a_1, a_2, a_3 are obtained from (11).

By analogy, taking into account equation (16) and introducing new unknowns denoted by X_c, Y_c, Z_c, we have

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix} = \begin{bmatrix} \Delta X_{12} & \Delta Y_{12} & \Delta Z_{12} \\ \Delta X_{13} & \Delta Y_{13} & \Delta Z_{13} \\ \Delta X_{14} & \Delta Y_{14} & \Delta Z_{14} \end{bmatrix}^{-1} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \tag{25}$$

where c_1, c_2, c_3, according to (12), after introducing squares of the distances obtained from the difference of the coordinates given in XYZ coordinate system, we establish the expressions given below

$$
\begin{aligned}
c_1 &= (p_1^2-p_2^2+\Delta X_{12}^2+\Delta Y_{12}^2+\Delta Z_{12}^2)/2 \\
c_2 &= (p_1^2-p_3^2+\Delta X_{13}^2+\Delta Y_{13}^2+\Delta Z_{13}^2)/2 \\
c_3 &= (p_1^2-p_4^2+\Delta X_{14}^2+\Delta Y_{14}^2+\Delta Z_{14}^2)/2
\end{aligned}
\tag{26}
$$

If now we denote the distances between the satellite 1 and the satellites 2, 3, 4 by s_{12}, s_{13}, s_{14}, respectively, we write

$$
\begin{aligned}
c_1 &= (p_1^2-p_2^2+s_{12}^2)/2 \\
c_2 &= (p_1^2-p_3^2+s_{13}^2)/2 \\
c_3 &= (p_1^2-p_4^2+s_{14}^2)/2
\end{aligned}
\tag{27}
$$

where

$$
\begin{aligned}
s_{12}^2 &= \Delta X_{12}^2+\Delta Y_{12}^2+\Delta Z_{12}^2 \\
s_{13}^2 &= \Delta X_{13}^2+\Delta Y_{13}^2+\Delta Z_{13}^2 \\
s_{14}^2 &= \Delta X_{14}^2+\Delta Y_{14}^2+\Delta Z_{14}^2
\end{aligned}
\tag{28}
$$

which we can obtain from the matrix (24) or (25) finding the sum of element squares of the first, the second, and the third row.

Taking into account the obtained expressions X_a, Y_a, Z_a and X_c, Y_c, Z_c, given by (24) and (25), respectively, as well as (15), (16), we can write equations being invariants of the orthogonal transformation

$$
\begin{aligned}
x_a^2+y_a^2+z_a^2 &= X_a^2+Y_a^2+Z_a^2 \\
x_a x_c+y_a y_c+z_a z_c &= X_a X_c+Y_a Y_c+Z_a Z_c \\
x_c^2+y_c^2+z_c^2 &= X_c^2+Y_c^2+Z_c^2
\end{aligned}
\tag{29}
$$

Then, in accordance with (19) and (27), we have

$$
\begin{aligned}
A &= X_a^2+Y_a^2+Z_a^2 \\
B &= X_a X_c+Y_a Y_c+Z_a Z_c \\
C &= X_c^2+Y_c^2+Z_c^2
\end{aligned}
\tag{30}
$$

where

$$\begin{bmatrix} X_a & X_c \\ Y_a & Y_c \\ Z_a & Z_c \end{bmatrix} = \begin{bmatrix} \Delta X_{12} & \Delta Y_{12} & \Delta Z_{12} \\ \Delta X_{13} & \Delta Y_{13} & \Delta Z_{13} \\ \Delta X_{14} & \Delta Y_{14} & \Delta Z_{14} \end{bmatrix}^{-1} \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \\ a_3 & c_3 \end{bmatrix} \tag{31}$$

as a result of putting together (24) and (25).

Taking into account (20) and denoting

$$b = cdt \tag{32}$$

after conversion, we obtain

$$(1-A)b^2 + 2(p_1 - B)b + p_1^2 - C = 0 \tag{33}$$

Note that when $1-A \neq 0$ we have the quadratic equation, if $1-A=0$ we get linear equation. Then we have to consider two cases.

