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Properties and analysis of the accuracy of estimation results obtained by the DiSTFA method in monitoring displacements and strains

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Abstract: The paper presents the results of research on the **DiSTFA** method (**Di**splacements and **S**trains using **T**ransformation and **F**ree **A**djustment) for the determination of displacement and strains of a surface determined in unstable reference systems. Additionally, covariance matrices were introduced to assess the accuracy of estimation results. The theoretical discussion includes an example of its application in a simulated, three-dimensional geodetic network. The obtained results encourage further, more detailed analysis of real geodetic networks.

Keywords: displacements, strains, free adjustment, transformation, unstable reference system, covariance matrices

1. Introduction

The determination of the displacements and strains of engineering structures remains a problem in a contemporary surveying engineering. In order to carry out such measurements and to interpret the results, the problem of the correct positioning of points of the geodetic network has to be solved. As a matter of principle, the points of the network, at which measurements will be made are situated outside the zone affected by the area being displaced and strained. In addition, before the control measurement is performed the condition of the stability of the network points should be checked. It may turn out from the test that all or some of the network points do not meet the condition of stability of their position. Proposals for checking and solving the problem have been presented by various authors (e.g. Chen et al., 1990). The stability of points may be examined by the similarity transformation of displacement components d_i with the condition $\sum |d_i| = \min$. Another method, used in full automatic monitoring of an engineering object with a robotic total station (RTS) with automatic target recognition and GPS observation, is the combined adjustment of both types of observations (Bond et al., 2005). Other methods which involve statistical tests in the verification of stability of network points have also been used in practical analyses.

The significance of the problem is indicated by the works presented during international symposia (Chrzanowski and Wilkins, 2006; Chrzanowski and Whitaker, 2008).

This paper presents a novel method of determining the displacement and strains of the surface on an unstable base and an accuracy analysis of the adjustment results, represented by covariance matrices. It develops further the issue discussed in previous papers (Kamiński 2008a, 2008b). Considering the application of the method, it has been named **DiSTFA**, (**Di**splacements and **S**trains using **T**ransformation and **F**ree **A**djustment). **DiSTFA** can be used for both determination of displacement and strains of the surface established in unstable reference systems, as well as in examining the stability of geodetic network points.

The **DiSTFA** method derived for a 3D network is a further development of the concept of vertical movements monitoring, presented by Wiśniewski (1989).

Some numerical tests with simulated observations were used for presenting the features of the method. It was assumed that the observations were performed by the GPS positioning technique. The results provide encouragement for further, more detailed research on real engineering objects.

2. The DiSTFA method

Discussions concerning the theoretical background of the **DiSTFA** method can be found in the papers by Kamiński (2008a, 2008b). This paper presents the **DiSTFA** method only in a form necessary for proper understanding of the issues raised in this study.

Let us define a plane which is the reference surface for the measurements in the assumed, local coordinate (X,Y,Z) system as Ω (Fig. 1) (Kamiński 2008 a, 2008b).

Measurements aiming to determine displacements and strains are performed at certain known moments "j", ($j = 0, 1, 2, 3, \ldots$) and referred (typically) to the original measurement "$j = 0$" treated as input data. Control measurements ($j = 1, 2, 3, \ldots$) are determined by surfaces Θ^j and corresponding plane Ω^j. Let us assume that s_i^j is a distance between any i point, ($i = 1, 2, 3, \ldots$) of irregular surface Θ^j and a projection of a point situated on an optimal surface. The vector $\mathbf{s}_i^j = [s_X^j, s_Y^j, s_Z^j]^T$ can be identified with a vector normal to Ω^j plane determining the temporary location of this plane in the unstable reference frame X^j, Y^j, Z^j.

Let us further assume that observations of vector components $\Delta X_{i,k}^{obs}$, $\Delta Y_{i,k}^{obs}$,$\Delta Z_{i,k}^{obs}$, (i, $k = 1, 2, \ldots, m$; m – number of observed points) were carried out using GPS. Observation equations at the measurement moment "j" have thus the following form

$$
\begin{aligned}
(\Delta X_{i,k})^j &= (\Delta X_{i,k}^{obs})^j + v_{i,k_{\Delta X}}^j \\
(\Delta Y_{i,k})^j &= (\Delta Y_{i,k}^{obs})^j + v_{i,k_{\Delta Y}}^j \qquad k \neq i \qquad (1)\\
(\Delta Z_{i,k})^j &= (\Delta Z_{i,k}^{obs})^j + v_{i,k_{\Delta Z}}^j
\end{aligned}
$$

