Mittag-Leffler stability analysis of a class of homogeneous fractional systems

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In this paper, we start by the research of the existence of Lyapunov homogeneous function for a class of homogeneous fractional Systems, then we shall prove that local and global behaviors are the same. The uniform Mittag-Leffler stability of homogeneous fractional time-varying systems is studied. A numerical example is given to illustrate the efficiency of the obtained results.

Key words: homogeneous fractional systems, Lyapunov homogeneous function, Mittag-Leffler stability

1. Introduction

The history of fractional systems is more than three centuries old, when Liouville asked the question how to find the fractional order derivative. Yet, it only receives much attention and interest in the past 20 years. This branch of mathematics has found applications in many areas, such as viscoelastic materials, dielectric polarization and signal processing. The advantage of fractional differential equations is that they can describe more systems better than ordinary differential equations. The reader may refer to [3, 9, 11, 12] for the theory and applications of fractional calculus.

Stability is the one of the most frequent terms used in literature whenever we deal with the dynamical systems and their behaviors. In mathematical terminology, stability theory addresses the convergence of solutions of differential or difference equations and of trajectories of dynamical systems under small perturbations of initial conditions. Same as classical differential or difference equations

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a lot of stress has been given to the stability and stabilization of the systems represented by fractional order differential equations. Indeed, in [4], the authors described an uniform stability for fractional order systems using general quadratic Lyapunov functions. In [8], Yan Li et al., presented the Mittag-Leffler stability of fractional order nonlinear dynamic systems. Furthermore, stability analysis of Hilfer fractional differential systems is shown in [15]. On the other hand, in [16], the authors described the asymptotical stability of nonlinear fractional differential system with Caputo derivative. In nonlinear systems, Lyapunov's direct method (also called the second method of Lyapunov) provides a way to analyze the stability of a system without explicitly solving the differential equations. The method generalizes the idea which shows the system is stable if there are some Lyapunov function candidates for the system. The Lyapunov direct method is a sufficient condition to show the stability of systems, which means the system may still be stable even one cannot find a Lyapunov function candidate to conclude the system stability property. There are many researches on Lypunov's second method for fractional differential equations (see [7] or [8]).

Homogeneous vector fields play also a prominent role in various aspects of nonlinear systems. In the framework of homogeneous dynamical systems, the behavior of the trajectories of the solution on a sphere suitably defined around the origin informs us about their global behavior [2,14]. This property has been found useful for stability analysis [1,2,5,13], approximation of system dynamics [6,10]. The main goal of this paper is to prove the existence of Lyapunov homogeneous function for homogeneous fractional systems. In addition, we shall prove that local and global behaviors are the same. The uniform Mittag-Leffler stability of homogeneous fractional time-varying systems is studied.

The outline of this work is as follows: the preliminary is given in Section 2. The construction of homogeneous Lyapunov function is given in Section 3. The property for some class of homogeneous time-varying system is presented in Section 4. In Section 5, an example is presented to illustrate the results.

2.   Preliminary

In this paper, we consider the Caputo definition of fractional derivative, which we will use next.

Definition 1 [3] *The uniform formula of a fractional integral with $0 < \alpha \leqslant 1$ is given by*

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \tau)^{\alpha-1} f(\tau) \, d\tau,$$

where $t \geqslant t_0$, $f(t)$ is an arbitrary integrable function, I_t^α is the fractional integral operator, $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \exp(-t) \, dt$ is the gamma function, and $\exp(\cdot)$ is the exponential function.

Definition 2 *(Caputo fractional derivative [3]) The Caputo fractional derivative of a function x of order $\alpha > 0$ is defined as*

$$^C D_{t_0}^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t (t-s)^{-\alpha} x'(s) \, ds,$$

where $0 < \alpha \leqslant 1$.

The Caputo fractional derivative of a n-dimensional vector function $x(t) = (x_1(t), \ldots, x_n(t))^T$ is defined component-wise as

$$^C D_{t_0}^\alpha x(t) = \left(^C D_{t_0}^\alpha x_1(t), \ldots, ^C D_{t_0}^\alpha x_n(t) \right)^T.$$

Definition 3 [3] *The Mittag-leffler function $E_\alpha(z)$ and the generalized Mittag-leffler function $E_{\alpha,\beta}(z)$ are defined as:*

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad \alpha > 0.$$

For $\alpha = 1$, we have the exponential series. Similarly,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha, \beta > 0.$$

For the convenience of mathematical analysis, the Caputo fractional order system is equally written as:

$$^C D_{t_0}^\alpha x(t) = f(x(t)), \tag{1}$$

where $\alpha \in (0, 1]$, $x(t) \in \mathbb{R}^n$ is the state vector; $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and $f(0) = 0$. Suppose that the function f is smooth enough to guarantee the existence of a global solution $x(t) = x(t, t_0, x_0)$ of system (1) for each initial condition (t_0, x_0).

