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POLSKA AKADEMIA NAUK

BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 67, No. 3, 2019
DOI: 10.24425/bpasts.2019.128611

On real order passivity

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Abstract. The aim of this paper is to show that a real order generalization of the dissipative concepts is a useful tool to determine the stability (in the Lyapunov and in the input-output sense) and to design control strategies not only for fractional order non-linear systems, but also for systems composed of integer and fractional order subsystems (mixed-order systems). In particular, the fractional control of integer order system (e.g. PI^{λ} control) can be formalized. The key point is that the gradations of dissipativeness, passivity and positive realness concepts are related among them. Passivating systems is used as a strategy to stabilize them, which is studied in the non-adaptive as well as in the adaptive case.

Key words: passivity, dissipativeness, adaptive, passivation, nonlinear mixed-order system, positive realness, control of nonlinear systems.

1. Introduction

Dissipative systems are those in which it is possible to define a function, called storage, satisfying a property relating their input and output through an integer order (IO) derivative or integral. Similarly to Lyapunov functions, storage ones are scalar functions from which properties of a system are drawn providing in addition a way to deal with transference into/from the environment. Accordingly, the concept of Lyapunov stability is replaced by input-to-output stability concepts such as finite gain or BIBO stability [1,2], and storage functions become Lyapunov functions when the input is of a feedback type or zero [3].

The way to consider the environment interaction on the system is to postulate that the storage function is affected by an additive scalar flow called supply. In this way, a system is dissipative if in the balance of storage, a fraction of it is dissipated. An important case – for its implications in stability [1,3–5] – are the passive systems which never generate a net storage to the outside. Since the environment interaction has a scalar nature, the interconnection of dissipative subsystems yields a dissipative system under mild assumptions. This allows the stability analysis of complex systems by considering the properties of their subsystems and interconnections in a simplified way as shown in multi-agent [6], network [7] or cooperative [8] problems.

The necessity of a real order generalization of the dissipation theory results from the fact that the application of (IO) passive theory to fractional order systems (FOS) has important weaknesses. First, in [10] stability properties of passive IO nonlinear time-invariant systems were employed in a local result for fractional systems. However, the latter are not dynamical in the pseudo state variable [9, Proposition 2], an essential property to obtain those stability results [4, 5]. Second, in [11] IO passivity was obtained but restricted to Riemann-Liouville systems, which have unbounded initial conditions [12] and stability concepts become unsuited. Third, FOS have polynomial rather than exponential rate of convergence [13], so an integer integral of a fractional system's output will often diverge [14]. Fourth, though diffusive representation of fractional order (FO) operators define internal variables (null initialized) which hold an IO dissipative relation [15], the relationships between the initial conditions of fractional Caputo or Riemann-Liouville systems and their diffusive realizations are unclear or yield infinite storage. In addition, the capability of fractional systems to model complex phenomena [13, 16] imposes a need to develop proper control tools for them. Our contributions and their relevance are commented in the following paragraphs.

In Section 2 the concepts of real order dissipativeness, passivity and positive realness are introduced in terms of the FO integral. In this way, the definitions become independent of the specific fractional derivative used (in [17] passivity was defined in terms of Caputo derivative and in [18] fractional positive realness was defined in the Laplace domain which restricts its application to linear systems) – a fact particularly important since there are many ways to generalize fractional derivative but just one commonly accepted fractional integral [19]. Then, relationships among these concepts (in the sense of [20]) are obtained by appealing to this common base of fractional integrals, and are a contribution regarding the above mentioned references. The main result asserts the fractional dissipativeness of an interconnected system by requiring the dissipativeness (possibly of different orders) of its subsystems. In addition, it is a generalization of the IO results in [2, 3, 21, 22] in proving a converse statement.

In Section 3, stability results in the Lyapunov and the input-output sense are provided for fractional dissipative systems. In comparison with IO results [1, 3–5], our proof is based on systems described by non-state internal variables. In comparison

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Manuscript submitted 2017-12-04, revised 2018-04-26, initially accepted for publication 2018-05-16, published in June 2019.

with fractional stability results [9,24], which dependent on using the same order of derivation, we allow systems composed of subsystems, each of them defined by possibly different order derivation.

In Section 4 we show how to turn passive non-linear fractional systems in order to apply the stability results of Section 3, generalizing the procedure in [4]. The adaptive control of an unknown system by using passive approach is finally done to show the usefulness of the results proposed in most realistic settings, generalizing the results in [25, 26].

In Section 5 examples are presented showing FOS having the fractional dissipation property. Finally, in Section 6 we provide the main conclusions.

2. Real order dissipativeness

The dissipativeness, passivity and positive realness concepts generalized to real order and relationships among them are presented in this section.

2.1. Notation. The Riemann-Liouville fractional integral of a function $f : [0,T] \to \mathbb{C}$ is given by [12],

$$I^{\alpha} f(t) := [I^{\alpha} f(\cdot)](t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \qquad (1)$$

where without loss of generality we have fixed the initial time of the fractional integral at $t = 0$ and $\alpha \in \mathbb{R}_{>0}$, is directly generalizing the Cauchy formula for repeated integration. It is well defined for locally integrable functions.

The Riemann-Liouville fractional derivative of order α is given by ${}^{R}D^{\alpha} f := D^m I^{m-\alpha} f$ where $m = \lceil \alpha \rceil$ and the Caputo derivative of order α is given by ${}^{C}D^{\alpha} f := I^{m-\alpha} D^m f$ (see [12, §2, 3] for formal definitions). To specify other lower i ntegration limit, we write ${}_{a}I^{\alpha} f(t)$ and ${}_{a}D^{\alpha} f(t)$; e.g. in the above cases $a = 0$.

For fixed $T > 0$, the fractional integral defines an internal product on the set of locally integrable functions from $[0,T]$ to $\mathbb{C}$ (the proof is similar to that for $\alpha = 1$). We denote this product as $\langle f,g \rangle_{T,\alpha} := [I^{\alpha} f^* g](T)$ where $*$ denotes the complex conjugate and $\|f\|^2_{T,\alpha} := \langle f,f \rangle_{T,\alpha}$.

An input-output system will be denoted by $\Sigma = \Sigma(u,y,x)$ where $u \in \mathscr{U} := \{u : [t_0,t_1) \to U \,|t_0,t_1 \in \mathbb{R}\}$ is the input to the system, $y \in \mathscr{Y} := \{y : [t_0,t_1) \to Y \,|t_0,t_1 \in \mathbb{R}\}$ is its output, U, Y are vector spaces and $x \in \mathscr{X} := \{x : [t_0,t_1) \to X \,|t_0,t_1 \in \mathbb{R}\}$, is an internal variable that allows to define a map from $\mathscr{U} \times X$ to $\mathscr{Y}$, where X is a vector space. An alternative notation [5] is $y := G(x_0)u$ where $G(x_0) : \mathscr{U} \to \mathscr{Y}$ and x_0 is the initial condition or the state at $t = 0$.