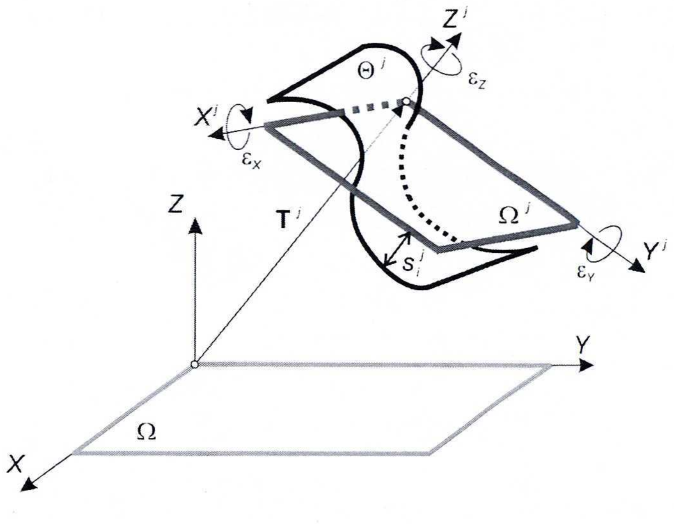

Fig. 1. Reference systems

Assuming that $\mathbf{\Delta X} = [\Delta X_{i,k}, \Delta Y_{i,k}, \Delta Z_{i,k}]^{\mathrm{T}}$, $\mathbf{l} = \left[\Delta X_{i,k}^{obs}, \Delta Y_{i,k}^{obs}, \Delta Z_{i,k}^{obs}\right]^{\mathrm{T}}$ and $\mathbf{v} = \left[v_{i,k_{\Delta X}}, v_{i,k_{\Delta Y}}, v_{i,k_{\Delta Z}}\right]^{\mathrm{T}}$, the relationship (1) can be presented using matrix notation in the following form

$$\mathbf{\Delta X} = \mathbf{l} + \mathbf{v} \tag{2}$$

To perform analyses, the point coordinates determined in a local, unstable reference system X^j, Y^j, Z^j, should be transformed to the theoretical reference system X, Y, Z (Kamiński, 2008a). The 6-parameter transformation of spatial coordinates, of the following form

$$\mathbf{X}_i^* = \mathbf{T}_i^j + \mathbf{R}^j \mathbf{x}_i^j \tag{3}$$

can be performed, where

$\mathbf{x}_i^j = \left[X_i^j,\ Y_i^j,\ Z_i^j\right]^{\mathrm{T}}$ – the vector of coordinates of an i-th point in an unstable, instantaneous coordinates system "j",

$\mathbf{T}^j = [T_X^j,\ T_Y^j,\ T_Z^j]^{\mathrm{T}}$ – the vector of translation of the origin of coordinates system,

$\mathbf{R}^j = \begin{bmatrix} 1 & \varepsilon_Z^j & -\varepsilon_Y^j \\ -\varepsilon_Z^j & 1 & \varepsilon_X^j \\ \varepsilon_Y^j & -\varepsilon_X^j & 1 \end{bmatrix}$ – the matrix of rotation of the coordinate system X^j, Y^j, Z^j.

The relationship (3) may also be written as follows (for a single point)

$$\begin{aligned} X_i &= T_X^j + X_i^j + \varepsilon_Z^j Y_i^j - \varepsilon_Y^j Z_i^j \\ Y_i &= T_Y^j - \varepsilon_Z^j X_i^j + Y_i^j + \varepsilon_X^j Z_i^j \\ Z_i &= T_Z^j + \varepsilon_Y^j X_i^j - \varepsilon_X^j Y_i^j + Z_i^j \end{aligned} \tag{4}$$

Substituting (4) into (1), gives the following system of observation equations (Kamiński 2008a)

$$\begin{aligned} v^j_{i,k_{\Delta X}} &= s^j_{Xk} - s^j_{Xi} + (Y^{j=0}_k - Y^{j=0}_i)\varepsilon^j_Z - (Z^{j=0}_k - Z^{j=0}_i)\varepsilon^j_Y - (\Delta X^{obs}_{i,k})^j \\ v^j_{i,k_{\Delta Y}} &= s^j_{Yk} - s^j_{ki} - (X^{j=0}_k - X^{j=0}_i)\varepsilon^j_Z + (Z^{j=0}_k - Z^{j=0}_i)\varepsilon^j_X - (\Delta Y^{obs}_{i,k})^j \\ v^j_{i,k_{\Delta Z}} &= s^j_{Zk} - s^j_{Zi} + (X^{j=0}_k - X^{j=0}_i)\varepsilon^j_Y - (Y^{j=0}_k - Y^{j=0}_i)\varepsilon^j_X - (\Delta Z^{obs}_{i,k})^j \end{aligned} \tag{5}$$

or in the matrix form

$$\mathbf{v}^j = \mathbf{A}_1\mathbf{s}^j + \mathbf{A}_2\boldsymbol{\varepsilon}^j - \mathbf{l}^j = \mathbf{AX} - \mathbf{l} \tag{6}$$

where $\mathbf{v}^j$ – vector of residuals, $\mathbf{A} = [\mathbf{A}_1, \mathbf{A}_2]$ – known design matrices, $\mathbf{X} = [\mathbf{s}^j, \boldsymbol{\varepsilon}^j]^{\mathrm{T}}$ – vector of unknown parameters of the model, $\mathbf{s} = [s_{X_i}, s_{Y_i}, s_{Z_i}]^{\mathrm{T}}$ – vector of the distances between the examined plane and the optimal plane ($i = 1, 2, \ldots, m$; m – number of points in which GPS measurements were carried out), and $\boldsymbol{\varepsilon} = [\varepsilon_X, \varepsilon_Y, \varepsilon_Z]^{\mathrm{T}}$– vector of rotations about the coordinate axes.