2.1.   Notion of stability

Some basic definitions of stability are introduced, which will be used in the following sections.

Definition 4 *The equilibrium point $x = 0$ of fractional nonautonomous system (1) is said to be:*

i) *Uniformly Mittag-Leffler stable (**UMLS**) if there exists a neighborhood of the origin $U \subset \mathbb{R}^n$ such that each solution of system (1) satisfies:*

$$\|x(t, t_0, x_0)\| \leqslant [m(x_0)E_\alpha(-\delta(t - t_0)^\alpha)]^b \quad \forall t \geqslant t_0, \ x_0 \in U$$

with $b > 0$, $\lambda > 0$, $m(0) = 0$, $m(x) \geqslant 0$ and m is locally Lipschitz.

ii) *Globally uniformly Mittag-Leffler (**GUMLS**) stable if the trajectory of (1) passing through any initial state x_0 at any initial time t_0 evaluated at time t satisfies:*

$$\|x(t, t_0, x_0)\| \leqslant [m(x_0)E_\alpha(-\delta(t - t_0)^\alpha)]^b \quad \forall t \geqslant t_0$$

with $b > 0$, $\lambda > 0$, $m(0) = 0$, $m(x) \geqslant 0$ and m is locally Lipschitz.

2.2.   Homogeneity

Definition 5 *For any $r = (r_1, \ldots, r_n) \in \mathbb{R}^n$ with $r_i > 0$, $i \in \{1, \ldots, n\}$, and $\lambda > 0$, the dilation vector of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ associated with weight r is defined as*

$$\Delta_\lambda(x) = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n).$$

The homogeneous norm of $x \in \mathbb{R}^n$ associated with weight r is defined as

$$\|x\|_r = \left(\sum_{i=1}^{n} |x|^{\frac{\varrho}{r_i}} \right)^{\frac{1}{\varrho}}, \qquad \varrho = \prod_{i=1}^{n} r_i.$$

An important property is that

$$\|\Delta_\lambda(x)\|_r = \lambda \|x\|_r.$$

The homogeneous norm is not a standard norm, because the triangle inequality is not satisfied. However, there exists $\overline{\sigma}, \underline{\sigma}$ such that

$$\underline{\sigma}\|x\|_r \leqslant \|x\| \leqslant \overline{\sigma}\|x\|_r.$$

Definition 6

   i) *A continuous function $h\colon R^n \to \mathbb{R}$ is r-homogeneous of degree k if $h(\Delta_\lambda(x)) = \lambda^k h(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n$.*

   ii) *We say that a continuous function $f\colon \mathbb{R}^n \to R^n$ is r-homogeneous of degree k if each f_i, $i \in \{1, \ldots, n\}$, is r-homogeneous of degree $k+r_i$. i.e. $f(\Delta_\lambda(x)) = \lambda^k \Delta_\lambda(f(x))$ for all $\lambda > 0$ and $x \in \mathbb{R}^n$.*

Definition 7 *The system (1) is said to admit a r-homogeneous Lyapunov function of degree k, if there exists a continuously differentiable function $V\colon \mathbb{R}^n \to [0, +\infty)$ such that:*

   i) *$V(0) = 0 \iff x = 0$.*

   ii) *$^C D_{t_0}^\alpha V(x) < 0$.*

   iii) *V is r-homogeneous of degree k: i.e. $V(\Delta_\lambda(x)) = \lambda^k V(x)$, for all $\lambda > 0$ and $x \in \mathbb{R}^n$.*

Remark 1 *The derivative in the sense of Caputo in the case of a dilation vector check the following property*

$$^C D_{t_0}^\alpha (\Delta_\lambda(x)) = \left(\lambda^{r_1} {}^C D_{t_0}^\alpha x_1, \ldots, \lambda^{r_n} {}^C D_{t_0}^\alpha x_n \right)^{\mathrm{T}} = \Delta_\lambda({}^C D_{t_0}^\alpha x).$$

Remark 2 *The homogeneity of the functions $\overline{V}\colon \mathbb{R}^n \to \mathbb{R}$ is verified for the Caputo derived $^C D_{t_0}^\alpha \overline{V}$. We have,*

$$\forall x = (x_i)_{i=1,n} \in \mathbb{R}^n\backslash\{0\} \quad \forall \lambda > 0 \quad \overline{V}(\Delta_\lambda(x)) = \overline{V}(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) = \lambda^k \overline{V}(x),$$

where $r_1, \ldots, r_n$ are some positive real numbers, and k is a non-negative real number.