Σ is finite-gain stable if there exist a constant $\gamma \in \mathbb{R}$ and a function $\beta : X \to \mathbb{R}$ such that for any initial condition x_0, for any $u \in \mathscr{U}$ and for any $T > 0$, the following inequality is satisfied

$$\|y\|_{T,\alpha} \leq \gamma \|u\|_{T,\alpha} + \beta(x_0). \qquad (2)$$

We assume continuity of all functions involved, which is required to apply properties of fractional derivative and to pass from integral to derivative form. Conditions for this can be obtained in [12].

2.2. Dissipative systems. A dissipative system is characterized by the existence of a scalar function which dissipates as time goes [3, Definition 2]. We generalize to non-negative real order this concept,

Definition 1. A system $\Sigma = \Sigma(u,y,x)$ is α-dissipative for a continuous function $w : \mathbb{R}_{\geq 0} \to \mathbb{R}$, called supply rate, if there exists a non-negative continuous function $V : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, called storage function, such that, $\forall u \in \mathscr{U}$

$$V(t) - V(0) \leq [I^{\alpha} w](t) \quad \forall t \geq 0 \qquad (3)$$

called dissipation inequality. In particular, system Σ is α-loseless if it is α-dissipative and the defining inequality (3) becomes an equality for every $t \geq 0$.

In general, the storage is a time function of type $V(t) = V(x(t),t)$ and the supply rate gives account on the input-output behavior, $w(t) = w(u(t),y(t))$. As y depends on the initial condition, w will have this dependence too. The inequality (3) relates the right hand side which is a purely input-output observable term, with the left hand side coming from an internal variable model and possibly not observable.

For memory-less systems, i.e. those defined for maps $\mathscr{F} : \mathscr{U} \to \mathscr{Y}$, dipativeness is compatible with Definition 1, by seen them as $\alpha = 0$-dissipative, since $I^{\alpha=0} w = w \geq 0$. It follows that any memory-less disipative system is α-dissipative for every $\alpha > 0$.

System Σ is *strictly* α-dissipative if

$$V(t) - V(0) \leq [I^{\alpha} w](t) - \varepsilon [I^{\alpha} u^T u](t) - \delta [I^{\alpha} y^T y](t) - \rho [I^{\alpha} \psi(x)](t), \qquad (4)$$

where $\rho\psi(x) \geq 0$ for every $x \in X$, $\varepsilon > 0$ (*strictly input*) and/or $\delta > 0$ (*strictly output*). Note that strictly α-dissipative implies α-dissipative. By taking $V(0) := \beta(x(0))$, $\rho = 0$ and using the no negativity of V, strictly dissipativeness implies the one obtained by generalizing [5, Definition 2] to FO, namely

$$[I^{\alpha} w](t) - [I^{\alpha} \varepsilon u^T u] - [I^{\alpha} \delta y^T y] + \beta(x_0) \geq 0. \qquad (5)$$

When $\delta = \varepsilon = 0$, (5) will be refereed as *weakly* α-dissipativeness following the notation in [5, Definition 2], since α-dissipativeness implies weakly α-dissipativeness. Note the input-output character of the weakly α-dissipativeness definition where non-internal variable is required to establish it. The following remark points out a subtle distinction with the usual concept of dissipativeness [3].

Remark 1. From α-integration, if $[{}^{C}D^{\alpha} V](t) \leq w(t)$ for all $t > 0$, then (3) holds for $\alpha \in (0,1]$. The converse is not necessarily true unless $\alpha = 1$. This differential condition has practical advantages since one can arrive to it from a mathematical model of the system. However, when $\alpha \neq 1$, this implication is valid for a fixed initial time equal to the initial time of the fractional derivative. Therefore, for fractional systems, we assume fixed initial time at (3).

For the next result, we define the available α-storage function given by

$$V_a(x;\alpha) := \sup_{u\in\mathscr{U},t\geq 0} \{-I^\alpha w(u,G(x)u)(t)\} \tag{6}$$

and the required supply given by

$$V_r(x;\alpha) := \inf_{t\geq 0,u_{x^*\to x}} \{I^\alpha w(u,G(x^*)u)(t)\}, \tag{7}$$

where $u_{x^*\to x}$ is any input that transfer the system from x^* at 0 to x at t. For the latter, we assume that there exists x^* such that $V(x^*) \leq V(x)$ for any $x \in X$, where without loss of generality, we fix the storage zero level such that $V(x^*) = 0$ and that any x is reachable from x^*.

Proposition 1. If a system $\Sigma(u,y,x)$ is α-dissipative with storage function $V = V(x)$ and supply w, then

$$0 \leq V_a(x;\alpha) \leq V(x) \leq V_r(x;\alpha) < \infty, \quad \forall x. \tag{8}$$

Moreover, if Σ is α-lose-less system then V_a and V_r are storage functions. Conversely, if V_a (or V_r) is a finite available (required) storage function, then Σ is α-dissipative.

Proof. (see Appendix)

Remark 2. For α-lose-less systems, the second part possibilities to determine the dissipative property from pure input-output experiments; i.e. it is a input-output property.

We will show in the next proposition relationships among different order of dissipativeness.

Proposition 2. Let Σ be an α-dissipative system with storage V and supply rate w. For any $\beta < \alpha$, Σ is β-dissipative with storage function V and rate $\tilde{w} := \tilde{w}(t) := I^{\alpha-\beta}[w(u,y)](t)$.

Proof. (see Appendix)

2.3. Passivity and positive real systems. We develop some closely related concepts to dissipativeness, which use a supply rate function given by $w = y^T u$, where it is assumed that the space $Y = U$ has an internal product.

Definition 2. System Σ is α-positive real if $(\forall u \in \mathscr{U})$, $(\forall t \geq 0)$, $[I^\alpha y^T u](t) \geq 0$ whenever $x(0) = 0$

This definition extends the one given in [4, Definition 2.8] for $\alpha = 1$. It follows that if Σ is α-positive real and $\beta > \alpha$ then Σ is β-positive real (by the semi group property of fractional integrals [12, Theorem 2.2]).

Definition 3. System Σ is (strictly or weakly) α-passive if Σ is (strictly or weakly) α-dissipative for the supply rate $w = y^T u$ and a storage function $V = V(x)$ such that $V(0) = 0$.

We have the following extension to FO of the equivalence between passivity and positive real for linear system [4, Proposition 2.12],

Proposition 3. If system Σ is α-passive then it is α-positive real. Conversely, if Σ is a linear commensurate fractional system for $\alpha \leq 1$ is α-positive real then it is α-passive.