Vectors $\Delta\mathbf{s}^j = [\Delta s^j_{X_{i,k}}, \Delta s^j_{Y_{i,k}}, \Delta s^j_{Z_{i,k}}]^{\mathrm{T}}$ obtained for the pair of points (i, k) can further be applied for the determination of the linear strains using the following relationship

$$\Delta\mathbf{S}^{(j+1)-j} = \Delta\mathbf{s}^{j+1} - \Delta\mathbf{s}^j \tag{7}$$

On the other hand, mutual displacements (represented only by rotation angles) can be determined as follows

$$\Delta\boldsymbol{\varepsilon}^{(j+1)-j} = \boldsymbol{\varepsilon}^{j+1} - \boldsymbol{\varepsilon}^j \tag{8}$$

The following adjustment problem corresponds to the **DiSTFA** method

$$\left.\begin{aligned} &\varphi(\hat{\mathbf{s}}^j, \hat{\boldsymbol{\varepsilon}}^j) = (\mathbf{v}^j)^{\mathrm{T}}\mathbf{P}(\mathbf{v}^j) = \min \\ &\mathbf{v}^j = [\mathbf{A}_1, \mathbf{A}_2]\begin{bmatrix} \hat{\mathbf{s}}^j \\ \hat{\boldsymbol{\varepsilon}}^j \end{bmatrix} - \mathbf{l}^j = \mathbf{A}\hat{\mathbf{X}}^j - \mathbf{l}^j \end{aligned}\right\} j = 0, 1, 2, \ldots \tag{9}$$

where $\varphi(\bullet)$ is the objective function of the adjustment problem.

Let $\mathbf{C}_l = m_0^2\mathbf{Q} = m_0^2\mathbf{P}^{-1}$ be the covariance matrix of observations, where m_0^2 is the unknown variance coefficient, $\mathbf{Q} = \mathbf{P}^{-1}$ is the cofactor matrix, and $\mathbf{P}$ is the weight matrix. The solution of (9) is an estimator of the unknown parameters vector $\hat{\mathbf{X}}$

$$\hat{\mathbf{X}}^j = \begin{bmatrix} \hat{\mathbf{s}}^j \\ \hat{\boldsymbol{\varepsilon}}^j \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1^{\mathrm{T}}\mathbf{PA}_1 & \mathbf{A}_1^{\mathrm{T}}\mathbf{PA}_2 \\ \mathbf{A}_2^{\mathrm{T}}\mathbf{PA}_1 & \mathbf{A}_2^{\mathrm{T}}\mathbf{PA}_2 \end{bmatrix}^{-} \begin{bmatrix} \mathbf{A}_1^{\mathrm{T}}\mathbf{Pl} \\ \mathbf{A}_2^{\mathrm{T}}\mathbf{Pl} \end{bmatrix} \tag{10}$$

The notation $[\bullet]^-$ means g – inverse of a matrix.

The results obtained by using the **DiSTFA** method should be assessed with an accuracy analysis, using the law of covariance propagation

$$\mathbf{C} = \mathbf{D}\mathbf{C}_l\mathbf{D}^{\mathrm{T}} \tag{11}$$

where **D** is the transformation matrix.

In the accuracy analysis of the results obtained, one can use, e.g. covariance matrices $\mathbf{C}_{\hat{\mathbf{X}}}$ of estimated parameters of the model, the vector $\mathbf{C}_{\mathbf{v}}$ of residuals, as well as the adjusted components of GPS vectors $\mathbf{C}_{\Delta\hat{\mathbf{X}}}$.

The covariance matrix of the estimated parameters of the model has the following form (Kamiński, 2008a)

$$\mathbf{C}_{\hat{\mathbf{X}}} = m_0^2 \begin{bmatrix} \mathbf{A}_1^{\mathrm{T}}\mathbf{P}\mathbf{A}_1 & \mathbf{A}_1^{\mathrm{T}}\mathbf{P}\mathbf{A}_2 \\ \mathbf{A}_2^{\mathrm{T}}\mathbf{P}\mathbf{A}_1 & \mathbf{A}_2^{\mathrm{T}}\mathbf{P}\mathbf{A}_2 \end{bmatrix}^{-} = m_0^2 \mathbf{Q}_{\hat{\mathbf{X}}} \tag{12}$$

where the cofactor matrix $\mathbf{Q}_{\hat{\mathbf{X}}}$ is as follows

$$\mathbf{Q}_{\hat{\mathbf{X}}} = \begin{bmatrix} \mathbf{A}_1^{\mathrm{T}}\mathbf{P}\mathbf{A}_1 & \mathbf{A}_1^{\mathrm{T}}\mathbf{P}\mathbf{A}_2 \\ \mathbf{A}_2^{\mathrm{T}}\mathbf{P}\mathbf{A}_1 & \mathbf{A}_2^{\mathrm{T}}\mathbf{P}\mathbf{A}_2 \end{bmatrix}^{-} \tag{13}$$

and $m_0^2 = (\mathbf{v}^j)^{\mathrm{T}}\mathbf{P}(\mathbf{v}^j)/f$; $f = n - r + d$ – number of degrees of freedom; n – number of observations, r – number of unknowns, d – network defect.