   Hence, it is clear that

$$\forall x = (x_i)_{i=1,n} \in \mathbb{R}^n\backslash\{0\}, \forall \lambda > 0 \quad {}^C D_{t_0}^\alpha \overline{V}(\Delta_\lambda(x)) = \lambda^{k} {}^C D_{t_0}^\alpha \overline{V}(x).$$

This property implies that the global behavior of trajectories could be evaluated based on their behavior on S^{n-1}, where $S^{n-1} := \{x \in \mathbb{R}^n\backslash\{0\} \mid \|x\|_r = 1\}$.

3.   Homogeneous Lyapunov function

In this section, we will establish the existence of a homogeneous Lyapunov function for an asymptotically stable homogeneous system (1). To do this, we need the following theorem which gives an upper bound of the fractional derivative of a composite function.

Theorem 1 [15] *For a given $x_0 \in \mathbb{R}^n$, let $x \in \{x_0\} + I_{0+}^{\alpha} C([0,T],\mathbb{R}^n)$ and $V \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following conditions:*

i) *The function V is convex on $\mathbb{R}^n$ and $V(0) = 0$.*

ii) *The function V is differentiable on $\mathbb{R}^n$.*

Then, the following inequality holds for all $t \in [0,T]$:

$$^{C}D_{0+}^{\alpha}V(x(t)) \leqslant \left\langle \nabla V(x(t)), {}^{C}D_{0+}^{\alpha}x(t) \right\rangle$$

where ∇V is the gradient of the function V.

Lemma 1 *Let the system (1) be a r-homogeneous system with degree k. If there exists a scalar function $V \in C^{\infty}(\mathbb{R}^n,\mathbb{R})$, such that:*

i) *$V(0) = 0$, $V(x) > 0$ for all $x \neq 0$ and $V(x) \to +\infty$ as $\|x\| \to +\infty$,*

ii) *$\forall\, x \neq 0 \; \langle \nabla V(x), f(x) \rangle < 0$.*

Then, there exists a r-homogeneous Lyapunov function of degree k $\overline{V} \in C^p(\mathbb{R}^n \, \mathbb{R})$ where

$$p < \frac{k}{\max\{r_i \mid 1 < i < n\}}, \text{ such that:}$$

a) *$\overline{V}(0) = 0$, $\overline{V}(x) > 0$ for all $x \neq 0$ and $\overline{V}(x) \to +\infty$ as $\|x\| \to +\infty$,*

b) *$^{C}D_{t_0}^{\alpha}\overline{V}(x) < 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$.*

Proof. Let $a \in C^{\infty}(\mathbb{R},\mathbb{R})$ such that

$$a = \begin{cases} 0 & \text{on } (-\infty, 1], \\ 1 & \text{on } [2, \infty) \end{cases} \quad \text{and } a' \geqslant 0 \text{ on } \mathbb{R}.$$

We consider the following function

$$\overline{V}(x) := \begin{cases} \displaystyle\int_{0}^{+\infty} \frac{1}{\lambda^{k+1}} (a \circ V)(\Delta_{\lambda}(x)) \, \mathrm{d}\lambda & \text{if } x \in \mathbb{R}^n \backslash \{0\}, \\[4pt] 0 & \text{if } x = 0. \end{cases}$$

Let $s > 0$, $\overline{V}(\Delta_s(x)) = \displaystyle\int_{0}^{+\infty} \frac{1}{\lambda^{k+1}} (a \circ V)((s\lambda)^{r_1}x_1, \ldots, (s\lambda)^{r_n}x_n) \, \mathrm{d}\lambda$ with the change of variable $\tau = s\lambda$, we obtain:

$$\overline{V}(\Delta_s(x)) = \int_{0}^{+\infty} \frac{s^k}{\tau^{k+1}} (a \circ V)(\tau^{r_1}x_1, \ldots, \tau^{r_n}x_n) \, \mathrm{d}\tau = s^k \overline{V}(x).$$

The proof of a) it is easy and for $\overline{V} \in C^p(\mathbb{R}^n, \mathbb{R})$ see [14].