Proof. (see Appendix)

Given this equivalence and the fact that (integer or fractional) linear systems are characterized by their transfer functions, it would be useful to have a frequency-domain characterization of α-positive real, generalizing [27, Theorem 1]. We restrict our analysis to the first Riemann sheet i.e. $arg(s) \in (-\pi,\pi)$

Proposition 4. Consider a system described by a proper transfer function $G(s)$

(i) The system is α-positive real for $\alpha \geq 1$ if $[G(jw) + G(jw)^*] \geq 0$ for all real w and G has not poles in the complex open right half-plane.

(ii) If the system is α-positive real for $\alpha \leq 1$ and G has not poles in the open right hand complex plane, then $G(jw) + G(jw)^* \geq 0$ and $s = \sigma + jw$ is not a pole of $G(s)$ for all $\sigma \geq 0$ and all real w.

Proof. (see Appendix)

Remark 3. (i) Definition 2 is equivalent to condition $[G(jw) + G(jw)^*] \geq 0$, for $\alpha = 1$. Writing $G(s) = A(s) + jB(s)$ where $A(s), B(s) \in \mathbb{R}$ for any $\mathfrak{Re}(s) \geq 0$, it follows that $G(jw) + G(jw)^* = A(jw) + A^T(jw) + j(B(jw) - B^T(jw))$. Hence, the condition $[G(jw)+G(jw)^*] \geq 0$ is equivalent to $\mathfrak{Re}(G(jw)) \geq 0$ and $B(jw) = B^T(jw)$ for any $w \in \mathbb{R}$, which implies that Nyquist plots help to analyze this condition. Moreover, since $\mathfrak{Re}(G)$ is analytic on the right hand plane, the minimum modulus theorem implies that $\mathfrak{Re}(G(s)) \geq 0$ for $\mathfrak{Re}(s) \geq 0$.

(ii) Since the condition for 1-positive real in the frequency-domain (or for 1-passive, by Proposition 3) is the same whether G comes for a fractional or integer system, theorems to determine the positive real property involving algebraic manipulations in Laplace domain are automatically true for fractional linear systems. For example, high gain feedback to turn positive real a system (see e.g. [31]). However, in the next section we will show that not all the stability results associated to positive real systems hold by the non-local character of fractional derivative, the diffusive approach to non-integer derivative being an exception.

(iii) In [18] it was suggested to define a fractional passive system by the extension of the concept of positive real system, in the following sense. If a rational transfer function $H(s)$ is positive real, then the resulting function from replacing s by s^α, $H(s^\alpha)$, is positive real when $s^\alpha \geq 0$; the irrational function $G(s) := H(s^\alpha)$ was called fractional positive real or passive. This strategy, although not fully developed on the passive part, is restricted to linear system and its generalization is captured in the 1-positive real and 1-passive concepts proposed here. From [32, Figure 5] where $G(s)$ is not positive real even though $H(s)$ is passive for $\alpha > 1$, it follows that Proposition 3 does not hold $\alpha > 1$.

Consider the non-linear Caputo fractional systems defined by

$$\begin{cases} D^{\alpha}x = f(x) + g(x)u, \\ y = h(x), \end{cases} \tag{9}$$

where $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $u, y : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$.

Definition 4. System (9) has the KYP property if there exists a smooth positive definite function $V = V(x)$ such that for all $x \in X$,

(i) $\frac{\partial V}{\partial x}^T f(x) \leq 0$,

(ii) $\frac{\partial V}{\partial x}^T g(x) = h^T(x)$.

This property is independent of the order α of system (9). For a linear system and considering $V = x^T Px$, we have (i) implies $PA + A^T P \leq 0$ and (ii) implies $PB = C$. We have the following result generalizing [4, Proposition 2.12],

Proposition 5. If system (9) has the KYP property, then it is α-passive for any $\alpha \leq 1$.

Proof. (see Appendix)

2.4. Large scale system. The scalar nature of the storage makes it possible to study complex systems by examining their subsystems. In particular, the dissipative property can be asserted if each subsystem is dissipative for a suited connection.

Consider a finite family of systems $(\Sigma_\lambda)_{\lambda\in\Lambda}$. Each system interacts with the ambient through w^e_λ and with the other systems of the family through w^i_λ. The interconnected system is defined as $\Sigma := (\Sigma_\lambda)_{\lambda\in\Lambda}$ with internal variables $x = (x_\lambda)_{\lambda\in\Lambda}$ where x_λ is the internal variable of Σ_λ.

The interconnecting system has input $(y^i_\lambda)_{\lambda\in\Lambda}$ and output $(u^i_\lambda)_{\lambda\in\Lambda}$. The interconnecting system is passive if $\sum_\lambda w^i_\lambda \leq 0$ and neutral or lose-less if $\sum_\lambda w^i_\lambda \equiv 0$. A feedback (or parallel) system resulting from connecting $\Sigma_1(y_1,u_1)$ and $\Sigma_2(y_2,u_2)$ such that $u_1 = e_1 - y_2, u_2 = e_2 + y_1$ where e_1, e_2 are their external input and $-y_2, y_1$ are their internal input respectively, is an example of interconnected neutral system.

The following result generalizes [3, Theorem 5] in allowing interconnection of different order of derivation systems and in a converse result which are compatible by Proposition 2.

Theorem 1. (i) Consider a family of α_λ-dissipative systems $(\Sigma_\lambda)_{\lambda\in\Lambda}$ with storage V_λ and such that the interconnecting system is passive. If $\alpha := \min_\lambda \alpha_\lambda > 0$, then the interconnected system is α-dissipative with $V := \sum_\lambda V_\lambda$ and $w := \sum_\lambda I^{\alpha_\lambda-\alpha} w^e_\lambda$. If the systems are α-(strictly output) passive systems and the interconnecting system is passive, then the interconnected system is α-(strictly output) passive.

(ii) Given a weakly α-dissipative (or passive) system with $\varepsilon = \delta = 0$ and supply $\sum_\lambda w^e_\lambda$, composed of subsystems neutrally interconnected such that $w_\lambda = 0$ whenever $u_\lambda = 0$ for each λ, then each subsystem is weakly α-dissipative.

Proof. (see Appendix)

3. Stability

In this section we present stability – in the Lyapunov and input-output sense – results for systems with passive properties. In particular, we will see that to stabilize a non-linear passive system is a simple problem (from Propositions 3 and 4 passive linear systems are stable).

In this section we consider the Caputo derivative, the differential sense of dissipative systems (see Remark 1) and $\alpha \in (0,1]$.

The main feature of dissipative properties is that they allow for considering input-output relationships.

Proposition 6. If system Σ is a strictly output α-dissipative system then Σ is finite-gain stable.

Proof. (see Appendix)

The following results show that stabilization of passive systems is a simple problem and then, passivation of systems is a relevant problem.