If $A \neq 1$, the roots of the equation, (33), can be written in the form

$$b = \frac{B - p_1 \pm \sqrt{E}}{1-A} \tag{34}$$

where the discriminant of the quadratic equation

$$E = (p_1 - B)^2 - (1-A)(p_1^2 - C) \tag{35}$$

or in determinant form

$$E = \begin{vmatrix} p_1 - B & 1-A \\ p_1^2 - C & p_1 - B \end{vmatrix} \tag{36}$$

and A, B, C are given by (30).

Taking into account (32), we write

$$dt = bc^{-1} \tag{37}$$

then, the clock error, after substituting (34) into (37), has the form

$$dt = \frac{B - p_1 \pm \sqrt{E}}{1-A} c^{-1} \qquad \text{for } A \neq 1 \tag{38}$$

where E we obtain from (35) or (36).

If $A = 1$, equation (33) has the form

$$2(p_1 - B)b + p_1^2 - C = 0 \tag{39}$$

hence

$$b = \frac{p_1^2 - C}{2(B - p_1)} \tag{40}$$

Substituting (40) into (37), we write

$$dt = \frac{p_1^2 - C}{2(B - p_1)} c^{-1} \qquad \text{for } A = 1 \tag{41}$$

which determines the clock error in case when $A = 1$.

In conclusion of the above considerations we state that the clock error is the quantity which can be determined independently of the position of the point.

4. *Determination of the positions*

The coordinates of the point P in xyz coordinate system, according to (7), have the form

$$\begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix} = \begin{bmatrix} u_x^0 & u_y^0 & u_z^0 \\ v_x^0 & v_y^0 & v_z^0 \\ w_x^0 & w_y^0 & w_z^0 \end{bmatrix} \begin{bmatrix} \Delta X_{1P} \\ \Delta Y_{1P} \\ \Delta Z_{1P} \end{bmatrix} \tag{42}$$

Premultiplying both sides of above equations by the matrix $\mathbf{M}^{-1}$, being the inverse of the matrix (5), we have

$$\begin{bmatrix} \Delta X_{1P} \\ \Delta Y_{1P} \\ \Delta Z_{1P} \end{bmatrix} = \begin{bmatrix} u_x^0 & u_y^0 & u_z^0 \\ v_x^0 & v_y^0 & v_z^0 \\ w_x^0 & w_y^0 & w_z^0 \end{bmatrix}^{-1} \begin{bmatrix} x_P \\ y_P \\ z_P \end{bmatrix} \tag{43}$$

and substituting (14) into (43), we write

$$\begin{bmatrix} \Delta X_{1P} \\ \Delta Y_{1P} \\ \Delta Z_{1P} \end{bmatrix} = \begin{bmatrix} u_x^0 & u_y^0 & u_z^0 \\ v_x^0 & v_y^0 & v_z^0 \\ w_x^0 & w_y^0 & w_z^0 \end{bmatrix}^{-1} \begin{bmatrix} x_2 & 0 & 0 \\ x_3 & y_3 & 0 \\ x_4 & y_4 & z_4 \end{bmatrix}^{-1} \begin{bmatrix} a_1 cdt + c_1 \\ a_2 cdt + c_2 \\ a_3 cdt + c_3 \end{bmatrix} \tag{44}$$

Notice that as a result of premultiplying both sides of the equation (22) by the matrix $\mathbf{M}^{-1}$, we obtain the following expressions

$$\begin{bmatrix} u_x^0 & u_y^0 & u_z^0 \\ v_x^0 & v_y^0 & v_z^0 \\ w_x^0 & w_y^0 & w_z^0 \end{bmatrix}^{-1} \begin{bmatrix} x_2 & 0 & 0 \\ x_3 & y_3 & 0 \\ x_4 & y_4 & z_4 \end{bmatrix}^{-1} = \begin{bmatrix} \Delta X_{12} & \Delta Y_{12} & \Delta Z_{12} \\ \Delta X_{13} & \Delta Y_{13} & \Delta Z_{13} \\ \Delta X_{14} & \Delta Y_{14} & \Delta Z_{14} \end{bmatrix}^{-1} \tag{45}$$