Let us use the notion of dilation and i), we may find for all $\|x\|_r \in [1-\varepsilon, 1+\varepsilon]$ with a small enough $\varepsilon > 0$ and two numbers $l > 0$ and $L > 0$ such that

$$V(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) \leqslant 1 \quad \text{for } \lambda \leqslant l,$$
$$V(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) \geqslant 2 \quad \text{for } \lambda \geqslant L.$$

Then, for all $x \in \mathbb{R}^n$ such that $\|x\|_r \in [1 - \varepsilon, 1 + \varepsilon]$,

$$\overline{V}(x) = \int_l^L \frac{1}{\lambda^{k+1}} (a \circ V)(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) \, d\lambda + \frac{1}{kL^k}.$$

We may write

$$\frac{\partial \overline{V}}{\partial x_i}(x) = \int_l^L \frac{\lambda^{r_i}}{\lambda^{k+1}} a'(V(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n)) \frac{\partial V}{\partial x_i}(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) \, d\lambda$$

and

$$\left\langle \nabla \overline{V}(x), f(x) \right\rangle = \sum_{i=1}^n \frac{\partial \overline{V}(x)}{\partial x_i} f_i(x)$$

$$= \int_l^L \frac{1}{\lambda^{\tau+k+1}} a'(V(\Delta_\lambda(x))) \left[\sum_{i=1}^n \left(f_i \frac{\partial V}{\partial x_i} \right)(\Delta_\lambda(x)) \right] d\lambda.$$

Then, using the Theorem 1, we have for all $x \in S^{n-1}$

$${}^C D_{t_0}^\alpha \overline{V}(x) \leqslant \left\langle \nabla \overline{V}(x), f(x) \right\rangle$$

$$\leqslant \int_l^L \frac{1}{\lambda^{\tau+k+1}} a'(V(\Delta_\lambda(x))) \left\langle \nabla V(\Delta_\lambda(x)), f(\Delta_\lambda(x)) \right\rangle d\lambda$$

$$< 0.$$

From Remark 2, we obtain ${}^C D_{t_0}^\alpha \overline{V}(x) < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

4. Mittag-Leffler stability of time-varying systems

Let us consider the following for time-varying systems

$$^C D_{t_0}^\alpha x(t) = f(t, x(t)), \qquad t \geqslant t_0. \tag{2}$$

where $\alpha \in (0, 1]$, $x(t) \in \mathbb{R}^n$ is the state vector; $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and $f(t, 0) = 0$. Suppose that the function f is smooth enough to guarantee the existence of a global solution $x(t) = x(t, t_0, x_0)$ of system (2) for each initial condition (t_0, x_0).

Lemma 2 *Let $x(t)$ be a solution of the r-homogeneous system (2) with the degree k for an initial condition $x_0 \in \mathbb{R}^n$. For any $\lambda > 0$.*

Then $y(t) = \Delta_\lambda(x(\lambda^{\frac{k}{\alpha}}t))$ for all $t \geqslant t_0$ with the initial condition $y_0 = \Delta_\lambda(x_0)$ is a solution of the following modification system

$$^C D_{t_0}^\alpha y(t) = f\left(\lambda^{\frac{k}{\alpha}}t, y(t)\right), \qquad t \geqslant t_0. \tag{3}$$

Proof. For $i = 1, \ldots, n$, $y_i(t) = \lambda^{r_i} x_i(\lambda^{\frac{k}{\alpha}}t)$, for all $t \geqslant t_0$. We have:

$$^C D_{t_0}^\alpha y_i(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t (t-s)^{-\alpha} y_i'(s)\,\mathrm{d}s$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t (t-s)^{-\alpha} \left(\lambda^{r_i} x_i\left(\lambda^{\frac{k}{\alpha}}s\right)\right)'\,\mathrm{d}s$$

$$= \frac{\lambda^{r_i+\frac{k}{\alpha}}}{\Gamma(1-\alpha)} \int_{t_0}^t (t-s)^{-\alpha} x_i'\left(\lambda^{\frac{k}{\alpha}}s\right)\,\mathrm{d}s.$$

