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Model reduction problem of linear discrete systems: Admissibles initial states

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Given a linear discrete system with initial state x_0 and output function y_i, we investigate a low dimensional linear system that produces, with a tolerance index ϵ, the same output function when the initial state belongs to a specified set, called ϵ-admissible set, that we characterize by a finite number of inequalities. We also give an algorithm which allows us to determine an ϵ-admissible set.

Key words: linear discrete systems, model order reduction

1. Introduction

The tendency to analyze and design systems of ever increasing complexity is becoming more and more a dominating factor in progress of chip design. Along with this tendency, the complexity of the mathematical models increases both in structure and dimension. Complex models are more difficult to analyze, and it is also harder to develop control algorithms. Therefore model order reduction (MOR) is of utmost importance [3, 8].

The problem of model reduction is to replace a given mathematical model of a system or process by a model that is much smaller than the original ones, yet still describes (at least approximately) certain aspects of the system or process (in control theory that is input-output behaviour of the system). If the approximation error is within a given tolerance, only the smaller system's model needs to be simulated, which will in general take much less time and computer memory than the original large-scale system would do. The reduced model might be used to replace the original system as a component in a large simulation, or it might be used to develop a low dimensional controller suitable for real-time applications [10].

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It must be stressed that achieving faster simulation and optimization times is not the only goal for applying the model reduction. Sometimes, it is most important to get the model with the lowest number of variables [13, 27].

Most methods of model reduction focus on linear systems, which, in many cases, provide accurate descriptions of the physical systems. Depending on the properties of the original system that are retained in the reduced model, there are different model reduction methodologies. Hence, there are techniques based on: singular perturbation analysis [16, 25], modal analysis [4, 6, 15, 18], singular value decomposition [13, 14, 19], moment matching [5, 20] and methods based on a combination of singular value decomposition and moment matching [1, 2, 11].

In this paper, we develop an original method for model order reduction problem, which takes into account the initial state.

Consider the class of discrete linear systems described by

$$\begin{cases} x_{i+1} = Ax_i + Bu_i, \\ x_0 \in \mathbb{R}^n \end{cases} \tag{1}$$

with the output

$$y_i = Cx_i,$$

where $x_i \in \mathbb{R}^n$, $u_i \in U \subset \mathbb{R}^p$, with U a given set of constraints, $y_i \in \mathbb{R}^q$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{q\times n}$. The model reduction problem we are interested in can be stated as follows: Given the matrices A, B and C, we investigate matrices $M \in \mathbb{R}^{m\times m}$, $P \in \mathbb{R}^{m\times n}$, and $L \in \mathbb{R}^{q\times m}$, where $m < n$, such that the output function $h_i = Lz_i$, $z_i \in \mathbb{R}^m$, of the low dimensional system

$$\begin{cases} z_{i+1} = Mz_i + PBu_i, \\ z_0 = Px_0 \end{cases} \tag{2}$$

satisfies, for some initial state x_0 and some tolerance index ϵ, the constraints

$$\|y_i - h_i\| \leqslant \epsilon, \ \ \forall i \geqslant 0, \ \ \forall u \in \mathcal{U}, \tag{3}$$

where $u \in \mathcal{U}$ means that $u_i \in U$, $\forall i \geqslant 0$ and $\|.\|$ denotes the ∞-norm, i.e., for $x = (x_i) \in \mathbb{R}^n$,

$$\|x\| = \max\{|x_i|, \ i = 1, \ldots, n\}.$$

The set of all x_0 which verify (3), called ϵ-admissible set, is denoted by $O_\infty^\epsilon(M, P, L)$, or simply O_∞, when the arguments are clear from context, i.e.,

$$O_\infty^\epsilon(M, P, L) = \{x_0 \in \mathbb{R}^n; \ \|y_i - h_i\| \leqslant \epsilon, \ \forall i \geqslant 0, \ \forall u \in \mathcal{U}\}.$$

Recursion and finite determination play a critical role in the characterization of the ϵ-admissible set [9, 17, 23, 24]. Indeed, let

$$O_i = \{x_0 \in \mathbb{R}^n; \ \|y_k - h_k\| \leqslant \epsilon, \ \forall k = 0, .., i, \ \forall u_k \in U, \ k = 0, .., i\}. \tag{4}$$

If there exists an $i \in \mathbb{N}$ such that $O_\infty = O_i$, we say that O_∞ is finitely determined. It will be shown that a necessary and sufficient condition for O_∞ to be finitely determined is that there exists an $i \in \mathbb{N}$ such that $O_{i+1} = O_i$. Thus if O_∞ is finitely determined it can be computed in a finite number of steps.