In order to determine the $\mathbf{C}_{\mathbf{v}}$ matrix, the vector **v** of residuals should be expressed as follows

$$\begin{aligned} \mathbf{v} = \mathbf{A}\hat{\mathbf{X}} - \mathbf{l} &= [\mathbf{A}_1, \mathbf{A}_2] \begin{bmatrix} \mathbf{A}_1^{\mathrm{T}}\mathbf{P}\mathbf{A}_1 & \mathbf{A}_1^{\mathrm{T}}\mathbf{P}\mathbf{A}_2 \\ \mathbf{A}_2^{\mathrm{T}}\mathbf{P}\mathbf{A}_1 & \mathbf{A}_2^{\mathrm{T}}\mathbf{P}\mathbf{A}_2 \end{bmatrix}^{-} \begin{bmatrix} \mathbf{A}_1^{\mathrm{T}}\mathbf{P} \\ \mathbf{A}_2^{\mathrm{T}}\mathbf{P} \end{bmatrix} \mathbf{l} - \mathbf{l} \\ &= \left([\mathbf{A}_1, \mathbf{A}_2] \begin{bmatrix} \mathbf{A}_1^{\mathrm{T}}\mathbf{P}\mathbf{A}_1 & \mathbf{A}_1^{\mathrm{T}}\mathbf{P}\mathbf{A}_2 \\ \mathbf{A}_2^{\mathrm{T}}\mathbf{P}\mathbf{A}_1 & \mathbf{A}_2^{\mathrm{T}}\mathbf{P}\mathbf{A}_2 \end{bmatrix}^{-} \begin{bmatrix} \mathbf{A}_1^{\mathrm{T}}\mathbf{P} \\ \mathbf{A}_2^{\mathrm{T}}\mathbf{P} \end{bmatrix} - \mathbf{E} \right) \mathbf{l} \\ &= -\left(\mathbf{E} - [\mathbf{A}_1, \mathbf{A}_2] \begin{bmatrix} \mathbf{A}_1^{\mathrm{T}}\mathbf{P}\mathbf{A}_1 & \mathbf{A}_1^{\mathrm{T}}\mathbf{P}\mathbf{A}_2 \\ \mathbf{A}_2^{\mathrm{T}}\mathbf{P}\mathbf{A}_1 & \mathbf{A}_2^{\mathrm{T}}\mathbf{P}\mathbf{A}_2 \end{bmatrix}^{-} \begin{bmatrix} \mathbf{A}_1^{\mathrm{T}}\mathbf{P} \\ \mathbf{A}_2^{\mathrm{T}}\mathbf{P} \end{bmatrix} \right) \mathbf{l} \end{aligned} \tag{14}$$

where **E** is a unit matrix.

Let us denote

$$\mathbf{D} = \left(\mathbf{E} - [\mathbf{A}_1, \mathbf{A}_2] \begin{bmatrix} \mathbf{A}_1^{\mathrm{T}}\mathbf{P}\mathbf{A}_1 & \mathbf{A}_1^{\mathrm{T}}\mathbf{P}\mathbf{A}_2 \\ \mathbf{A}_2^{\mathrm{T}}\mathbf{P}\mathbf{A}_1 & \mathbf{A}_2^{\mathrm{T}}\mathbf{P}\mathbf{A}_2 \end{bmatrix}^{-} \begin{bmatrix} \mathbf{A}_1^{\mathrm{T}}\mathbf{P} \\ \mathbf{A}_2^{\mathrm{T}}\mathbf{P} \end{bmatrix} \right) = (\mathbf{E} - \mathbf{A}\mathbf{Q}_{\hat{\mathbf{X}}}\mathbf{A}^{\mathrm{T}}\mathbf{P}) \tag{15}$$

The covariance matrix of residuals will be as follows

$$\mathbf{C}_{\mathbf{v}} = \mathbf{D}\mathbf{C}_l\mathbf{D}^{\mathrm{T}} = (\mathbf{E} - \mathbf{A}\mathbf{Q}_{\hat{\mathbf{X}}}\mathbf{A}^{\mathrm{T}}\mathbf{P}) m_0^2 \mathbf{P}^{-1} (\mathbf{E} - \mathbf{A}\mathbf{Q}_{\hat{\mathbf{X}}}\mathbf{A}^{\mathrm{T}}\mathbf{P})^{\mathrm{T}} = m_0^2 (\mathbf{P}^{-1} - \mathbf{A}\mathbf{Q}_{\hat{\mathbf{X}}}\mathbf{A}^{\mathrm{T}}) \tag{16}$$

and the cofactor matrix $\mathbf{Q}_{\mathbf{v}}$ becomes

$$\mathbf{Q_v} = (\mathbf{P}^{-1} - \mathbf{A}\mathbf{Q}_{\hat{\mathbf{X}}}\mathbf{A}^{\mathrm{T}}) \qquad (17)$$

Setting up the covariance matrix $\mathbf{C}_{\Delta\hat{\mathbf{X}}} = \mathbf{D}\mathbf{C_l}\mathbf{D}^{\mathrm{T}}$ of adjusted components of GPS vectors, one obtains (for $\mathbf{v} = \mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{l} - \mathbf{l}$)

$$\Delta\hat{\mathbf{X}} = \mathbf{l} + \mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{l} - \mathbf{l} = \mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{l} \qquad (18)$$