Using the change of variable $v = \lambda^{\frac{k}{\alpha}}s$ we get

$$^C D_{t_0}^\alpha y_i(t) = \frac{\lambda^{r_i+\frac{k}{\alpha}}}{\Gamma(1-\alpha)} \int_{\lambda^{\frac{k}{\alpha}}t_0}^{\lambda^{\frac{k}{\alpha}}t} \frac{\left(t - \lambda^{-\frac{k}{\alpha}}v\right)^{-\alpha} x_i'(v)\,\mathrm{d}v}{\lambda^{\frac{k}{\alpha}}}$$

$$= \frac{\lambda^{r_i}}{\Gamma(1-\alpha)} \int_{\lambda^{\frac{k}{\alpha}}t_0}^{\lambda^{\frac{k}{\alpha}}t} \lambda^k \left(\lambda^{\frac{k}{\alpha}}t - v\right)^{-\alpha} x_i'(v)\,\mathrm{d}v$$

$$= \frac{\lambda^{r_i+k}}{\Gamma(1-\alpha)} \int_{\lambda^{\frac{k}{\alpha}}t_0}^{\lambda^{\frac{k}{\alpha}}t} \left(\lambda^{\frac{k}{\alpha}}t - v\right)^{-\alpha} x_i'(v)\,\mathrm{d}v$$

$$= \lambda^{r_i+k} \, {}^C D_{\lambda^{\frac{k}{\alpha}}t_0}^\alpha x_i\left(\lambda^{\frac{k}{\alpha}}t\right), \quad \text{for all } i = 1, 2, \ldots, n.$$

Therefore,

$$
\begin{aligned}
{}^C D_0^\alpha y(t) &= \left({}^C D_{t_0}^\alpha y_1(t), \ldots, {}^C D_{t_0}^\alpha y_n(t) \right)^{\mathrm{T}} \\
&= \left(\lambda^{r_1+k}\, {}^C D_{\lambda^{\frac{k}{\alpha}} t_0}^\alpha x_1\left(\lambda^{\frac{k}{\alpha}} t\right), \ldots, \lambda^{r_n+k}\, {}^C D_{\lambda^{\frac{k}{\alpha}} t_0}^\alpha x_n\left(\lambda^{\frac{k}{\alpha}} t\right) \right)^{\mathrm{T}} \\
&= \left(\lambda^{r_1+k} f_1\left(\lambda^{\frac{k}{\alpha}} t, x\left(\lambda^{\frac{k}{\alpha}} t\right)\right), \ldots, \lambda^{r_n+k} f_n\left(\lambda^{\frac{k}{\alpha}} t, x\left(\lambda^{\frac{k}{\alpha}} t\right)\right) \right)^{\mathrm{T}} \\
&= \lambda^k \left(\lambda^{r_1} f_1\left(\lambda^{\frac{k}{\alpha}} t, x\left(\lambda^{\frac{k}{\alpha}} t\right)\right), \ldots, \lambda^{r_n} f_n\left(\lambda^{\frac{k}{\alpha}} t, x\left(\lambda^{\frac{k}{\alpha}} t\right)\right) \right)^{\mathrm{T}} \\
&= \lambda^k \Delta_\lambda \left(f\left(\lambda^{\frac{k}{\alpha}} t, x\left(\lambda^{\frac{k}{\alpha}} t\right)\right) \right) \\
&= f\left(\lambda^{\frac{k}{\alpha}} t, \Delta_\lambda \left(x\left(\lambda^{\frac{k}{\alpha}} t\right)\right)\right), \\
&= f\left(\lambda^{\frac{k}{\alpha}} t, y(t)\right).
\end{aligned}
$$

Then $y(t) = \Delta_\lambda(x(\lambda^{\frac{k}{\alpha}} t))$ is a solution of system (3). □

Lemma 3 *Let $y(t)$ be a solution of the r-homogeneous system (3) with the degree k for an initial condition $y_0 \in \mathbb{R}^n$. For any $\lambda > 0$, the system (2) has a solution $x(t) = \Delta_{\lambda^{-1}}\left(y\left(\lambda^{-\frac{k}{\alpha}} t\right)\right)$, for all $t \geqslant t_0$ with the initial condition $x_0 = \Delta_{\lambda^{-1}}(y_0)$.*

Proof. The proof is similar to the proof of Lemma 2. □

Remark 3 *An advantage of homogeneous systems described by fractional non-autonomous system is that any of its solution can be obtained from another solution under the dilation rescaling and a suitable time re-parametrization.*