and substituting (45) into (43), we have

$$\begin{bmatrix} \Delta X_{1P} \\ \Delta Y_{1P} \\ \Delta Z_{1P} \end{bmatrix} = \begin{bmatrix} \Delta X_{12} & \Delta Y_{12} & \Delta Z_{12} \\ \Delta X_{13} & \Delta Y_{13} & \Delta Z_{13} \\ \Delta X_{14} & \Delta Y_{14} & \Delta Z_{14} \end{bmatrix}^{-1} \begin{bmatrix} a_1 cdt + c_1 \\ a_2 cdt + c_2 \\ a_3 cdt + c_3 \end{bmatrix} \tag{46}$$

Taking into account of the parameter b, (32), and considering the equations bellow

$$\begin{bmatrix} \Delta X_{1P} \\ \Delta Y_{1P} \\ \Delta Z_{1P} \end{bmatrix} = \begin{bmatrix} X_P \\ Y_P \\ Z_P \end{bmatrix} - \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} \tag{47}$$

we get

$$\begin{bmatrix} X_P \\ Y_P \\ Z_P \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} + \begin{bmatrix} \Delta X_{12} & \Delta Y_{12} & \Delta Z_{12} \\ \Delta X_{13} & \Delta Y_{13} & \Delta Z_{13} \\ \Delta X_{14} & \Delta Y_{14} & \Delta Z_{14} \end{bmatrix}^{-1} \begin{bmatrix} ba_1 + c_1 \\ ba_2 + c_2 \\ ba_3 + c_3 \end{bmatrix} \tag{48}$$

If we now introduce the expressions (24) and (25), the coordinates of the point P, (48), can be also written in the following form

$$\begin{bmatrix} X_P \\ Y_P \\ Z_P \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} + \begin{bmatrix} bX_a + X_c \\ bY_a + Y_c \\ bZ_a + Z_c \end{bmatrix} \tag{49}$$

or

$$\begin{bmatrix} X_P \\ Y_P \\ Z_P \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} + b \begin{bmatrix} X_a \\ Y_a \\ Z_a \end{bmatrix} + \begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix} \tag{50}$$

where X_i, Y_i, Z_i, $i = a, c$, are obtained from (31), and b from (34) or (40).

Note that the speed of light does not appear in the expressions X_i, Y_i, Z_i, $i = a, c$, and the expression b, so for computation of the position of the point we do not need to know the speed of light. To avoid misunderstanding we should add that the speed of light is the basic quantity for pseudorange measurements as well as for computation of the clock error.

The obtained functions allow us to compute the position of the point without determination of the approximate coordinates.

5. *Cases of obtained positions*

The necessary condition of position determination, which is the result of finding the inverse of matrix, (48), has the following form

$$D = \begin{vmatrix} \Delta X_{12} & \Delta Y_{12} & \Delta Z_{12} \\ \Delta X_{13} & \Delta Y_{13} & \Delta Z_{13} \\ \Delta X_{14} & \Delta Y_{14} & \Delta Z_{14} \end{vmatrix} \neq 0 \qquad (51)$$

The above expression is a sixfold volume of tetrahedron which vertices are determined by satellites. It means in geometric terms that the satellites are not lying on the same plane.

Irrespective of the given condition, $D \neq 0$, which will be not mentioned in each consideration of the further part of the work, there are other conditions. These conditions are determined by the expressions A and E. The said expressions have definite geometric interpretation, but it is not the subject of our study.

Considering the problem of all solutions, we state the following.

1. One real position

If in the square equation, (33), coefficient $1 - A$ equals zero, then

$$A = 1 \qquad (52)$$

and we have one real position.

This position is obtained as a result of substituting (40) into (48).

2. Two real positions

In case when

$$A \neq 1, \quad E > 0 \qquad (53)$$

we get two roots of the equation and therefore we have two positions.

Note that when applying an iterative method, improper choice of the approximate coordinates when positions are convergent may result in a false position.

3. A real double position

For

$$A \neq 1, \quad E = 0 \qquad (54)$$

we have two equal roots of square equation. Then we obtain a real double position.