Assuming $\mathbf{D}=\mathbf{A}^{\mathrm{T}}\left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-}\mathbf{A}^{\mathrm{T}}\mathbf{P}$, a covariance matrix of adjusted components of the GPS vector becomes

$$\mathbf{C}_{\Delta\hat{\mathbf{X}}} = \mathbf{D}\mathbf{C_l}\mathbf{D}^{\mathrm{T}} = m_0^2\mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{P}^{-1}\mathbf{P}\mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-}\mathbf{A}^{\mathrm{T}} = m_0^2\mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A})^{-}\mathbf{A}^{\mathrm{T}} \qquad (19)$$

with the cofactor matrix $\mathbf{Q}_{\Delta\hat{\mathbf{X}}}$

$$\mathbf{Q}_{\Delta\hat{\mathbf{X}}} = \mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A})^{-}\mathbf{A}^{\mathrm{T}} \qquad (20)$$

3. An example of application

The practical features of the **DiSTFA** method have been examined based on the example of a simulated, 3D network established using GPS survey (Fig. 2). It was assumed that the network consists of four points (1, 2, 3, 4) of the following coordinates: $1(X = 0, Y = 0, Z = 0)$, $2(X = 0, Y = 800, Z = 0)$, $3(X = 700, Y = 800, Z = 0)$, $4(X = 700, Y = 0, Z = 0)$. The coordinates were used to calculate the theoretical components of vectors spanned on the network points. The components of those vectors were subsequently deformed by the values of the range of <-0.003 m; 0.003 m>, in the process of simulating random errors. This produced "primary measurement results" $\Delta X_{i,k}^{obs}$, $\Delta Y_{i,k}^{obs}$, $\Delta Z_{i,k}^{obs}$ at epoch $j = 0$.

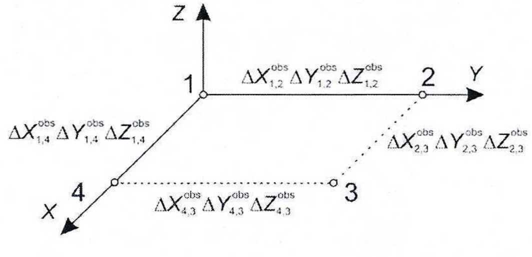

Fig. 2. The test network

Subsequently, simulating a movement of network points, a 6-parameter transformation of the primary measurement results was performed, yielding 5 variants of calculations. Table 1 presents the applied transformation parameters. The first three variants assume that the network points only subside. In variant IV it was assumed that the system rotated about the OX-axis by an angle ε_X while in variant V transformation consisted of translation and three rotations.

Table 1. Calculation variants and their transformation parameters

Variant number	T_X [m]	T_Y [m]	T_Z [m]	ε_X^{cc}	ε_Y^{cc}	ε_Z^{cc}
I	0.000	0.000	-0.001	0	0	0
II	0.000	0.000	-0.002	0	0	0
III	0.000	0.000	-0.030	0	0	0
IV	0.000	0.000	0.000	3.000	0	0
V	-0.002	-0.002	-0.002	2.387	2.387	2.387

Simulated "primary measurement results, $j = 0$" and "results of current measurements, j" obtained from particular variants are presented in Table 2.

Table 2. Simulated results of measurements "j" [m]

Vector components	Primary measurement	Variant I	Variant II	Variant III	Variant IV	Variant V
$\Delta X_{1,2}^{obs}$	0.002	0.002	0.002	0.002	0.002	0.007
$\Delta Y_{1,2}^{obs}$	799.997	799.997	799.997	799.997	799.997	799.999
$\Delta Z_{1,2}^{obs}$	0.001	0.000	-0.001	-0.029	-0.037	-0.004
$\Delta X_{2,3}^{obs}$	699.999	699.999	699.999	699.999	699.999	700.001
$\Delta Y_{2,3}^{obs}$	0.002	0.002	0.002	0.002	0.002	0.001
$\Delta Z_{2,3}^{obs}$	-0.003	-0.004	-0.005	-0.033	-0.003	-0.002
$\Delta X_{4,3}^{obs}$	0.003	0.003	0.003	0.003	0.003	0.008
$\Delta Y_{4,3}^{obs}$	799.998	799.998	799.998	799.998	799.998	800.000
$\Delta Z_{4,3}^{obs}$	0.001	0.000	-0.001	-0.029	-0.037	-0.004
$\Delta X_{1,4}^{obs}$	700.002	700.002	700.002	700.002	700.002	700.004
$\Delta Y_{1,4}^{obs}$	-0.002	-0.002	-0.002	-0.002	-0.002	-0.003
$\Delta Z_{1,4}^{obs}$	0.003	0.002	0.001	-0.027	0.003	0.004

Following design matrices were taken for calculations

$$\mathbf{A}_1 = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & -0.001 & 799.997 \\ 0.001 & 0 & -0.002 \\ -799.997 & 0.002 & 0 \\ 0 & 0.003 & 0.002 \\ -0.003 & 0 & -699.999 \\ -0.002 & 699.999 & 0 \\ 0 & -0.001 & 799.998 \\ 0.001 & 0 & -0.003 \\ -799.998 & 0.003 & 0 \\ 0 & -0.003 & -0.002 \\ 0.003 & 0 & -700.002 \\ 0.002 & 700.002 & 0 \end{bmatrix}$$

and the weight matrix **P**

$$\mathbf{P} = diag(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4) \text{ with } \mathbf{P}_l = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix}, \quad l = 1,2,3,4.$$

was applied.