For the next result – which generalizes to fractional systems [4, Theorem 3.2], consider the nonlinear fractional system

$$\begin{cases} D^{\alpha}x(t) = f(x,u,t), \\ y = h(x), \end{cases} \tag{10}$$

where $y(t), u(t) \in \mathbb{R}^m$, $x(t) \in \mathbb{R}^n$ for all $t \geq 0$. f, h, u are smooth enough such that the solution x is continuous (see [12]), $f(0,0,t) = 0$ for all $t \geq 0$ and $h(0) = 0$, so that $x = 0$ is an equilibrium point for $u \equiv 0$ (unforced system).

Theorem 2. Suppose that system (10) is α-passive with storage function $V = V(x)$ positive definite. Let $\phi : Y \to Y$ a function such that $y^T\phi(y) \geq 0$ and $\phi(0) = 0$. Then the control $u = -\phi(y)$ stabilizes the origin, makes $I^\alpha y^T\phi(y)$ a bounded function. If, in addition, V is radially unbounded, then x is bounded. Moreover, if $\phi(y) = -ky$, for any constant number $k > 0$, then the RMS value of y converges to zero and if y only vanishes at zero, then $x = 0$ is weakly asymptotically stable.

Proof. (see Appendix)

Remark 4. (i) If the storage is not positive definite the stability cannot be guaranteed; but the asymptotic properties of y and x still hold. (ii) Note that $u = 0$ hold condition of Theorem 2 i.e. an unforced α-passive system is stable.

Consider the autonomous instance of system (10), where f does not explicitly depend on time. In [4] it was stated that if f has the for $f'(x) + g(x)u$ and the system is passive and detectable then it can be asymptotically stabilized, when $\alpha = 1$. Let $\phi(t;x_0,u)$ the solution at time t given the initial condition x_0 and input u. System is called zero state observable if $h(\phi(t;x_0,0)) \equiv 0$ then $\phi(t;x_0,0) \equiv 0$; zero state detectable if $h(\phi(t,x_0,0)) \equiv 0$ implies $\lim_{t\to\infty}\phi(t,x_0,0) = 0$. The system is zero state asymptotically detectable if $h(\phi(t;x_0,0)) \to 0$ then $\phi(t;x_0,0) \to 0$ as $t \to \infty$ and zero state α-integrally detectable if $[I^\alpha h^T h] < C < \infty$ for constant C then $\phi(t;x_0,0) \to 0$ as $t \to \infty$.

Proposition 7. Let system (10) – where f does not explicitly depend on time – be β-passive, with positive definite storage function V and $u = -\phi(y)$ where ϕ is as in Theorem 2.

(i) If the system is strictly output (or input) β-passive with $\rho\psi(x) \geq \rho\|x\| \geq 0$, then the origin $x = 0$ is asymptotically stable for the unforced case.

(ii) If V is radially unbounded, h, f, ϕ are continuous functions, $\beta \geq 1$ and the system is zero state asymptotically detectable, then $x = 0$ is asymptotically stable.

(iii) If the system is zero state β-integrally detectable, then $x = 0$ is asymptotically stable.

Proof. (see Appendix)

The next result studies stability for a feedback interconnection of passive systems, generalizing [1, §10.3].

Theorem 3. Let Σ_i an strictly α-passive system with positive definite storage V_i and parameters $(\varepsilon_i,\, \delta_i,\, \rho_i\psi_i(x_i))$ for $i = 1,2$, given by

$$\begin{cases} D^\alpha x_i = f_i(x_i, e_i), \\ y_i = h_i(x_i, e_i), \end{cases} \tag{11}$$

where $x_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_i}$, $e_i, y_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, $f_i(0,0) = 0$ and $h_i(0,0) = 0$ for $i = 1,2$. Consider the interconnection given by $e_1 = u_1 - y_2$ and $e_2 = u_2 - y_1$. The interconnected system has input (u_1, u_2) and output (y_1, y_2).

(i) Assume that $\varepsilon_2 + \delta_1 > 0$ and $\varepsilon_1 + \delta_2 > 0$. Then, the interconnected system is finite gain stable. If the functions y_1, y_2 only vanish at $x_1 = 0$ and $x_2 = 0$ respectively, then the origin is weakly asymptotically stable and the output has bounded $\|\cdot\|_\alpha$ norm.

(ii) Assume that $\varepsilon_2 + \delta_1 \geq 0$ and $\varepsilon_1 + \delta_2 \geq 0$. Then, the interconnected system is stable. If the function ψ only vanishes at $x = (x_1, x_2) = 0$ then the origin is weakly asymptotically stable and the output has bounded $\|\cdot\|_\alpha$ norm. If in addition $V1$ are V_2 are radially unbounded, then (x_1, x_2) remains bounded.

Proof. (see Appendix)

Remark 5. Taking u_2 as additive noise in the output measurement and Σ_2 as a control system of Σ_1, Theorem 3 provides conditions to guarantee robust perform.

4. Passivation

By the results of Section 3, the problem of input passivation of a given system i.e. to render the system passive by using a new input v for the same output function, has an equivalent importance to the stabilization problem of the system. If such input exists, the system is said feedback equivalent to a passive system. We consider the cases where the full internal variables are available (feedback passivation) and where the parameters of the realization in internal variables of the system are unknown (adaptive passivation).

A necessary condition assuring that there exist $u = \eta(x) + v$ and $h(x)$ that makes system (10) passive with storage function positive definite is that the system can be stabilized by feedback input (Theorem 2). A sufficient condition is by defining a new output $h(x) := g^T x$ and requiring KYP (Proposition 5). The linear case can be solved as a particular case of those or by employing Proposition 3 and Remark 3(iii) for a frequency domain technique.

In this section we consider the Caputo derivative, the differential sense of dissipative systems (see Remark 1) and $\alpha \in (0,1]$.

4.1. Feedback passivation. An IO dynamical system is rendered passive by smooth state feedback if and only it has relative degree one and is weakly minimum phase [4]. Essentially, the relative degree one allows to cancel terms with the feedback input and the weakly minimum phase allows to get the passive inequality. These operations will play a similar role in the next result for the following class of systems

$$\begin{cases} D^\alpha z = f^*(z) + p(z,y)y + \left[\sum_i q_i(z,y)y_i\right]v, \\ D^\alpha y = v, \end{cases} \tag{12}$$

where $y(t), v(t) \in \mathbb{R}^m$, $z(t) \in \mathbb{R}^{n-m}$ for all $t \geq 0$ and $0 < \alpha \leq 1$. Note that the system in normal form

$$\begin{cases} D^\alpha \eta = c(\eta,y) + d(\eta,y)u, \\ D^\alpha y = b(z,y) + a(z,y)u \end{cases}$$

can be written as (12) through the feedback $u = a^{-1}[-b+v]$, provided that a^{-1} exists. Note also that the feedback autonomous instance of system (10), where f does not explicitly depend on time, can putted in normal form, by infinitesimal expansion locally around the origin, provided that $u(x)$ are small when x is small.