The set definition (4), and others which appear later on, can be expressed compactly in terms of a set operation called the P-difference. Suppose $U, V \subset \mathbb{R}^n$, then the P-difference of V from W is

$$V \sim W = \{z \in \mathbb{R}^n,\ \ z + w \in V,\ \forall w \in W\}.$$

The prefix P acknowledges Pontryagin who in the context of game theory [21], seems to have originated the difference. The difference also apears in the book by Demyanov and Rubinov [7].

The paper is organized as follows, section 2 contains the material on P-difference. Basic results are considered in section 3. The computation of O_∞ is treated in section 4. An example is given in section 5.

We conclude this section with notations. We mean by $\|x\|$ the ∞-norm of a vector x. The superscript T indicates matrix transpose. $A \in \mathbb{R}^{m\times n}$ means that A is a matrix of m rows and n column of real scalar. The interior, closure and convex hull of a set are denoted respectively by int, cl, co. The set V is symmetric if $V = -V$. The support function of V, evaluated at $\eta \in \mathbb{R}^n$, is $h_V(\eta) = \sup\limits_{v\in V} \eta^T v$.

2. P-difference

Basic properties of P-difference are summarized in the following theorem. See for example, [12, 22, 26]

Theorem 1 *Let $V, W \subset \mathbb{R}^n$ and assume that $V \sim W = \{z;\ \ z + W \subset V\} \neq \emptyset$. Then the following results hold.*
(i) $V \sim W = \bigcap\limits_{w\in W} (V - w)$.
(ii) $(V \sim W) + W \subset V$.
(iii) $0 \in W$ *implies* $V \sim W \subset V$.
(iv) suppose $W = W_1 + W_2$. *Then,* $V \sim W = (V \sim W_1) \sim W_2$.
(v) Suppose $V = V_1 \cap V_2$. *Then,* $V \sim W = (V_1 \sim W) \cap (V_2 \sim W)$.
(vi) For $\alpha \in \mathbb{R}$, $\alpha V \sim \alpha W = \alpha(V \sim W)$.
(vii) If V *is (bounded)[closed]{convex},* $(V \sim W)$ *is (bounded)[closed]{convex}.*
(viii) If V, W *are symmetric,* $(V \sim W)$ *is symmetric.*
(ix) If V, W *are symmetric and convex,* $0 \in V \sim W$.
(x) Suppose V *is convex. Then,* $(V \sim W) = V \sim coW$.

In certain cases it is possible to obtain a concrete characterization of $V \sim W$.

Theorem 2 *[17] Suppose V is a polyedron given by,*

$$V = \left\{ z \in \mathbb{R}^n;\ \ s_i^T z \leqslant r_i,\ i = 1,..,N \right\},$$

where $s_i \in \mathbb{R}^n$, $s_i \neq 0$, and $r_i \in \mathbb{R}$, $i = 1,..,N$. Assume $h_W(s_i)$ is defined for $i = 1,..,N$. Then,

$$V \sim W = \left\{ z \in \mathbb{R}^n;\ \ s_i^T z \leqslant r_i - h_W(s_i),\ \ i = 1,..,N \right\}.$$

3. Basic results

The output y_k of system (1) is given by

$$y_k = CA^k x_0 + \sum_{j=1}^{k} CA^{k-j} B u_{j-1}, \quad k \geqslant 1$$

and the output h_i of system (2) is given by

$$h_k = LM^k P x_0 + \sum_{j=1}^{k} LM^{k-j} P B u_{j-1}, \quad k \geqslant 1.$$

It follows that

$$y_k - h_k = (CA^k - LM^k P) x_0 + \sum_{j=1}^{k} (CA^{k-j} - LM^{k-j} P) B u_{j-1}.$$

Let's define the matrix H_i by

$$H_i = CA^i - LM^i P \tag{5}$$

then

$$y_k - h_k = H_k x_0 + \sum_{j=1}^{k} H_{k-j} B u_{j-1}. \tag{6}$$

On the other hand

$$H_i A = CA^{i+1} - LM^i PA \tag{7}$$

and from (5) and (7) we deduce that

$$\begin{aligned} H_{i+1} - H_i A &= LM^i PA - LM^{i+1} P \\ &= LM^i (PA - MP). \end{aligned}$$

Hence,

$$\begin{cases} H_{i+1} &= H_i A + LM^i (PA - MP), \\ H_0 &= C - LP. \end{cases} \tag{8}$$