The parameters to be determined are

$$\mathbf{s} = [s_{X_1},\ s_{Y_1},\ s_{Z_1},\ s_{X_2},\ s_{Y_2},\ s_{Z_2},\ s_{X_3},\ s_{Y_3},\ s_{Z_3},\ s_{X_4},\ s_{Y_4},\ s_{Z_4}]^{\mathrm{T}},\ \text{and}\ \boldsymbol{\varepsilon}^{\mathrm{T}} = [\varepsilon_X,\ \varepsilon_Y,\ \varepsilon_Z]^{\mathrm{T}}$$

Table 3 presents the values of parameters $\hat{\varepsilon}^{cc}_{\hat{X}}, \hat{\varepsilon}^{cc}_{\hat{Y}}, \hat{\varepsilon}^{cc}_{\hat{Z}}$ determined with the use of the **DiSTFA** method, and the differences between the results obtained from particular variants "j" and the primary measurement "j = 0". Since this is a purely theoretical example, the estimated rotation angles ε (for $j=0$) indicate very low torsion of the system with respect to all coordinate axes, that is at the level of 10^{-7}.

Table 3. Estimation results

Variant number	$(\hat{\varepsilon}^{cc}_{\hat{X}})^j$ / $\hat{\varepsilon}^j_{\hat{X}} - \hat{\varepsilon}^{j=0}_{\hat{X}}$	$(\hat{\varepsilon}^{cc}_{\hat{Y}})^j$ / $\hat{\varepsilon}^j_{\hat{Y}} - \hat{\varepsilon}^{j=0}_{\hat{Y}}$	$(\hat{\varepsilon}^{cc}_{\hat{Z}})^j$ / $\hat{\varepsilon}^j_{\hat{Z}} - \hat{\varepsilon}^{j=0}_{\hat{Z}}$
Primary measurement	0.0000118	–0.0000070	0.0000062
	–	–	–
I	0.7958	–0.9095	–0.0000068
	0.7957881	–0.909493	–0.0000006
II	1.5915	–1.8189	–0.0000074
	1.5914881	–1.818893	–0.0000136
III	23.8732	–27.2836	–0.0000255
	23.8731881	–27.283593	–0.0000317
IV	30.2395	–0.0000070	–0.0000062
	30.2394881	0.0000	–0.0000124
V	3.9789	0.9094	2.6479
	3.9788881	0.909407	2.6478938

j = I, II, III, IV, V

The analysis of the adjustment results (Table 3) shows that the **DiSTFA** method indicated displacement of the network points in all variants considered. Differences of parameters $\Delta\hat{\varepsilon} = \hat{\varepsilon}^j_m - \hat{\varepsilon}^{j=0}_m$, ($m$: X, Y, Z); obtained are within the range of $< -27.283593^{cc}; 30.2394881^{cc} >$.

Based on the adjusted components of the vector $\hat{\mathbf{s}}$, increments $\Delta\hat{s}$ between respective points were calculated (Table 4).

Table 4. Adjustment results [m]

Increments	Primary measurement	Variant I	Variant II	Variant III	Variant IV	Variant V
$\Delta\hat{s}_{X_{1,2}}$	0.0031	0.0031	0.0031	0.0030	0.0031	0.0047
$\Delta\hat{s}_{Y_{1,2}}$	799.9962	799.9962	799.9962	799.9962	799.9962	799.9982
$\Delta\hat{s}_{Z_{1,2}}$	0.0025	0.0025	0.0024	0.0024	0.0024	0.0024
$\Delta\hat{s}_{X_{2,3}}$	700.0000	699.9999	699.9999	700.0000	699.9999	700.0020
$\Delta\hat{s}_{Y_{2,3}}$	0.0013	0.0013	0.0013	0.0013	0.0013	0.0032
$\Delta\hat{s}_{Z_{2,3}}$	–0.0014	–0.0014	–0.0014	–0.0014	–0.0014	–0.0014
$\Delta\hat{s}_{X_{4,3}}$	0.0021	0.0020	0.0020	0.0021	0.0020	0.0037
$\Delta\hat{s}_{Y_{4,3}}$	799.9988	799.9988	799.9988	799.9988	799.9988	800.0007
$\Delta\hat{s}_{Z_{4,3}}$	–0.0004	–0.0004	–0.0004	–0.0004	–0.0004	–0.0004
$\Delta\hat{s}_{X_{1,4}}$	700.0010	700.0010	700.0001	700.0001	700.0010	700.0030
$\Delta\hat{s}_{Y_{1,4}}$	–0.0013	–0.0013	–0.0013	–0.0013	–0.0013	0.0007
$\Delta\hat{s}_{Z_{1,4}}$	0.0015	0.0015	0.0014	0.0014	0.0014	0.0014