Theorem 2 *We assume that the systems (2) and (3) are r-homogenous with degree k. Then, the system (2) is same GUMLS if and only if the system (3) is GUMLS.*

Proof. Let $x(t, t_0, x_0)$ is a solution of system (2) which is GUMLS. So,

$$
\|x(t, t_0, x_0)\|_r \leqslant [m(x_0) E_\alpha(-\delta(t - t_0)^\alpha)]^b
$$

From Lemma 2 we have $y(t, t_0, y_0) = \Delta_\lambda(x(\lambda^{\frac{k}{\alpha}} t, \lambda^{\frac{k}{\alpha}} t_0, x_0))$ with $y_0 = \Delta_\lambda(x_0)$ is a solution of the system (3). We have

$$
\begin{aligned}
\|y(t, t_0, y_0)\|_r &= \lambda \left\| x\left(\lambda^{\frac{k}{\alpha}} t, \lambda^{\frac{k}{\alpha}} t_0, x_0\right) \right\|_r \\
&\leqslant \lambda \left[m(x_0) E_\alpha(-\delta(\lambda^{\frac{k}{\alpha}} t - \lambda^{\frac{k}{\alpha}} t_0)^\alpha) \right]^b \\
&\leqslant \lambda \left[m(\Delta_{\lambda^{-1}}(y_0)) E_\alpha(-\delta \lambda^k (t - t_0)^\alpha) \right]^b \\
&\leqslant \left[\widetilde{m}(y_0) E_\alpha\left(-\widetilde{\delta}(t - t_0)^\alpha\right) \right]^b
\end{aligned}
$$

with $\widetilde{m}(y_0) = \lambda^{\frac{1}{b}} m(\Delta_{\lambda^{-1}}(y_0))$ and $\widetilde{\delta} = \lambda^k \delta$. It's clear that $\widetilde{m}(0) = 0$ and $\widetilde{m}(y) \geqslant 0$ for all $y \in \mathbb{R}^n$. So, the function $\widetilde{m}$ is locally lipschitz with constant $\widetilde{m}_0 = \lambda^{\frac{1}{b}-1} m_0$, indeed: we pose $y_1 = \Delta_\lambda(x_1)$ and $y_2 = \Delta_\lambda(x_2)$, therefore, we have

$$\begin{aligned}\|\widetilde{m}(y_1) - \widetilde{m}(y_1)\|_r &= \lambda^{\frac{1}{b}} \left\| m(\Delta_{\lambda^{-1}}(y_1)) - m(\Delta_\lambda^{-1}(y_2)) \right\|_r \\ &\leqslant \lambda^{\frac{1}{b}} m_0 \|\Delta_{\lambda^{-1}}((y_1 - y_2))\|_r \\ &\leqslant \lambda^{\frac{1}{b}-1} m_0 \|y_1 - y_2\|_r \\ &\leqslant \widetilde{m}_0 \|y_1 - y_2\|_r . \end{aligned}$$

It follow that the system (3) is GUMLS.

Inversely, using the same technique, we proved that, if the system (3) is GUMLS, then, the system (2) is also GUMLS. □

Theorem 3 *We suppose that the system (2) is r-homogeneous with degree k and $\forall x_0 \in B_\rho = \{x \in \mathbb{R}^n, \|x\|_r \leqslant \rho\}$ with $0 < \rho < \infty$, the system (2) is UMLS. Then, the system (2) is GUMLS.*

Proof. For all $x_0 \in B_\rho$, and $t \geqslant t_0$, we have:

$$\|x(t, t_0, x_0)\| \leqslant [m(x_0) E_\alpha(-\delta(t-t_0)^\alpha)]^b, \ \forall t \geqslant t_0$$

Let $\overline{x}_0 \notin B_\rho$, then there is $x_0 \in B_\rho$ such that $\|x_0\|_r = \rho$ and $\overline{x}_0 = \Delta_\lambda(x_0)$ with $\lambda = \|\overline{x}_0\|_r \rho^{-1}$. By using theorem (2), we have

$$\overline{x}(t, t_0, \overline{x}_0) = \Delta_\lambda \left(x\left(\lambda^{\frac{k}{\alpha}} t, \lambda^{\frac{k}{\alpha}} t_0, x_0 \right) \right)$$