If we choose the matrices P and M such that $PA = MP$, then equatiom (8) becomes

$$\begin{cases} H_{i+1} &= H_i A, \\ H_0 &= C - LP \end{cases} \tag{9}$$

which gives,

$$H_i = H_0 A^i = (C - LP)A^i, \quad i \geqslant 0.$$

It follows from (6) that

$$\begin{aligned} y_k - h_k &= H_0 A^k x_0 + \sum_{j=1}^{k} H_0 A^{k-j} B u_{j-1} \\ &= H_0 (A^k x_0 + \sum_{j=1}^{k} A^{k-j} B u_{j-1}) \\ &= H_0 x_k, \quad k \geqslant 1 \end{aligned}$$

which is also true for $k = 0$. Hence

$$y_k - h_k = H_0 x_k, \ \forall k \geqslant 0.$$

Let $M = (m_{ij})_{1 \leqslant i,j \leqslant m}$, $P = (p_{ij})$, $i = 1,\ldots,m$, $j = 1,\ldots,n$ and Define the matrice H_M by

$$H_M = \begin{bmatrix} A^T - m_{11} I_n & -m_{12} I_n & \ldots & -m_{1m} I_n \\ -m_{21} I_n & A^T - m_{22} I_n & \ldots & -m_{2m} I_n \\ \vdots & \vdots & \vdots & \vdots \\ -m_{m1} I_n & -m_{m2} I_n & \ldots & A^T - m_{mm} I_n \end{bmatrix},$$

where I_n is the identity matrix of order n, then we have the following result.

Proposition 1 *The equation $MP = PA$ has nonzero solution P if and only if* $\det(H_M) = 0$.

Proof. Denote by $\mathcal{P}$ the vector

$$\mathcal{P} = (p_{11},\ldots,p_{1n},p_{21},\ldots,p_{2n},\ldots,p_{m1},\ldots,p_{mn}).$$

Then $PA = MP$ is equivalent to $H_M \mathcal{P} = 0$. Hence there exists a matrix $P \neq 0$ such that $PA = MP$ if and only if $\det(H_M) = 0$.

Proposition 2 *If the matrix A^T has at least one nonzero real eigenvalue then there exists a matrix $M \in \mathbb{R}^{m\times m}$ such that* $\det(H_M) = 0$.

Proof. Define a nonzero diagonal matrix $M = diag(m_{ii}), i = 1,\ldots,m$, where m_{ii} is a real eigenvalue of A^T, then $\det(H_M) = \prod_{i=1}^{m} \det(A^T - m_{ii}I) = 0$.

Denote

$$\overline{C} = C - LP$$

and B_ϵ the closed ball of radius ϵ, i.e.,

$$B_\epsilon = \{y \in \mathbb{R}^q \,.\, \|y\| \leqslant \epsilon\}\,.$$

Then inequalities (3) are equivalente to

$$\overline{C}x_k \in B_\epsilon,\ \forall k \geqslant 0,\ \forall u_i \in U$$

which implies that

$$\begin{aligned} O_\infty &= \left\{x_0 \in \mathbb{R}^n,\ \overline{C}x_k \in B_\epsilon,\ \forall k \geqslant 0,\ \forall u_i \in U\right\}, \\ O_i &= \left\{x_0 \in \mathbb{R}^n,\ \overline{C}x_k \in B_\epsilon,\ \forall k = 0\ldots i,\ \forall u_j \in U\right\}. \end{aligned} \tag{10}$$

Define the set

$$\Gamma = \left\{\phi \in \mathbb{R}^n,\ \overline{C}\phi \in B_\epsilon\right\}.$$

Then, from (10) and the defintion of the P-substraction, it is easy to see that

$$O_i = \{x_0 \in \Gamma;\ Ax_0 \in O_i \sim BU\}\,. \tag{11}$$

Define, as in [17], the sequence of sets by

$$\begin{aligned} T_0 &= B_\epsilon, \\ T_i &= B_\epsilon \sim \overline{C}BU \sim \overline{C}ABU \sim \ldots \overline{C}A^{i-1}BU, \quad i \geqslant 1. \end{aligned} \tag{12}$$

Then O_i can be described by

$$O_i = \left\{x_0 \in \mathbb{R}^n,\ \overline{C}A^k x_0 \in T_k,\ k = 0\ldots i\right\} \tag{13}$$

and we have

$$\begin{aligned} T_{i+1} &= T_i \sim \overline{C}A^iBU, \\ T_0 &= B_\epsilon\,; \end{aligned} \tag{14}$$

$$\begin{aligned} O_{i+1} &= O_i \cap \left\{\phi \in \mathbb{R}^n,\ \overline{C}A^{i+1}\phi \in T_{i+1}\right\}, \\ O_0 = \Gamma &= \left\{\phi \in \mathbb{R}^n,\ \overline{C}\phi \in B_\epsilon\right\}. \end{aligned} \tag{15}$$

Remark 1 $O_\infty = \bigcap_{i \geqslant 0} O_i$.