Data presented in Table 4 was used for the determination of the linear strain vector components. The obtained results are given in Table 5

Table 5. Strain vector components [mm]

Increments	Variants				
	I	II	III	IV	V
$\Delta\hat{S}^{j-(j=0)}_{X_{1,2}}$	0	0	–0.1	0	1.6
$\Delta\hat{S}^{j-(j=0)}_{Y_{1,2}}$	0	0	0	0	2
$\Delta\hat{S}^{j-(j=0)}_{Z_{1,2}}$	0	–0.1	–0.1	–0.1	–0.1
$\Delta\hat{S}^{j-(j=0)}_{X_{2,3}}$	–0.1	–0.1	0	–0.1	2
$\Delta\hat{S}^{j-(j=0)}_{Y_{2,3}}$	0	0	0	0	1.9
$\Delta\hat{S}^{j-(j=0)}_{Z_{2,3}}$	0	0	0	0	0
$\Delta\hat{S}^{j-(j=0)}_{X_{4,3}}$	–0.1	–0.1	0	–0.1	1.6
$\Delta\hat{S}^{j-(j=0)}_{Y_{4,3}}$	0	0	0	0	1.9
$\Delta\hat{S}^{j-(j=0)}_{Z_{4,3}}$	0	0	0	0	0
$\Delta\hat{S}^{J-(j=0)}_{X_{1,4}}$	0	0	0	0	2
$\Delta\hat{S}^{j-(j=0)}_{Y_{1,4}}$	0	0	0	0	2
$\Delta\hat{S}^{j-(j=0)}_{Z_{1,4}}$	0	–0.1	–0.1	–0.1	–0.1

Data given in Table 5 can be used for calculating the linear strains (Table 6) caused by a change of the length of the measured sides ($d = \sqrt{\Delta\hat{S}^2_X + \Delta\hat{S}^2_Y + \Delta\hat{S}^2_Z}$).

Table 6. Linear strains [mm]

Side length	Variants				
	I	II	III	IV	V
$d_{1,2}$	0	0	0	0	3
$d_{2,3}$	0	0	0	0	3
$d_{4,3}$	0	0	0	0	2
$d_{1,4}$	0	0	0	0	3

The results presented in Table 6 show no strain in variants: I, II, III, IV. The strains were detected only in variant V; they are within the range of 2 – 3 mm.

Corrections to simulated observations given in Table 2 are shown in Table 7. Table 8 presents the adjusted vector components $\Delta\hat{X}, \Delta\hat{Y}, \Delta\hat{Z}$.

The adjusted observations shown in Table 8 were used for the determination of the displacement vectors. The components of those vectors are shown in Table 9.

Results shown in Table 9 indicate that the displacement of network points occurred in all analysed variants. In variants I and II, a subsidence of 1 mm and 2 mm, respectively, took place, while in variant III a subsidence of 30 mm occurred. The results obtained are consistent with the theoretical values shown in Table 2. It is noteworthy that the results of adjustment obtained in variants I, II, if presented without additional information contained in Table 3, may suggest only inaccuracies resulting from a measurement, and not displacements of network points.

Table 7. Estimated corrections [mm]

Correction	Variants					
	Primary measurement	I	II	III	IV	V
v_1	1	1	1	1	1	1
v_2	–0.7	–0.7	–0.8	–0.8	–0.7	–0.7
v_3	1.5	1.5	1.5	1.5	1.5	1.5
v_4	1	1	1	1	1	1
v_5	–0.7	–0.7	–0.7	–0.8	–0.7	–0.8
v_6	1.5	1.5	1.5	1.5	1.5	1.5
v_7	–1	–1	–1	–1	–1	–1
v_8	0.8	0.8	0.8	0.8	0.8	0.8
v_9	–1.5	–1.5	–1.5	–1.5	–1.5	–1.5
v_{10}	–1	–1	–1	–1	–1	–1
v_{11}	0.8	0.8	0.8	0.8	0.7	0.8
v_{12}	–1.5	–1.5	–1.5	–1.5	–1.5	–1.5

Table 8. Adjusted observations [m]

Vector components	Variants					
	Primary measurement	I	II	III	IV	V
$\Delta\hat{X}_{1,2}$	0.003	0.003	0.003	0.003	0.003	0.008
$\Delta\hat{Y}_{1,2}$	799.996	799.996	799.996	799.996	799.996	799.998
$\Delta\hat{Z}_{1,2}$	0.003	0.002	0.001	–0.027	–0.035	–0.002
$\Delta\hat{X}_{2,3}$	700.000	700.000	700.000	700.000	700.000	700.002
$\Delta\hat{Y}_{2,3}$	0.001	0.001	0.001	0.001	0.001	0
$\Delta\hat{Z}_{2,3}$	–0.001	–0.002	–0.003	–0.031	–0.001	0
$\Delta\hat{X}_{4,3}$	0.002	0.002	0.002	0.002	0.002	0.007
$\Delta\hat{Y}_{4,3}$	799.999	799.999	799.999	799.999	799.999	800.001
$\Delta\hat{Z}_{4,3}$	–0.001	–0.002	–0.003	–0.031	–0.039	–0.006
$\Delta\hat{X}_{1,4}$	700.001	700.001	700.001	700.001	700.001	700.003
$\Delta\hat{Y}_{1,4}$	–0.001	–0.001	–0.001	–0.001	–0.001	–0.002
$\Delta\hat{Z}_{1,4}$	0.001	0	-0.001	-0.029	0.001	0.002

