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Jerzy Balcerzak
Jan Panasiuk

Institute of Photogrammetry and Cartography
Warsaw University of Technology
(00-661 Warszawa, Plac Politechniki 1)

Indicator of deformations of mutual projection of two arbitrary, regular, three-dimensional spaces

In theory of regular projection of a surface into another surface, the term of the indicator of projection deformations, as a topological substitute of an infinitesimal circle of a unit radius is known. In the case of regular projection of a three-dimensional space into a three-dimensional space, a three-axis ellipsoid is the indicator of projection deformations, being the topological substitute of an infinitesimal unit sphere. This paper presents the attempt to analytically describe the infinitesimal unit sphere in the original space and its topological substitute in the projected space.

By analogy to projection of a surface into another surface [1], identity projection of the three-dimensional space into the same space may be in general described by means of the system of two functions:

$$\begin{cases} \vec{r} = \vec{r}(u,\ \upsilon,\ w) \\ \vec{\tau} = \vec{\tau}(u,\ \upsilon,\ w)' \end{cases} \qquad (u,\ \upsilon,\ w) \in \omega, \qquad \omega - \text{area} \tag{1}$$

The local scale of length for such projection is expressed by the quotient:

$$\frac{d\vec{r}}{|d\vec{r}|} = \frac{\dfrac{d\vec{r}}{dw}}{\left|\dfrac{d\vec{r}}{dw}\right|} = \frac{\left(\vec{r}_u \dfrac{du}{d\upsilon} + \vec{r}_\upsilon\right)\dfrac{d\upsilon}{dw} + \vec{r}_w}{\left|\left(\vec{r}_u \dfrac{du}{d\upsilon} + \vec{r}_\upsilon\right)\dfrac{d\upsilon}{dw} + \vec{r}_w\right|} \tag{2}$$

In the formula (2) the following relation occur:

$$\frac{du}{dv} \overset{def}{=} \frac{H\cot A - F}{E} = \frac{H\cos A - F\sin A}{E\sin A},$$

$$E = |\vec{r}_u|^2,\ F = \vec{r}_u \cdot \vec{r}_v,\ G = |\vec{r}_u|^2,\ H = |\vec{r}_u \times \vec{r}_v| = \sqrt{EG - F^2}, \tag{3}$$

$$\frac{dv}{d\omega} \overset{def}{=} \frac{\hat{H}\cot\hat{A} - \hat{F}}{\hat{E}} = \frac{\hat{H}\cos\hat{A} - \hat{F}\sin\hat{A}}{\hat{E}\sin\hat{A}},$$

$$\hat{E} = |\hat{\vec{r}}_u|^2,\ \hat{F} = \hat{\vec{r}}_u \cdot \hat{\vec{r}}_v,\ \hat{G} = |\hat{\vec{r}}_u|^2,\ \hat{H} = |\hat{\vec{r}}_u \times \hat{\vec{r}}_v| = \sqrt{\hat{E}\hat{G} - \hat{F}^2}, \tag{4}$$

where A and $\hat{A}$ mean directional angles, and:

$$\hat{\vec{r}}_u = \left(\vec{r}_u \frac{du}{dv} + \vec{r}_v\right) = \vec{r}_u \left(\frac{H\cos A - F\sin A}{E\sin A} + \vec{r}_v\right) =$$

$$= \frac{H\vec{r}_u\cos A + (E\vec{r}_v - F\vec{r}_u)\sin A}{E\sin A} \tag{5}$$

$$\hat{\vec{r}}_v \equiv \vec{r}_w$$

are corresponding partial vectors of tangency to parametric lines u = const, v = const and w = const.

The ratio $\frac{d\vec{r}}{|d\vec{r}|}$ may be presented in the form of linear combination

$$\vec{\rho} = \frac{d\vec{r}}{|d\vec{r}|} = \frac{\hat{\vec{r}}_u \frac{dv}{dw} + \vec{r}_w}{\left|\hat{\vec{r}}_u \frac{dv}{dw} + \vec{r}_w\right|} = \frac{\hat{\vec{r}}_u \frac{\hat{H}\cos\hat{A} - \hat{F}\sin\hat{A}}{\hat{E}\sin\hat{A}} + \vec{r}_w}{\left|\hat{\vec{r}}_u \frac{\hat{H}\cos\hat{A} - \hat{F}\sin\hat{A}}{\hat{E}\sin\hat{A}} + \vec{r}_w\right|} =$$