System (12) is weakly minimum phase if there exists a smooth positive definite function $V_0 : \mathbb{R}^{n-m} \to \mathbb{R}$ such that for $v \equiv y \equiv 0$ (zero dynamic), $D^\alpha V(z(t)) \leq 0$ for all $t > 0$. For a linear system given by the transfer function N/D, the zero dynamic is given by $N(s)u \equiv 0$ and minimal phase is equivalent to stable zeros.

Theorem 4. Consider system (12) weakly minimum phase. Then Σ is locally feedback equivalent to an α-passive system.

Proof. (see Appendix)

In the following proposition, we show how passivation can help us to determine properties of systems up to feedback equivalence.

Proposition 8. Consider the system

$$\begin{cases} D^\alpha \xi = f_0(\xi) + f_1(\xi,y)y, \\ D^\beta y = f(\xi,y) + g(\xi,y)u, \end{cases} \tag{13}$$

where $y(t), u(t) \in \mathbb{R}^m, \xi(t) \in \mathbb{R}^n$ for all $t \geq 0$ and g is nonsingular around zero. Assume that the ξ-system holds that ξ is bounded whenever y is bounded. Then, (13) is feedback equivalent to a bounded system.

Proof. (see Appendix)

4.2. Adaptive passivation. We consider systems described by equations containing unknown parameters. The input u will depend besides on the new input v and the internal variables, on the adjustable parameters θ.

Theorem 5. Consider the weakly minimum phase (with V_0 radially unbounded) system

$$\begin{cases} D^\alpha z = \Lambda_0 f_0(z,y) + p(z,y)\Lambda_p y, \\ D^\alpha y = \Lambda_b b(z,y) + \Lambda_a a(z,y)u, \end{cases} \tag{14}$$

where $0 < \alpha < 1$, $y(t), u(t) \in \mathbb{R}^m$, $z(t) \in \mathbb{R}^n$ for all $t \geq 0$. Λ_0, Λ_p, Λ_b, Λ_a are real constant unknown matrices of suited dimensions, Λ_a and a are non-singular. Then the input

$$u = a^{-1}\left[\theta_3\theta_1 a + \theta_3\theta_2 p^T \frac{\partial V_0}{\partial z}\right], \tag{15}$$

where

$$\begin{cases} D^\alpha \theta_1 = -yb^T, \\ D^\alpha \theta_2 = -y\dfrac{\partial V_0}{\partial z}^T p, \\ D^\alpha \Theta = -yu^T a^T \end{cases} \tag{16}$$

and $\Theta = \theta_3^{-1}$, makes the solutions of system (14) bounded. Moreover, the input

$$u = a^{-1}\left[\theta_3\theta_1 a + \theta_3\theta_2 p^T z - \theta_3 y\right] \tag{17}$$

guarantees that $I^\alpha y^T y$ is bounded and the RMS value of y converges to zero. In particular for $\alpha = 1$, $\lim_{t\to\infty} y(t) = 0$ and if in addition $\dfrac{\partial V_0}{\partial z}^T \Lambda_0 f_0(z,y) < 0$ and p is a finite function then $\lim_{t\to\infty} z(t) = 0$.

Proof. (see Appendix)

Remark 6. It was proved that the system (Φ, z, y) is passive but the system (z, y) was not necessarily passivized because $V(z=0, y=0, \Phi) = trace(\Phi\Phi^T) \neq 0$ is not positive definite restricted to (y, z). Moreover, if the system (z, y) is zero state detectable, $y \equiv 0$ implies that $\Phi = \Phi(0)$, that is, the detectable property is lost.

5. Examples

The first example shows a model of a fractional passive system.

Example 1. Some fractional models have been proposed for capacitors. Consider the following model of capacitor

$$CD^\alpha v = i, \tag{18}$$

where v, i are the voltage and current, respectively. Note that its Laplacian relationship, $Cs^\alpha \hat{v}(s) = \hat{i}(s)$, is what has been postulated for a more exact model of a capacitor [13, equation (10.56)]. Defining the storage function $2V = Cv^2$, we have $D^\alpha V \leq CvD^\alpha v = vi$. Hence, the capacitor is α-passive for the supply rate $w = vi$, i.e. the electric power.

Consider the fractional circuit with C an α-passive capacitor like element connected in parallel with a resistor R element

$$\begin{cases} CD^\alpha v + v/R = i, \\ u = v, \\ y = i. \end{cases}$$

Defining the storage function $2V = Cv^2$, we have $D^\alpha V \leq CvD^\alpha v = v(i - v/R)$. Hence, the system is α-strictly passive for the usual electric power supply $w = uy = vi$. This result follows alternatively from Theorem 1 since a 0-dissipative element is also α-dissipative and the connection is neutral. It is remarked that the study of passive positive real systems and the recent attention on fractional calculus had came both from the analysis of electric circuit impedance.

The second example studies conditions for KYP property and passivity in the linear case.

Example 2. Consider a Caputo commensurate linear fractional system

$$\begin{cases} D^\alpha x = Ax + Bu, \\ y = Cx + Du \end{cases}$$

with $\alpha \leq 1$, $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $y : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ and real constant matrices A, B, C, D of suited dimensions. Suppose that the system has the following property: $\exists P > 0, w, L, \varepsilon > 0$ constant matrices of adequate dimensions such that

$$\begin{cases} PA + A^T P = -L^T L - \varepsilon P, \\ PB = C^T - L^T w, \\ w^T w = D + D^T. \end{cases}$$

This property is a particular instance of KYP. By choosing $2V = x^T Px$, we have that $D^\alpha V \leq x^T PD^\alpha x$. Using this property we have

$$\begin{aligned} u^T y - D^\alpha V &\geq u^T(Cx + Du) - x^T P(Ax + Bu) \\ &= u^T Cx + 1/2u^T(D + D^T)u - 1/2x^T(PA + A^T P)x \\ &\quad - x^T PBu \\ &= 1/2(Lx + wu)^T(Lx + wu) + 1/2\varepsilon x^T Px. \end{aligned}$$

Therefore

$$D^\alpha V + 1/2\varepsilon\lambda_{min}(P)x^T x \leq u^T y, \tag{19}$$

i.e. the system is strictly passive with $\rho\psi(x) := \lambda_{min}(P)x^T x$. By α-integration of (19) and making $x(0) = 0$, we have

$$0 \leq V(t) + I^\alpha[1/2\varepsilon\lambda_{min}(P)x^T x](t) \leq I^\alpha[u^T y](t), \tag{20}$$

i.e. the system is α-positive real. A necessary and sufficient condition to have the KYP property, is that the quadruple (A, B, C, D) defines a 1-positive real integer system ([27, Theorem 1]).