We have the following result.

Proposition 3

i) The sets T_i are convex, symetric and compact.
ii) If U is symetric then T_i is symetric.

Proof.

i) The set B_ϵ is bounded, closed and convex, then by theorem (1) we deduce that T_1 is bounded, closed and convex, and by recurence we deduce that T_i is bounded, closed and convex for all i.

ii) suppose that U is symetric then, since B_ϵ is symetric it follows from theorem (1) that T_1 is symetric and by recurence we prove that T_i is symetric for all i.

Remark 2 *If U is symetric and $T_i \neq \emptyset$ for all i, then it follows from proposition (3) that T_i is convex and symetric and since T_i is not empty then $0 \in T_i$ for all i. This and (13) implies that $0 \in O_\infty$.*

4. Algorithmique determination of O_∞

Suppose that $O_i = \emptyset$ for some $i \in \mathbb{N}$, then $O_\infty = \bigcap_{i \geqslant 0} O_i = \emptyset$. Also, if there exists an i such that $T_i = \emptyset$, then it follows from (13) that $O_\infty = \emptyset$. Now, if there exists an i such that $O_{i+1} = O_i$, then it follows from (11) that $O_{i+2} = O_{i+1}$ and $O_\infty = O_i$. This observation is the basis for the following conceptual algorithm.

Algorithm

step 1: $i = 0$, if $O_0 = \Gamma$, then stop, set $O_\infty = \emptyset$ and $i^* = 0$

step 2: determine T_{i+1} by (14)
if $T_{i+1} = \emptyset$, then stop, set $O_\infty = \emptyset$, $i^* = i$.

step 3: determine O_{i+1} by (15)
if $O_{i+1} = \emptyset$, then stop, set $O_\infty = \emptyset$, $i^* = i$.

step 4: if $O_{i+1} = O_i$, then stop, set $O_\infty = O_i$, $i^* = i$.

step 5: replace i by i+1 and return to step 2.

To make algorithm practical we need to describe how the sets O_i and T_i can be calculated, and also how to test if $O_i = \emptyset$, $T_i = \emptyset$ and $O_{i+1} = O_i$.

Suppose that h_U can be evaluated, then relation (14) can be implemented as follows: The set B_ϵ can be described by

$$B_\epsilon = \{y \in \mathbb{R}^q,\ \ Sy \leqslant r\},$$

where $S = [s_1, \ldots, s_{2q}]^T \in \mathbb{R}^{2q\times q}$ is given by

$$\begin{cases} s_{2i-1} & = & e_i, \\ s_{2i} & = & -e_i, \end{cases} \qquad i = 1, \ldots, 2q$$

with (e_i) the canonical basis of $\mathbb{R}^q$, and $r = [\epsilon, \ldots, \epsilon]^T \in \mathbb{R}^{2q}$. Then T_i is given by

$$T_i = \{y \in \mathbb{R}^p,\ \ Sy \leqslant r_i\}, \tag{16}$$

where $r_i \in \mathbb{R}^{2q}$ is given recursively by

$$\begin{cases} r_0^j & = & \epsilon, \\ r_{i+1}^j & = & r_i^j - h_U\left((\overline{C}A^iB)^T s_j\right), \quad j = 1, \ldots, 2q \end{cases}$$

with r_i^j the j-th component of r_i.
We have $T_i \neq \emptyset$ if and only if $r_i^j \geqslant 0$, $\forall j = 1, \ldots, 2q$.

Recursion (15) allows us to construct the set O_i by

$$O_i = \{x_0 \in \mathbb{R}^n;\ R_i x_0 \leqslant g_i\},$$

where $n_i = 2(i+1)q$ and $R_i \in \mathbb{R}^{n_i\times n}$, $g_i \in \mathbb{R}^{n_i}$ are given by

$$\begin{cases} R_0 & = & S\overline{C}, \\ R_{i+1} & = & \begin{bmatrix} R_i \\ S\overline{C}A^{i+1} \end{bmatrix}, \end{cases} \qquad \begin{cases} g_0 & = & r = [\epsilon, \ldots, \epsilon]^T, \\ g_{i+1} & = & \begin{pmatrix} g_i \\ r_{i+1} \end{pmatrix}. \end{cases}$$