is a solution of system (2). Moreover, we have

$$\begin{aligned}\|\overline{x}(t, t_0, \overline{x}_0)\|_r &= \left\| \Delta_\lambda \left(x\left(\lambda^{\frac{k}{\alpha}} t, \lambda^{\frac{k}{\alpha}} t_0, x_0 \right) \right) \right\|_r \\ &= \lambda \left\| x\left(\lambda^{\frac{k}{\alpha}} t, \lambda^{\frac{k}{\alpha}} t_0, x_0 \right) \right\|_r \\ &\leqslant \lambda \left[m(x_0) E_\alpha \left(-\delta \left(\lambda^{\frac{k}{\alpha}} t - \lambda^{\frac{k}{\alpha}} t_0 \right)^\alpha \right) \right]^b \\ &\leqslant \left[\lambda^{\frac{1}{b}} m(\Delta_{\lambda^{-1}}(\overline{x}_0)) E_\alpha(-\lambda^k \delta(t-t_0)^\alpha) \right]^b \\ &\leqslant \left[\widetilde{m}(\overline{x}_0) E_\alpha(-\widetilde{\delta}(t-t_0)^\alpha) \right]^b . \end{aligned}$$

Therefore, for all $t \geqslant t_0$, the system (2) is GUMLS. □

Theorem 4 *Let the system (3) be r-homogeneous with degree k and UMLS for all $x_0 \in B_\rho$ for a fixed $0 < \rho < \infty$, for any $w > 0$. Then the system (3) is GUMLS, for any $w > 0$.*

Proof. We suppose that the system (3) is UMLS. Using the theorem (2), the system (2) is UMLS. It follows that the system (2) is GUMLS. Therefore, the system (3) is GUMLS. □

5. Example

Consider the homogeneous fractional order system

$$
\begin{aligned}
{}^{C}D_0^{\alpha} x_1(t) &= -x_1(t) - \frac{1}{1+t^2} x_2(t), \\
{}^{C}D_0^{\alpha} x_2(t) &= x_1(t) - x_2(t), \\
{}^{C}D_0^{\alpha} x_3(t) &= -x_3(t),
\end{aligned}
\tag{4}
$$

where $0 < \alpha \leqslant 1$. The system (4) is of the form

$$
{}^{C}D_0^{\alpha} x(t) = f(t, x(t)),
$$

where

$$
f(t, x(t)) = \left(-x_1(t) - \frac{1}{1+t^2} x_2(t), \ x_1(t) - x_2(t), \ -x_3(t) \right).
$$

Let us choose the function

$$
V(t, x) = x_1^2 + \frac{2+t^2}{1+t^2} x_2^2 + x_3^2 .
$$

Note that V is a homogeneous function and

$$
x_1^2 + x_2^2 + x_3^2 \leqslant V(t, x) \leqslant x_1^2 + 2x_2^2 + x_3^2, \quad \forall x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 .
$$

Then, the Caputo fractional derivative of $V(t, x)$ along the solution $x(t)$ to (4) as follows

$$
\begin{aligned}
{}^{C}D_0^{\alpha} V(t, x(t)) &\leqslant \left[-2x_1^2(t) - \frac{2}{1+t} x_1(t) x_2(t) \right] \\
&\quad + \left[\left(\frac{2+t^2}{1+t^2} x_2 \right) x_1(t) x_2(t) - \left(\frac{2+t^2}{1+t^2} \right) x_2^2(t) \right] - 2x_3^2(t) \\
&= -2x_1^2(t) + 2x_1(t) x_2(t) - \left(2 + \frac{2}{1+t} \right) x_2^2(t) - 2x_3^2(t) \\
&\leqslant -2x_1^2(t) + 2x_1(t) x_2(t) - 2x_2^2(t) - 2x_3^2(t) \\
&\leqslant -x_1^2(t) - (x_1(t) - x_2(t))^2 - x_2^2(t) - 2x_3^2(t) \leqslant -\|x(t)\|^2 .
\end{aligned}
$$

We can obtain

$$V(t, x(t)) \leqslant E_\alpha \left(-\frac{1}{2} t^\alpha \right) V(0, x(0)), \quad \forall t \geqslant 0.$$

Thus, it follows that

$$\|x(t)\| \leqslant \sqrt{2 E_\alpha \left(-\frac{1}{2} t^\alpha \right) \|x(0)\|^2}, \quad \forall t \geqslant 0.$$