Finally, we showa passivity approach to PI^{α} control of integer systems.

Example 3. Consider the fractional integral system

$$y(t) = K_I [I^{\alpha} u](t), \tag{21}$$

where $u, y : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$ and $K_I \in \mathbb{R}_{>0}$.

6. Conclusions

We have shown that the concepts of positive real, passivity and dissipation, admit a generalization by considering real order rather than integer one in their defining relationships.

By this generalization, a methodology to stabilize – in the Lyapunov and input-output sense – linear and non-linear, known and unknown fractional systems is proposed, which relies in the development of an (adaptive) fractional passivation technique and the stated fact that fractionally passive systems can be easily stabilized.

We have shown further that those generalized to real order concepts are related among them, which allows the analysis of systems consisting of subsystems with different order of dissipation. In particular, the stability of a system composed of subsystems defined by integer and fractional derivatives can be asserted. It is exemplified for a PI^{λ} control of an integer system.

Acknowledgements. The authors thank to CONICYT Chile for the support under Grants FONDECYT 1150488 "Fractional Error Models in Adaptive Control and Applications", FONDECYT 1190959 "Development of fractional order tools for stability, estimation and control of systems and applications" and FB0809 "Advanced Mining Technology Center".

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Appendix

Proof of Proposition 1. Since V is non negative, we have $-[I^{\alpha}w(u,G(x(0)u))](t) \leq V(x(0))$, $\forall u \in \mathscr{U}$, $\forall t \geq 0$. Hence, $V_a(x(0);\alpha) = \sup_{u,t}[-I^{\alpha}w(u,G(x(0)))(t)] \leq \sup_{u,t}[V(x(0))] \leq V(x(0))$. Since $x(0)$ is arbitrary and V_a is non negative since for $t=0$, the integral in (6) is zero because w,y,u are continuous, V_a is finite and $0 \leq V_a(x;\alpha) \leq V(x)$. Also, by the α-dissipative inequality, $V(x) \leq I^{\alpha}w(u,G(x^*))(t)$ for any u,t and, by similar reasons, it follows $V(x) \leq V_r(x;\alpha)$.

Clearly $V_r(x^*;\alpha) = 0$ and by continuity of w, $V_r(x;\alpha) < \infty$.

For α-lose-less systems, taking any trajectory from x to x^* (arriving at time t), we have from inequality (8), $V(x(t)) = -[I^{\alpha}w](t) \geq V_a(x) = \sup_{t,u} -I^{\alpha}w(t)$. Hence, $V(x) = V_a(x)$. Similarly, for any trajectory from x^* to x (arriving at time t), we have $V(x(t)) = [I^{\alpha}w](t) \leq V_r(x) = \inf_{t,u} I^{\alpha}w(t)$. Hence, $V(x) = V_r(x)$. Therefore, for α-lose-less system $V_a = V_r$ (there is a unique storage function), V_a and V_r are storage functions. □

Proof of Proposition 2. Since Σ is α-dissipative, we have $V(t) - V(0) \leq I^{\beta}[I^{\alpha-\beta}w](t)$, which was used for any $\alpha_1,\ \alpha_2 > 0$ and any continuous function x, $I^{\alpha_1}I^{\alpha_2}x = I^{\alpha_1+\alpha_2}x$ (see e.g. [12, Theorem 2.2]). By defining $\tilde{w}(t) := I^{\alpha-\beta}w(u(\cdot),y(\cdot))(t)$, we have that Σ is β-dissipative with storage function V and time-varying supply $\tilde{w}$. □

Proof of Proposition 3. From inequality (8) and passivity, we have $0 \leq V_a(0;\alpha) = \sup_{t,u} -I^{\alpha}y^Tu \leq V(0) = 0$, which implies that $(\forall t > 0)$, $(\forall u \in \mathscr{U})$, $I^{\alpha}y^Tu(t) \geq 0$. The converse follows using the already noted fact that if a system Σ is α-positive real and $\beta > \alpha$ then Σ is β-positive real, together with Example 2. □

Proof of Proposition 4. (i) Consider the functions u_t, y_t identical to u,y in $[0,t]$ and zero otherwise. Then, by the semi-group property of the fractional integrals [12], we can write

$$
{}_0I^{\alpha}y^Tu(t) = {}_0I^{\alpha-1}\,{}_0I^1 y_t^T u_t(t) = \int_0^t K_{\alpha}(t-\tau)\int_0^{\tau} y_{\tau}^T u_{\tau}\,\mathrm{d}\tau\,\mathrm{d}t
$$

$$
= \int_0^t K_{\alpha}(t-\tau)\int_{-\infty}^{\infty} y_{\tau}^T u_{\tau}\,\mathrm{d}\tau\,\mathrm{d}t.
$$

Since G has not unstable poles, the impulse response associated to G is L_1 [30, Theorem 3.1], whereby $G * u_t$ is also L_1 for any t. Then, by the linear relation in Laplace domain $y = Gu$ for $x(0) = 0$, we can apply Parseval's theorem to obtain,

$$
{}_0I^{\alpha}y^Tu(t) = (2)^{-1}\,{}_0I^{\alpha}[y^*u + u^*y](t) = (4\pi)^{-1}\int_0^t K_{\alpha}(t-\tau)
$$

$$
\int_{-\infty}^{\infty} u_{\tau}^*(jw)[G(jw)+G(jw)^*]u_{\tau}(jw)\,\mathrm{d}w\,\mathrm{d}t \geq 0,
$$

where the last inequality is due to the hypothesis.

(ii) Since $\int_0^t y^Tu\,\mathrm{d}t = {}_0I^{1-\alpha}\left[{}_0I^{\alpha}y^Tu(\cdot)\right](t) \geq 0$ we have from Parseval's theorem as above $\int_{-\infty}^{\infty} u_{\tau}^*(jw)G(jw)^*u_{\tau}(jw)\,\mathrm{d}w \geq 0$ which implies $G(jw),\ G(jw)^* \geq 0$ since u is arbitrary. □

Proof of Proposition 5. By using the KYP properties and [28, Theorem 3], we have the inequality, $D^{\alpha}V \leq \frac{\partial V}{\partial x}^T f(x) + \frac{\partial V}{\partial x}^T g(x)u$, where we have used that since V is positive definite and smooth it is convex around the origin. Using the KYP hypotheses, $D^{\alpha}V \leq y^Tu$. By α-integrating, $V(x(t)) - V(x(0)) \leq I^{\alpha}y^Tu(t)$. □