In variant IV, only the rotation about the OX-axis of 3^{cc} was considered. This value along a distance of 800 m gives a transverse deviation of 38 mm. The value obtained from calculations is consistent with the adopted theoretical assumptions. Rotation by an angle of 3^{cc} caused only a change in vertical components $\Delta\hat{Z}_{1,2}$ and $\Delta\hat{Z}_{4,3}$ of the two vectors. In variant V there are horizontal as well as vertical displacements of the network points.

Table 9. Displacement vector components [mm]

Increments	Variants				
	I	II	III	IV	V
$\Delta\hat{X}_{1,2}^{j-(j=0)}$	0	0	0	0	5
$\Delta\hat{Y}_{1,2}^{j-(j=0)}$	0	0	0	0	2
$\Delta\hat{Z}_{1,2}^{j-(j=0)}$	–1	–2	–30	–38	–5
$\Delta\hat{X}_{2,3}^{j-(j=0)}$	0	0	0	0	2
$\Delta\hat{Y}_{2,3}^{j-(j=0)}$	0	0	0	0	–1
$\Delta Z_{2,3}^{j-(j=0)}$	–1	–2	–30	0	1
$\Delta\hat{X}_{4,3}^{j-(j=0)}$	0	0	0	0	5
$\Delta\hat{Y}_{4,3}^{j-(j=0)}$	0	0	0	0	2
$\Delta\hat{Z}_{4,3}^{j-(j=0)}$	–1	–2	–30	–38	–5
$\Delta\hat{X}_{1,4}^{j-(j=0)}$	0	0	0	0	2
$\Delta\hat{Y}_{1,4}^{j-(j=0)}$	0	0	0	0	–1
$\Delta Z_{1,4}^{j-(j=0)}$	–1	–2	–30	0	1

Presented in the paper assessment of accuracy of the adjustment results concerns only mean errors of residuals ($m_{v_i} = \sqrt{[\mathbf{C_v}]_{i,i}}$). Mean errors m_{v_i} are particularly important in analysing the measurements designed to detect observations contaminated with systematic or gross errors. The following criterion can be applied

$$|v_i|/m_{v_i} \leq a$$

where a is a width of the acceptable interval, reserved for random measurement errors. All observations for which $|v_i|/m_{v_i} > a$, should be checked in detail. The obtained mean errors m_{v_i} are all equal to m_{v_i} = 1.1 mm. That result should not be surprising, since it is the consequence of applying the same weights for all simulated observations. On the other hand, results of standardisation $|v_i|/m_{v_i}$ calculated on the basis of data presented in Tables 7 and 10 are within the interval of $< 0.64; 1.36 >$.

4. Conclusions

This paper presents a new method for determining displacements and strains of objects in unstable reference systems, conventionally referred to as **DiSTFA**.

Analysis of the results obtained in the numerical experiments on simulated observations enable to draw a few valuable conclusions.

Parameters of displacements represented by rotation angles $\hat{\varepsilon}_{\hat{X}}^{cc}, \hat{\varepsilon}_{\hat{Y}}^{cc}, \hat{\varepsilon}_{\hat{Z}}^{cc}$ determined in different measurement epochs as well as their differences (Table 3) provide information on stability of the reference system used. The results of numerical tests indicate possible displacements of network points in the investigated area.

The **DiSTFA** method can also be applied for the determination of the deformation of objects. This ability of the method is illustrated with the results presented in Table 6.

The estimated residuals (Table 7) obtained for all variants investigated do not substantially differ; their effect on the adjusted GPS-derived vector components is of the range of single millimetres, reaching up to 3 cm (Tables 2 and 8). The distortion of the reference system is thus not easily observable on the basis of obtained results. More reliable indication of possible existence of displacement and strains in the area investigated can be obtained extending the analysis on the deformation parameters.

The **DiSTFA** method provides the possibility to locate observations suspected of being affected with gross errors even if the observations are performed with only one measurement technology, e.g. GPS. It is an important advantage of the method.

The results of adjustment obtained with the **DiSTFA** method indicate it's applicability and encourage for further detailed research.

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Własności i analiza dokładności wyników estymacji metodą DiSTFA w monitoringu przemieszczeń i odkształceń

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Streszczenie

W niniejszej pracy przedstawiono wyniki badań metody **DiSTFA** (**Di**splacements and **S**trains with usage **T**ransformation and **F**ree **A**djustnent) wyznaczania przemieszczeń i odkształceń powierzchni wyznaczanych w niestabilnych układach odniesienia. Wyprowadzono także macierze kowariancji umożliwiające ocenę dokładności wyników estymacji. Rozważania teoretyczne uzupełniono przykładem zastosowania na symulowanej, trójwymiarowej sieci geodezyjnej. Uzyskane wyniki zachęcają do przeprowadzenia dalszych, bardziej szczegółowych analiz na rzeczywistych sieciach geodezyjnych.