4. Complex conjugate positions
If

$$A \neq 1, \quad E < 0 \tag{55}$$

then there is no real solution of the quadratic equation and so we obtain two positions expressed by complex conjugate roots. Because in the determined positions there are complex conjugate roots, we call them complex conjugate positions.

The fact that there are other solutions, beside the set of real numbers, indicates the existance of space regions in which it is not possible to establish positions in real number domain. This limitation has special meaning when the positions of the flying objects are determined, which are in any relation to navigation satellites.

6. *Numerical considerations*

We now discuss how the functions obtained as a result of the solution of pseudorange equations, given in literature, work in numerical computations. Bearing in mind the general character of considerations we take arbitrary satellite constellation. Moreover, the pseudorange observations are carried out from any position in relation to navigation satellites.

In the examples below we use the same ephemerides only changing the quantities of measured pseudoranges. For computation we apply full numerical expressions, without approximations, which leads to absolute control of the obtained results.

Example 1

Geocentric coordinates of four satellites

$$1(3, 4, 4), \quad 2(5, 3, 4), \quad 3(5, 4, 5), \quad 4(4, 5, 4)$$

and pseudoranges measured simultaneously to four GPS satellites

$$p_1 = 2, \quad p_2 = 3, \quad p_3 = 3, \quad p_4 = 2$$

give the following pseudorange equation system

$$2 = \sqrt{(3 - X_P)^2 + (4 - Y_P)^2 + (4 - Z_P)^2} - cdt$$
$$3 = \sqrt{(5 - X_P)^2 + (3 - Y_P)^2 + (4 - Z_P)^2} - cdt$$
$$3 = \sqrt{(5 - X_P)^2 + (4 - Y_P)^2 + (5 - Z_P)^2} - cdt$$
$$2 = \sqrt{(4 - X_P)^2 + (5 - Y_P)^2 + (4 - Z_P)^2} - cdt$$

Compute the clock error, dt, and the coordinates $P(X_P, Y_P, Z_P)$.

Solution

Stating that

$$D = \det\begin{bmatrix} \Delta X_{12} & \Delta Y_{12} & \Delta Z_{12} \\ \Delta X_{13} & \Delta Y_{13} & \Delta Z_{13} \\ \Delta X_{14} & \Delta Y_{14} & \Delta Z_{14} \end{bmatrix} = \det\begin{bmatrix} 2 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = -3 \neq 0$$

and finding the expressions

$$\begin{aligned} a_1 &= 2-3 = -1 & c_1 &= (4-9+5)/2 = 0 \\ a_2 &= 2-3 = -1 & c_2 &= (4-9+5)/2 = 0 \\ a_3 &= 2-2 = \ \ 0 & c_3 &= (4-4+2)/2 = 1 \end{aligned}$$

we obtain

$$\begin{bmatrix} X_a & X_c \\ Y_a & Y_c \\ Z_a & Z_c \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 2 \\ -2 & 3 & -2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 & 1 \\ 1 & 2 \\ -1 & -2 \end{bmatrix}$$

hence

$$\begin{aligned} A &= X_a^2 + Y_a^2 + Z_a^2 = 1/3 \\ B &= X_a X_c + Y_a Y_c + Z_a Z_c = 1/3 \\ C &= X_c^2 + Y_c^2 + Z_c^2 = 1 \end{aligned}$$

Because $A = 1/3 \neq 1$ we determine

$$E = \begin{vmatrix} p_1 - B & 1-A \\ p_1^2 - C & p_1 - B \end{vmatrix} = \frac{1}{9}\begin{vmatrix} 5 & 2 \\ 9 & 5 \end{vmatrix} = \frac{7}{9} > 0$$

and state that we have two real positions, $E > 0$.

Then, setting coefficients of square equations, $1 - A = 2/3$; $2(p_1 - B) = 10/3$; $p_1^2 - C = 3$, we write

$$2b^2 + 10b + 9 = 0$$

thus

$$b = \left(-5 \pm \sqrt{7}\right)/2$$

what gives two solutions.