Proof of Theorem 1. (i) By hypothesis,

$$
V_{\lambda}(x_{\lambda}(t)) \leq \left[I^{\alpha_{\lambda}}w_{\lambda}^i + w_{\lambda}^e\right](t) + V_{\lambda}(x_{\lambda}(0)).
$$

Adding and using passivity of the interconnecting system

$$
\sum_{\lambda} V_{\lambda}(x_{\lambda}(t)) \leq \sum_{\lambda}\left[I^{\alpha_{\lambda}}w_{\lambda}^e\right](t) + \sum_{\lambda} V_{\lambda}(x_{\lambda}(0)).
$$

Since the set Λ is finite, w are continuous and $\alpha = \min_{\lambda}\alpha_{\lambda}$, we have $\sum_{\lambda} V_{\lambda}(x_{\lambda}(t)) \leq I^{\alpha}\left[\sum_{\lambda} I^{\alpha_{\lambda}-\alpha}w_{\lambda}^e\right](t) + \sum_{\lambda} V_{\lambda}(x_{\lambda}(0))$. For the passive claim, we have similarly the following steps

$$
V_{\lambda}(x_{\lambda}(t)) \leq V_{\lambda}(x_{\lambda}(0)) + \left[I^{\alpha}u_{\lambda}^Ty_{\lambda} - \varepsilon_{\lambda}\|y_{\lambda}\|^2\right](t)
$$

$$
\sum_{\lambda} V_{\lambda}(x_{\lambda}(t)) \leq \sum_{\lambda} V_{\lambda}(x_{\lambda}(0)) + \sum_{\lambda}\left[I^{\alpha}u_{\lambda}^{e\,T}y_{\lambda} - \varepsilon_{\lambda}\|y_{\lambda}\|^2\right](t).
$$

Defining the vectors $u^e = (u_{\lambda})_{\lambda\in\Lambda}$ and $y = (y_{\lambda})_{\lambda\in\Lambda}$, we can write

$$
\sum_{\lambda} V_{\lambda}(x_{\lambda}(t)) \leq \sum_{\lambda} V_{\lambda}(x_{\lambda}(0)) + I^{\alpha}u^{eT}y(t) - \sum_{\lambda} I^{\alpha}\left[\varepsilon_{\lambda}\|y_{\lambda}\|^2\right](t).
$$

Then, the system $u^e \to y$ is strictly output passive for $V(x(t)) = \sum_{\lambda} V_{\lambda}(x_{\lambda}(t))$

(ii) From weak α-dissipativeness, we can write for every $t \geq 0$ and every $u_{\lambda} \in \mathscr{U}$

$$
\left[I^{\alpha}\sum_{\lambda} w_{\lambda}^e\right](t) \geq \beta(x_0)
$$

and by neutrality of the interconnection

$$
\left[I^{\alpha}\sum_{\lambda} w_{\lambda}^i + w_{\lambda}^e\right](t) \geq \beta(x_0).
$$

By choosing $u_\lambda \equiv 0$ whenever $\lambda \neq \lambda_0$ and fixing arbitrarily the components x_λ whenever $\lambda \neq \lambda_0$, we obtain

$$\left[I^\alpha w^i_{\lambda_0} + w^e_{\lambda_0}\right](t) \geq \beta(x_{\lambda_0}),$$

that is, each subsystem is weakly α-dissipative. □

Proof of Proposition 6. From the α-dissipative definition and algebraic manipulations we have,

$$D^\alpha V \leq u^T y - \delta y^T y \leq (2\delta)^{-1} u^T u - (\delta/2) y^T y.$$

By α-integrating,

$$I^\alpha y^T y(t) \leq (\delta^2)^{-1} I^\alpha u^T u(t) - (2/\delta)(V(t) - V(0)).$$

Using that $\sqrt{a^2+b^2} \leq a+b$ and $V \geq 0$, we have for all $t > 0$

$$\|y^T y\|_{t,\alpha} \leq \delta^{-1}\|u^T u\|_{t,\alpha} + \sqrt{2V(0)/\delta}. \quad □$$

Proof of Theorem 2. (i) From α-passivity definition and using the hypothesis,

$$V(x(t)) - V(x(0)) \leq I^\alpha y^T u(t) = -\left[I^\alpha y^T \phi(y)\right](t) \leq 0.$$

Therefore, $V(x(t)) - V(x(0)) \leq 0$. Since V is positive definite and $0 \leq V(x(t)) \leq V(x(0))$, stability of the origin follows from standard arguments ([9, 28]). On the other hand, $\left[I^\alpha y^T \phi(y)\right](t) \leq V(x(0)) - V(x(t)) < \infty$. Since $V(x)$ is bounded and radially unbounded, it follows that x is bounded.

Choosing $u = -ky$, we have $I^\alpha y^T y < \infty$ and we obtain that the RMS value of y converges to zero from similar arguments of [33, Proposition 1(iii)]. Since $y(\cdot)$ is continuous and x is bounded, y is bounded. Similarly $f(x, -ky, t)$ is bounded, whereby x is uniformly continuous [24, Proposition 1] and therefore, $y^T y$ is uniformly continuous. Asymptotic results follows from [9, 28] since $D^\alpha y \leq k y^T y = -\gamma(x)$, with γ a class K function. □

Proof of Proposition 7. (i) By definition and setting $u \equiv 0$, we have

$$D^\alpha V(x) \leq -\rho\psi(x) - \delta y^T y \leq -\rho\psi(x).$$

Since $\psi(x)$ is a positive definite function locally around $x = 0$, by [9, Theorem 1], $x = 0$ is locally asymptotically stable (the proof for strictly input passive being similar).

(ii) From Theorem 2 and radially unbounded of V, x is bounded and since f, g, h, ϕ are continuous, then $f(x), g(x), h(x), \phi(h(x))$ are bounded. Hence, x is uniformly continuous [24, Proposition 1]. By continuity, $y(x)$ and $\phi(y(x))$ are uniformly continuous as time functions. From the proof of Theorem 2(i), we have for all $t > 0$

$$\left[I^\beta y^T \phi(y)\right](t) \leq V(x(t_0)) - V(x(t)) \leq V(x(t_0)) < \infty.$$

By Barbalat's lemma ([23]), $\lim_{t\to\infty} y^T(t)\phi(y(t)) = 0$. By hypothesis on ϕ, $\lim_{t\to\infty} y(t) = 0$. By the detectable hypothesis, $\lim_{t\to\infty} x(t) = 0$.