$$= \frac{\dfrac{\hat{H}\hat{\vec{r}}_u \cos\hat{A} + (\hat{E}\hat{\vec{r}}_w - \hat{F}\hat{\vec{r}}_u)\sin\hat{A}}{\hat{E}\sin\hat{A}}}{\left|\dfrac{\hat{H}\hat{\vec{r}}_u \cos\hat{A} + (\hat{E}\hat{\vec{r}}_w - \hat{F}\hat{\vec{r}}_u)\sin\hat{A}}{\hat{E}\sin\hat{A}}\right|} = \frac{\hat{H}\hat{\vec{r}}_u \cos\hat{A} + (\hat{E}\hat{\vec{r}}_w - \hat{F}\hat{\vec{r}}_u)\sin\hat{A}}{\hat{H}\sqrt{\hat{E}}} \equiv$$

$$\equiv \hat{\vec{\rho}}_1 \cos\hat{A} + \hat{\vec{\rho}}_2 \sin\hat{A} \tag{6}$$

of two vectors

$$\hat{\vec{\rho}}_1 = \frac{\hat{\vec{r}}_u}{\sqrt{\hat{E}}}, \qquad \hat{\vec{\rho}}_2 = \frac{\hat{E}\hat{\vec{r}}_w - \hat{F}\hat{\vec{r}}_u}{\hat{H}\sqrt{\hat{E}}} \tag{7}$$

vectors (7) are unit and orthogonal vectors

$$|\hat{\vec{\rho}}_1| = 1, \qquad \hat{\vec{\rho}}_1 \cdot \hat{\vec{\rho}}_2 = 0, \qquad |\hat{\vec{\rho}}_2| = 1, \qquad |\hat{\vec{\rho}}_1 \times \hat{\vec{\rho}}_2| = 1 \tag{8}$$

The vector $\hat{\vec{\rho}}_1$ is expressed by the linear combination, which is known from theory of projections of surfaces:

$$\hat{\vec{\rho}}_1 = \frac{\hat{\vec{r}}_u}{\sqrt{\hat{E}}} = \frac{H\vec{r}_u \cos A + (E\vec{r}_v - F\vec{r}_u)\sin A}{E\sin A}\frac{E\sin A}{H\sqrt{E}} =$$

$$= \frac{\vec{r}_u}{\sqrt{E}}\cos A + \frac{E\vec{r}_v - F\vec{r}_u}{H\sqrt{E}}\sin A = \vec{\rho}_1 \cos A + \vec{\rho}_2 \sin A \tag{9}$$

where

$$\vec{\rho}_1 = \frac{\vec{r}_u}{\sqrt{E}}, \qquad \vec{\rho}_2 = \frac{E\vec{r}_v - F\vec{r}_u}{H\sqrt{E}} \tag{10}$$

Basing on (7) and (9) the vector $\vec{\rho}$ may have the more simple form:

$$\vec{\rho} = \frac{d\vec{r}}{|d\vec{r}|} = \hat{\vec{\rho}}_1 \cos\hat{A} + \hat{\vec{\rho}}_2 \sin\hat{A} = (\vec{\rho}_1 \cos A + \vec{\rho}_2 \sin A)\cos\hat{A} + \vec{\rho}_3 \sin\hat{A} \tag{11}$$

The vector $\vec{\rho}_3$ is the linear combination

$$\vec{\rho}_3 = \hat{\vec{\rho}}_2 = \frac{\hat{E}\vec{r}_w - \hat{F}\hat{\vec{r}}_u}{\hat{H}\sqrt{\hat{E}}} \tag{12}$$

The equation (11), as the relation of the vector $\vec{\rho}$ and directional angles A and $\hat{A}$ describes, in general, the unit sphere of the unit radius $|\vec{\rho}| = 1$.

If $\hat{F} = 0$, the expression, which describes $\vec{\rho}_3$ in (12) is simplified to the form:

$$\vec{\rho}_3 = \frac{\vec{r}_w}{|\vec{r}_w|} = \vec{\rho}_w \tag{13}$$

This happens only when the vector $\vec{r}_w$ is orthogonal to vectors $\vec{\rho}_1$ and $\vec{\rho}_2$.