For the first solution the clock error is

$$dt_1 = \left(-5 + \sqrt{7}\right)/2c$$

and the coordinates

$$\begin{bmatrix} X_{P_1} \\ Y_{P_1} \\ Z_{P_1} \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 5-\sqrt{7}+2 \\ -5+\sqrt{7}+4 \\ 5-\sqrt{7}-4 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 25-\sqrt{7} \\ 23+\sqrt{7} \\ 25-\sqrt{7} \end{bmatrix}$$

For the second solution, we have

$$dt_2 = \left(-5 - \sqrt{7}\right)/2c$$

and the following coordinates

$$\begin{bmatrix} X_{P_2} \\ Y_{P_2} \\ Z_{P_2} \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 5-\sqrt{7}+2 \\ -5+\sqrt{7}+4 \\ 5-\sqrt{7}-4 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 25+\sqrt{7} \\ 23-\sqrt{7} \\ 25+\sqrt{7} \end{bmatrix}$$

what determines the second position of the point.

Control

Substituting the obtained solutions into the right sides of initial equations, and introducing the sign of the square root, for the first solution plus and for the second minus, we obtain sequently 2, 3, 3, 2. Therefore, we conclude that the left and the right sides of equations are equal, what is the control of the solution.

Example 2

Geocentric coordinates of four satellites

$$1(3, 4, 4), \quad 2(5, 3, 4), \quad 3(5, 4, 5), \quad 4(4, 5, 4)$$

and pseudorange measured simultaneously

$$p_1 = 2, \quad p_2 = 2, \quad p_3 = 3, \quad p_4 = 2$$

determine the pseudorange equations

$$2 = \sqrt{(3-X_P)^2+(4-Y_P)^2+(4-Z_P)^2}-cdt$$
$$3 = \sqrt{(5-X_P)^2+(3-Y_P)^2+(4-Z_P)^2}-cdt$$
$$3 = \sqrt{(5-X_P)^2+(4-Y_P)^2+(5-Z_P)^2}-cdt$$
$$2 = \sqrt{(4-X_P)^2+(5-Y_P)^2+(4-Z_P)^2}-cdt$$

Compute the clock error, *dt,* and the coordinates $P(X_P, Y_P, Z_P)$.

S o l u t i o n

Stating that

$$D = \det\begin{bmatrix} \Delta X_{12} & \Delta Y_{12} & \Delta Z_{12} \\ \Delta X_{13} & \Delta Y_{13} & \Delta Z_{13} \\ \Delta X_{14} & \Delta Y_{14} & \Delta Z_{14} \end{bmatrix} = \det\begin{bmatrix} 2 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = -3 \neq 0$$

and establishing expressions

$$a_1 = 2-2 = 0 \qquad c_1 = (4-4+5)/2 = 5/2$$
$$a_2 = 2-3 = -1 \qquad c_2 = (4-9+5)/2 = 0$$
$$a_3 = 2-2 = 0 \qquad c_3 = (4-4+2)/2 = 1$$

we obtain

$$\begin{bmatrix} X_a & X_c \\ Y_a & Y_c \\ Z_a & Z_c \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 2 \\ -2 & 3 & -2 \end{bmatrix}\begin{bmatrix} 0 & 5 \\ -2 & 0 \\ 0 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 0 & 7 \\ 0 & -1 \\ -6 & -14 \end{bmatrix}$$

Determining

$$A = X_a^2+Y_a^2+Z_a^2 = 1$$
$$B = X_aX_c+Y_aY_c+Z_aZ_c = 7/3$$
$$C = X_c^2+Y_c^2+Z_c^2 = 41/6$$

we state that $A = 1$, thus we have only one real solution.

Coefficients of linear equations are $2(p_1-B) = -2/3$; $p_1^2-C = -17/6$, then the equation has the form

$$4b+17 = 0$$

and hence

$$b = -17/4$$

For the clock error, we have

$$dt = -17/4c$$

and finding the coordinates, we write

$$\begin{bmatrix} X_P \\ Y_P \\ Z_P \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} + \frac{1}{12}\begin{bmatrix} 14 \\ -2 \\ 23 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 50 \\ 46 \\ 71 \end{bmatrix}$$

what determines the position of the point.

Control

Substituting the obtained unknowns into the right sides of equation, and taking the square root of sign minus, we obtain sequently 2, 2, 3, 2 what checks the solution.