(iii) Boundedness of $I^\beta y^T \phi(y)$ follows from the fact that passivity implies $V(x(t)) - V(x(0)) \leq -I^\beta y^T \phi(y) \leq 0$ and from the fact that V is positive definite. The claim follows from zero state β-integrally detectable. □

Proof of Theorem 3. (i) From passivity, $e_i^T y_i \geq D^\alpha V_i + \varepsilon_i e_i^T e_i + \delta_i y_i^T y_i$. Defining $V := V_1 + V_2$, we have

$$D^\alpha V \leq -y^T L y - u^T M u + u^T N y,$$

where

$$y := (y_1^T,\, y_2^T)^T, \quad u := (u_1^T,\, u_2^T)^T,$$

$$L := \begin{pmatrix} (\varepsilon_2+\delta_1)I & 0 \\ 0 & (\varepsilon_1+\delta_2)I \end{pmatrix}, \quad M := \begin{pmatrix} \varepsilon_1 I & 0 \\ 0 & \varepsilon_2 I \end{pmatrix},$$

$$N := \begin{pmatrix} I & 2\varepsilon_1 I \\ -2\varepsilon_2 I & I \end{pmatrix}.$$

Following a similar procedure as in the proof of Proposition 6, we obtain

$$D^\alpha V \leq b^2(2a)^{-1}\|u\|_2^2 - c^2\|y\|_2^2,$$

where $a = \lambda_{min}(L) > 0$, $b = \|N\|_2$. Therefore

$$\|y\|_\alpha \leq b(a)^{-1}\|u\|_\alpha + \sqrt{2V(x(0))/c}.$$

If $u_1 = u_2 = 0$, we have

$$D^\alpha V \leq -\rho_1\psi(x_1) - \rho_2\psi(x_2) - (\varepsilon_1+\delta_2)y_2^T y_2 - (\varepsilon_2+\delta_1)y_1^T y_1 \leq 0 \tag{22}$$

then $(x_1, x_2) = (0,0)$ is a stable point, since $V = V(x)$ vanishes only at zero (see proof of Theorem 2). If $V(x)$ is radially unbounded then (x_1, x_2) is bounded, by definition since V is bounded. Since $\rho_1\psi(x_1) + \rho_2\psi(x_2) \geq 0$, we have $D^\alpha V \leq -(\varepsilon_1+\delta_2)y_2^T y_2 - (\varepsilon_2+\delta_1)y_1^T y_1$. Where the right hand term only vanishes at origin. From similar arguments than in Theorem 2, (y_1, y_2) has vanishing RMS value. Asymptotic results follows from [9, 28].

(ii) The proof follows the same arguments of part (i) and therefore is omitted. □

Proof of Theorem 4. We rewrite (12) as

$$\begin{cases} D^\alpha z = f^*(z) + p(z,y)y + Q^T yv, \\ D^\alpha y = v. \end{cases}$$

Defining $V := V_0(z) + 2^{-1}y^T y$ and using inequality in [28] (since V is positive definite and smooth, it is convex around $(y,z) = 0$), it follows that

$$D^\alpha V \leq \frac{\partial V_0}{\partial z}^T \left[f^*(z) + p(z,y)y + Q^T yv\right] + y^T v.$$

From weakly minimum phase, we have

$$D^{\alpha}V \leq y^T p^T(z,y)\frac{\partial V_0}{\partial z} + \left[\frac{\partial V_0}{\partial z}^T Q^T y + y^T\right] v$$

and thus,

$$D^{\alpha}V \leq y^T p^T(z,y)\frac{\partial V_0}{\partial z} + y^T\left[Q\frac{\partial V_0}{\partial z} + I\right] v.$$

Around $(z,y) = (0,0)$, $\left(I + Q\frac{\partial V_0}{\partial z}\right)$ is invertible since $Q\frac{\partial V_0}{\partial z}z = 0$ at $z = 0$. Therefore, by defining $v := (I + QPz)^{-1}\left(-p^T\frac{\partial V_0}{\partial z} + w\right)$ where w is the new input, we obtain $D^{\alpha}V \leq y^T w$ and since V is positive definite, the system (y,w) is passive. □

Proof of Proposition 8. By choosing $u = g^{-1}(-f+v)$, the system $D^{\alpha}y = v$ is passive for $2V = y^T y$ and therefore, stabilizable by Theorem 2, using $v = -\phi(y)$. With this input and this storage, y is bounded. Hence, from hypothesis, ξ is bounded. □

Proof of Theorem 5. We will show that the input

$$u = a^{-1}\left[\theta_3\theta_1 a + \theta_3\theta_2 p^T\frac{\partial V_0}{\partial z} + \theta_3 w\right] \quad (23)$$

passivizes the system (z,y,Φ) where $\Phi_1(t) = \theta_1(t) + \Lambda_b$, $\Phi_2(t) = \theta_2(t) + \Lambda_p^T$ and $\Phi_3(t) = \theta_3(t) - \Lambda_a^{-1}$. Define the positive definite function $V := V_0 + 2^{-1}y^T y + 2^{-1}trace\left(\sum_{i=1}^{3}\Phi_i\Phi_i^T\right)$. Using [28] and the minimum phase hypothesis, it follows that

$$D^{\alpha}V \leq \frac{\partial V_0}{\partial z}^T p\Lambda_p y + y^T\Lambda_b b + y^T\Lambda_a a u + trace\left(\sum_{i=1}^{3}\Phi_i[D^{\alpha}\Phi_i]^T\right).$$

By replacing u and regrouping terms

$$D^{\alpha}V \leq y^T(\Lambda_b + \Lambda_a\theta_3\theta_1)a + y^T(\Lambda_p + \Lambda_a\theta_3\theta_2)p\frac{\partial V_0}{\partial z} + y^T(I - I + \Lambda_a\theta_3) + trace\left(\sum_{i=1}^{3}\Phi_i[D^{\alpha}\Phi_i]^T\right).$$

Working the expression of the right hand side, we have

$$D^{\alpha}V \leq y^T\Phi_1 a + y^T\Phi_2 p\frac{\partial V_0}{\partial z} + y^T\Phi_3 a u + y^T w + trace\left(\sum_{i=1}^{3}\Phi_i[D^{\alpha}\Phi_i]^T\right)$$

and hence

$$D^{\alpha}V \leq y^T w + trace\left(\Phi_1 b y^T + \Phi_2 p\frac{\partial V_0}{\partial z}y^T + \Phi_3 b u y^T + \sum_{i=1}^{3}\Phi_i[D^{\alpha}\Phi_i]^T\right).$$

By noting that $D^{\alpha}\Phi_i = D^{\alpha}\theta_i$ and using equation (16) we get

$$D^{\alpha}V \leq y^T w.$$

If $w = 0$ then $D^{\alpha}V \leq 0$, i.e. $V(t) \leq V(0)$ for all $t > 0$. In particular, $V_0(z) + 2y^T y \leq V(0)$, whereby z, y remains bounded. The rest of the proof is similar to the proof of Theorem 2 by choosing the input $w = -y$; the case of $\alpha = 1$ following from [24, Theorem 6]. □