As a result, the zero solution to the system (4) is Mittag-Leffler stable. Thus, the zero solution is asymptotically stable. The numerical solution shown in the Fig. 1 indicates that the non-trivial solutions approach to the zero solution. The value of fractional order $\alpha = 0.99$ starting from initial values $x_1(0) = -0.3$, $x_1(0) = 0.3$ and $x_1(0) = 0.1$.
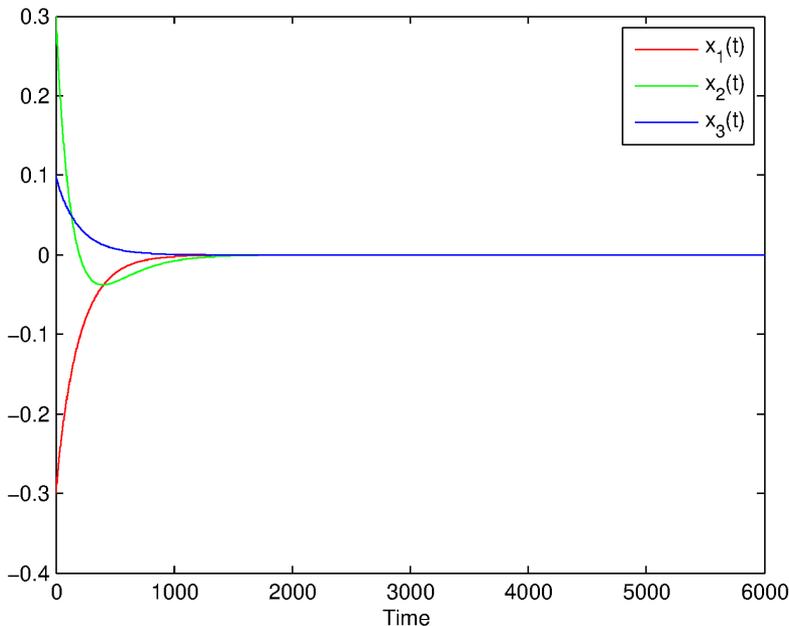

Figure 1: Trajectories of solutions of system (4)

Figure 2 corresponding the numerical solution of the system $^C D_0^\alpha x(t) = f(\lambda^{\frac{k}{\alpha}} t, x(t))$ where $\lambda = 10^4$ and $k = 1$. The value of fractional order $\alpha = 0.99$ starting from initial values $y_1(0) = -3.10^3$, $y_1(0) = 3.10^3$ and $y_1(0) = 10^3$.
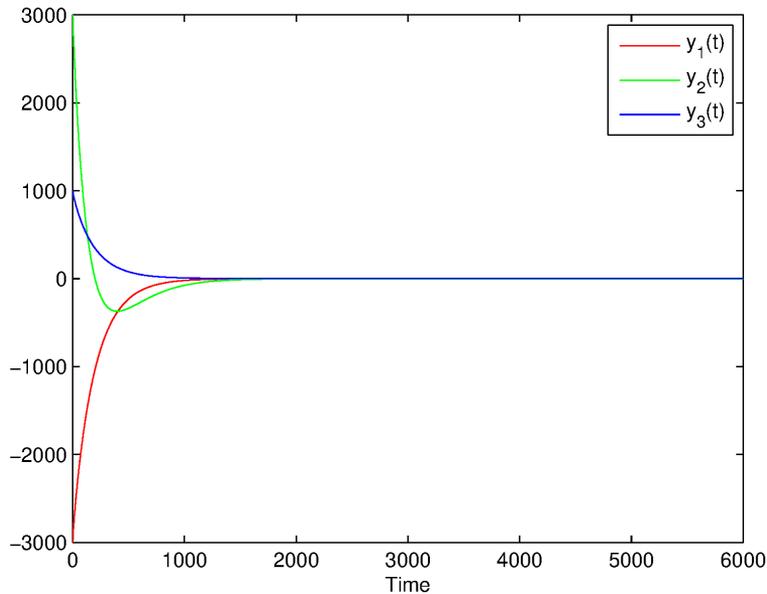
Figure 2: Trajectories of solutions of system $^{C}D_0^{\alpha} x(t) = f(\lambda^{\frac{k}{\alpha}} t, x(t))$

6.   Conclusion

In this paper, a class of homogeneous fractional systems is studied. The construction of Lyapunov homogeneous function is given and we proved that the global behavior of trajectories could be evaluated based on their behavior on a suitably defined sphere around the origin. A homogeneous time-varying system is studied, we proved that the uniform Mittag-Leffler stability of homogeneous fractional system and the modified system are the same. The construction of control homogeneous function for the fractional order homogeneous systems is another possible direction of future work.

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