Example 3

On the base of geocentric satellite coordinates

$$1(3, 4, 4), \quad 2(5, 3, 4), \quad 3(5, 4, 5), \quad 4(4, 5, 4)$$

and pseudorange measured simultaneously

$$p_1 = 2, \quad p_2 = 4, \quad p_3 = 4, \quad p_4 = 2$$

the following equations are obtained

$$2 = \sqrt{(3-X_P)^2 + (4-Y_P)^2 + (4-Z_P)^2} - cdt$$
$$4 = \sqrt{(5-X_P)^2 + (3-Y_P)^2 + (4-Z_P)^2} - cdt$$
$$4 = \sqrt{(5-X_P)^2 + (4-Y_P)^2 + (5-Z_P)^2} - cdt$$
$$2 = \sqrt{(4-X_P)^2 + (5-Y_P)^2 + (4-Z_P)^2} - cdt$$

Compute the clock error, *dt,* and the coordinates $P(X_P, Y_P, Z_P)$.

Solution

Stating that

$$D = \det\begin{bmatrix} \Delta X_{12} & \Delta Y_{12} & \Delta Z_{12} \\ \Delta X_{13} & \Delta Y_{13} & \Delta Z_{13} \\ \Delta X_{14} & \Delta Y_{14} & \Delta Z_{14} \end{bmatrix} = \det\begin{bmatrix} 2 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = -3 \neq 0$$

and finding the expressions

$$\begin{aligned} a_1 &= 2-4 = -2 \qquad & c_1 &= (4-16+5)/2 = -7/2 \\ a_2 &= 2-4 = -2 \qquad & c_2 &= (4-16+5)/2 = -7/2 \\ a_3 &= 2-2 = 0 \qquad & c_3 &= (4-4+2)/2 = 1 \end{aligned}$$

we have

$$\begin{bmatrix} X_a & X_c \\ Y_a & Y_c \\ Z_a & Z_c \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 2 \\ -2 & 3 & -2 \end{bmatrix}\begin{bmatrix} -4 & -7 \\ -4 & -7 \\ 0 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -4 & -5 \\ 1 & 11 \\ -4 & -11 \end{bmatrix}$$

hence

$$\begin{aligned} A &= X_a^2+Y_a^2+Z_a^2 = 4/3 \\ B &= X_aX_c+Y_aY_c+Z_aZ_c = 3 \\ C &= X_c^2+Y_c^2+Z_c^2 = 89/12 \end{aligned}$$

Establishing $A = 4/3 \neq 1$, and finding determinate

$$E = \begin{vmatrix} p_1-B & 1-A \\ p_1^2-C & p_1-B \end{vmatrix} = -\frac{1}{12}\begin{vmatrix} 12 & 4 \\ 41 & 12 \end{vmatrix} = -\frac{5}{3} < 0$$

we state that there are complex conjugate positions, $E < 0$.

Setting coefficients of square equations, $1-A = -1/3$; $2(p_1-B) = -2$; $p_1^2-C = -41/12$, we write

$$4b^2+24b+41 = 0$$

and from this

$$b = \left(-6 \pm i\sqrt{5}\right)/2$$

what gives two complex conjugate solutions.

For the first solution the clock error is

$$dt_1 = \left(-6+i\sqrt{5}\right)/2c$$

and the coordinates are

$$\begin{bmatrix} X_{P_1} \\ Y_{P_1} \\ Z_{P_1} \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 7 - i2\sqrt{5} \\ -1 + i2\sqrt{5} \\ 1 - i2\sqrt{5} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 25 - i2\sqrt{5} \\ 23 + i2\sqrt{5} \\ 25 - i2\sqrt{5} \end{bmatrix}$$

for the second solution the clock error is

$$dt_2 = \left(-6 - i\sqrt{5}\right)/2c$$

and the coordinates are as follows

$$\begin{bmatrix} X_{P_2} \\ Y_{P_2} \\ Z_{P_2} \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 7 + i2\sqrt{5} \\ -1 - i2\sqrt{5} \\ 1 + i2\sqrt{5} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 25 + i2\sqrt{5} \\ 23 - i2\sqrt{5} \\ 25 + i2\sqrt{5} \end{bmatrix}$$

what gives the solutions in complex conjugate form.