In general, when $\vec{\rho}_3$ is not identical with $\vec{\rho}_w$ in the interval $A \in \langle -\pi, \pi)$, $A' \in \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle$, an interesting property of the system A = const, $\hat{A}$ = const occurs. This property partially disappears, if the polar angle $\hat{A}$ is limited from the top by the value:

$$\hat{\Theta}' = \operatorname{arccot}\left(\frac{\hat{\vec{\rho}}_1 \cdot \vec{\rho}_3}{|\hat{\vec{\rho}}_1 \times \hat{\rho}_3|}\right) = \operatorname{arccot}\left(\frac{(\vec{\rho}_1 \cos A + \vec{\rho}_2 \sin A) \cdot \left(\frac{\hat{E}\vec{r}_w - \hat{F}\hat{\vec{r}}_u}{\hat{H}\sqrt{\hat{E}}}\right)}{\left|(\vec{\rho}_1 \cos A + \vec{\rho}_2 \sin A) \cdot \left(\frac{\hat{E}\vec{r}_w - \hat{F}\hat{\vec{r}}_u}{\hat{H}\sqrt{\hat{E}}}\right)\right|}\right) =$$

$$= \operatorname{arccot}\left(\frac{(\vec{\rho}_1 \cos A + \vec{\rho}_2 \sin A) \cdot (\hat{E}\vec{r}_w - \hat{F}\hat{\vec{r}}_u)}{|(\vec{\rho}_1 \cos A + \vec{\rho}_2 \sin A) \cdot (\hat{E}\vec{r}_w - \hat{F}\hat{\vec{r}}_u)|}\right) \tag{14}$$

This also results in limitation of the angle $\hat{A}$ from the bottom. Thus the standard analytical description of the unit sphere, with poles in the points $\vec{r}_w$ and $(-\vec{r}_w)$, is obtained.

Substitution of vectors $\vec{r}_u$, $\vec{r}_v$, $\vec{r}_w$ in the second space (1) by vectors $\vec{\tau}_u$, $\vec{\tau}_v$, $\vec{\tau}_w$ leads to affine transformation of the unit sphere into the ellipsoid:

$$\vec{\mu} = \frac{d\vec{\tau}}{|d\vec{r}|} = (\vec{\mu}_1 \cos A + \vec{\mu}_2 \sin A)\cos\hat{A} + \vec{\mu}_3 \sin\hat{A}$$

$$(A, \hat{A}) \in \omega, \qquad \omega = \left\{(A, \hat{A}) : A \in \langle -\pi, \pi), \qquad \hat{A} \in \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle\right\} \tag{15}$$

where

$$\vec{\mu}_1 = \frac{\vec{r}_u}{\sqrt{E}}, \qquad \vec{\mu}_2 = \frac{E\vec{\tau}_v - F\vec{\tau}_u}{H\sqrt{E}}, \qquad \vec{\mu}_3 = \frac{\hat{E}\vec{\tau}_w - \hat{F}\hat{\vec{\tau}}_u}{\hat{H}\sqrt{\hat{E}}} \tag{16}$$

The ellipsoid (15), spanned on structural vectors (16), may be considered as the indicator of projection deformations in the projection of the form:

$$\begin{cases} \vec{r} = \vec{r}(u, v, w) \\ \vec{\tau} = \vec{\tau}(u, v, w) \end{cases}, \qquad (u, v, w) \in \omega, \qquad \omega - \text{area} \tag{17}$$

of a three-dimensional space into a three-dimensional space.

REFERENCES

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Jerzy Balcerzak
Jan Panasiuk

Wskaźnica zniekształceń odwzorowawczych dwóch dowolnych regularnych przestrzeni trójwymiarowych na siebie

Streszczenie

W niniejszej pracy w formie najogólniejszej podany został, w ujęciu wektorowym, opis analityczny elipsoidy zniekształceń odwzorowawczych w regularnym odwzorowaniu przestrzeni trójwymiarowej w przestrzeń trójwymiarową. Wyprowadzony został opis analityczny infinitezymalnej kuli jednostkowej w otoczeniu różniczkowym punktu przestrzeni oryginału i rozpoznany jej oddpowiednik topologiczny w różniczkowym otoczeniu punktu przestrzeni obrazu.

Ежы Бальцежак
Ян Панасюк

Индикатриса отображающих деформаций двух любых регулярных трёхмерных пространств на себя

Резюме

В теории регулярных отображений поверхности на поверхность известным является понятие индикатрисы отображающих деформаций как эквивалента инфинитезимальной окружности с единичным радиусом. Зато в случае регулярного отображения трёхмерного пространства в трёхмерное пространство, индикатрисой отображающих деформаций является трёхосный эллипсоид, с точки зрения топологии соответствующий инфинитезимальной единичной сфере. В настоящей работе была предпринята попытка аналитического описания инфинитезимальной единичной сферы в пространстве оригинала и её топологического эквивалента в пространстве изображения.