Control

In order to control the obtained results we substitute the found unknowns to the right sides of equations getting 2, 4, 4, 2. Since the control requires operations on conjugates we introduce this procedure in Appendix.

8. CONCLUSIONS

The obtained general solution of nonlinear satellite pseudorange equation system leads to the following conclusions.

For computation of the position one does not need to know the speed of light, which is in pseudorange measurements and in computation of the clock error.

Determination of satellite positions does not require knowledge of approximate coordinates of the point.

The general solution of satellite pseudorange equations ensures getting all possible positions: one real position, two real positions, a real double position, and complex conjugate positions.

This solution of the pseudorange equation system shows that there exist some space regions in which it is not possible to determine positions in the domain of real numbers. It is most important for objects moving in the space which can take any position in relation to navigation satellites.

Appendix. Control of Solution

Checking of the solution of Example 3 has been given below.

The first equation

Substituting the unknowns obtained from the first solution, X_{P_1}, Y_{P_1}, Z_{P_1} and dt_1, into the first initial equation, we write

$$\sqrt{\left[3-\left(25-i2\sqrt{5}\right)/6\right]^2+\left[4-\left(23+i2\sqrt{5}\right)/6\right]^2+\left[4-\left(25-i2\sqrt{5}\right)/6\right]^2}-\left(-6+i\sqrt{5}\right)/2=$$

$$=\frac{1}{6}\sqrt{\left(-7+i2\sqrt{5}\right)^2+\left(1-i2\sqrt{5}\right)^2+\left(-1+i2\sqrt{5}\right)^2}+\left(6-i\sqrt{5}\right)/2=$$

$$=\frac{1}{6}\sqrt{-9-i36\sqrt{5}}+\left(6-i\sqrt{5}\right)/2=\frac{1}{2}\left(\sqrt{-1-i4\sqrt{5}}+6-i\sqrt{5}\right)$$

Because $(a+ib)^2=a^2-b^2+i2ab$ then $(-2+i\sqrt{5})^2=-1-i4\sqrt{5}$ and hence

$$\frac{1}{2}\left(\sqrt{-1-i4\sqrt{5}}+6-i\sqrt{5}\right)=\frac{1}{2}\left(\sqrt{\left(-2+i\sqrt{5}\right)^2}+6-i\sqrt{5}\right)=\frac{1}{2}\left(-2+i\sqrt{5}+6-i\sqrt{5}\right)=2$$

what is identical with the left side of initial equation.

The second equation

When substituting X_{P_1}, Y_{P_1}, Z_{P_1} and dt_1 into the second initial equation, we have

$$\sqrt{\left[5-\left(25-i2\sqrt{5}\right)/6\right]^2+\left[3-\left(23+i2\sqrt{5}\right)/6\right]^2+\left[4-\left(25-i2\sqrt{5}\right)/6\right]^2}-\left(-6+i\sqrt{5}\right)/2=$$

$$=\frac{1}{6}\sqrt{\left(5+i2\sqrt{5}\right)^2+\left(-5-i2\sqrt{5}\right)^2+\left(-1+i2\sqrt{5}\right)^2}+\left(6-i\sqrt{5}\right)/2=$$

$$=\frac{1}{6}\sqrt{-9-i36\sqrt{5}}+\left(6-i\sqrt{5}\right)/2=\frac{1}{2}\left(\sqrt{-1-i4\sqrt{5}}+6-i\sqrt{5}\right)=$$

$$=\frac{1}{2}\left(\sqrt{\left(2+i\sqrt{5}\right)^2}+6-i\sqrt{5}\right)=\frac{1}{2}\left(-2+i\sqrt{5}+6-i\sqrt{5}\right)=4$$

what is the same as in the initial equation.

The third equation

Taking X_{P_1}, Y_{P_1}, Z_{P_1} and dt_1, we obtain

$$\sqrt{\left[5-\left(25-i2\sqrt{5}\right)/6\right]^2+\left[4-\left(23+i2\sqrt{5}\right)/6\right]^2+\left[5-\left(25-i2\sqrt{5}\right)/6\right]^2}-\left(-6+i\sqrt{5}\right)/2=$$

$$=\frac{1}{6}\sqrt{\left(5+i2\sqrt{5}\right)^2+\left(1-i2\sqrt{5}\right)^2+\left(5+i2\sqrt{5}\right)^2}+\left(6-i\sqrt{5}\right)/2=$$

$$=\frac{1}{6}\sqrt{-9+i36\sqrt{5}}+\left(6-i\sqrt{5}\right)/2=\frac{1}{2}\left(\sqrt{-1-i4\sqrt{5}}+6-i\sqrt{5}\right)=$$

$$=\frac{1}{2}\left(\sqrt{\left(2+i\sqrt{5}\right)^2}+6-i\sqrt{5}\right)=4$$

what should be expected.

The fourth equation

On the basis X_{P_1}, Y_{P_1}, Z_{P_1} and dt_1, we write

$$\sqrt{\left[4-\left(25-i2\sqrt{5}\right)/6\right]^2+\left[5-\left(23+i2\sqrt{5}\right)/6\right]^2+\left[4-\left(25-i2\sqrt{5}\right)/6\right]^2}-\left(-6+i\sqrt{5}\right)/2=$$

$$=\frac{1}{6}\sqrt{\left(-1+i2\sqrt{5}\right)^2+\left(7-i2\sqrt{5}\right)^2+\left(-1+i2\sqrt{5}\right)^2}+\left(6-i\sqrt{5}\right)/2=$$

$$=\frac{1}{6}\sqrt{-9-i36\sqrt{5}}+\left(6-i\sqrt{5}\right)/2=\frac{1}{2}\left(\sqrt{-1-i4\sqrt{5}}+6-i\sqrt{5}\right)=$$

$$=\frac{1}{2}\left(\sqrt{\left(-2+i\sqrt{5}\right)^2}+6-i\sqrt{5}\right)=2$$

which is the desired result.

Considering the above we state that the left and the right sides of the equations are really the same, and equals 2, 4, 4, 2, respectively.

In case of the second solution, where the unknowns are X_{P_2}, Y_{P_2}, Z_{P_2} and dt_2, we have the same computing operations, so we do not give them here. We only state that we obtain the same quantities, 2, 4, 4 and 2, respectively.

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Ogólne rozwiązanie układu równań pseudoodległościowych dla wyznaczania pozycji GPS

S t r e s z c z e n i e

W pracy podano ogólne rozwiązanie nieliniowego układu równań pseudoodległościowych dla wyznaczania pozycji satelitarnych. Rozwiązanie otrzymano w wyniku zastosowania transformacji pozycyjnej, ustalającej względne położenia punktów w przestrzeni trójwymiarowej.

Podane rozwiązanie pozwala na obliczanie pozycji bez znajomości prędkości światła, występującej w pomiarach pseudoodległości, oraz ustalania współrzędnych przybliżonych.

Ogólne rozwiązanie pozwala na otrzymywanie wszystkich możliwych pozycji, łącznie z pozycjami w dziedzinie liczb urojonych.

Wskazano na istnienie pewnych obszarów przestrzeni w których nie można ustalić pozycji w dziedzinie liczb rzeczywistych. Ma to szczególne znaczenie w nawigacji obiektów poruszających się w przestrzeni.

Януш Мартусевич

Общее решение системы псевдодистанционных уравниений для определения позиции GPS

Р е з ю м е

В работе представлено общее решение нелинейной системы псевдодистанционных уравниений для определения позиции спутника. Решение получено в результате применения позиционной трансформации, определяющей относительное положение пунктов в трёхмерном пространстве.

Представленное решение даёт возможность вычисления позиции без знакомства скорости света, присутствующей в измерениях мнимых расстояний, а также без определения приблизительных координат.

Общее решение даёт возможность получения всех возможных позиций, вместе с позициями в области мнимых чисел.

Указано присутствие таких областей пространства, в которых является невозможным определение позиции в области действительных чисел. Это имеет особое значение в навигации объектов двигающихся в пространстве.