To avoid redundante inequalities in the definition of O_{i+1} we can proceed as follows: The process begins by checking the first, added scalar inequality, $s_1^T\overline{C}A^{i+1}$, for redundancy. For this, let $\bar{R}_{i+1}$ be the matrix obtained by removing the n_{i+1} row, $s_1^T\overline{C}A^{i+1}$, of R_{i+1}, and $\bar{g}_{i+1}$ the vector obtained by removing the n_{i+1} component of g_{i+1} and consider the linear programming

$$m_1 = \sup_{\bar{R}_{i+1}x_0 \leqslant \bar{g}_{i+1}} s_1^T\overline{C}A^{i+1}x_0$$

if $m_1 \leqslant r_{i+1}^1$, then the constraint $s_1^T\overline{C}A^{i+1}x_0 \leqslant r_{i+1}^1$ is redundante and R_{i+1} is updated to $\bar{R}_{i+1}$ and g_{i+1} is updated to $\bar{g}_{i+1}$, else, R_{i+1} and g_{i+1} are kept without change. We proceed similarly with the next constraint, $s_2^T \overline{C} A^{i+1} x_0 \leqslant r_{i+1}^2$, and so until the last constraint $s_{2q}^T \overline{C} A^{i+1} x_0 \leqslant r_{i+1}^{(2q)}$. The test of redundance alows us to test also if $O_{i+1} = O_i$, indeed if all the constraints $s_j^T \overline{C} A^{i+1} x_0 \leqslant r_{i+1}^{(j)}$ are eliminated, then $R_{i+1} = R_i$ and $O_{i+1} = O_i$, else, $O_{i+1} \neq O_i$.

Here after we will need the following known result.
Consider the set F_i of all possible states, of system (1) which can occur at time i, starting from $x_0 = 0$

$$F_i = \left\{ x_i = \sum_{j=0}^{i-1} A^{i-j-1} B u_j,\ u_j \in U \right\}, \quad i \geqslant 1,$$
$$F_0 = \{0\}$$

with some added conditions, the sequence of sets (F_i) has a limit.

Theorem 3 *Assume U is bounded and A is asymptotically stable. Then there exists a compact set, $F \subset \mathbb{R}^n$, with the following properties*

i) $F_i \subset F$, $\forall i \geqslant 0$;

ii) For every $\epsilon > 0$ there exist i $\geqslant 0$ such that $F \subset F_i + \epsilon\mathcal{B}$, where $\mathcal{B}$ is the closed unit ball of $\mathbb{R}^n$.

Now we can prove the following result.

Theorem 4 *If the pair $(\overline{C}, A)$ is observable, A is asymptotically stable and U is bounded, then for every $\epsilon > \theta\|\overline{C}\|$ we have $O_\infty^\epsilon \neq \emptyset$ and is finitely determined, where θ is the smallest real such that $F \subset B_\theta$ and $\|\overline{C}\|$ is the norm of $\overline{C}$ induced by the ∞-norm.*

Proof. Since U is bounded then clearly the sets F_i are also bounded and from ii) of theorem (3) we deduce that F is bounded. Let $\theta > 0$ be the smallest real such that $F \subset B_\theta$. It follows from (12) that

$$T_\epsilon = \left\{ x_0 \in \mathbb{R}^n;\ x_0 + \sum_{j=0}^{i-1} \overline{C} A^j B u_j \in B_\epsilon,\ \forall u_j \in U \right\}$$

suppose $x_0 \in B_\gamma$, then $x_0 + \sum_{j=0}^{i-1} \overline{C} A^j B u_j \in B_{\gamma + \|\overline{C}\|\theta}$, $\forall u_j \in U$, $\forall i \geqslant 1$. This implies that

$$B_\gamma \subset T_i^{\gamma + \|\overline{C}\|\theta}, \quad \forall i \geqslant 1. \tag{17}$$

Define the matrix $\Phi = \left(\overline{C}^{\mathrm{T}}, (\overline{C}A)^T, \ldots, (\overline{C}A^{n-1})^T\right)^T \in \mathbb{R}^{nq\times n}$. The observability of $(\overline{C}, A)$ implies that $rank\,\Phi = n$, hence the matrix $(\Phi\Phi^T)^{-1}$ is well defined. Let $\epsilon > \theta$ be fixed. Denote $\epsilon = \gamma + \|\overline{C}\|\theta$, with $\gamma > 0$. Then it follows from (13) that

$$O^{\epsilon}_{n-1} = \left\{x_0 \in \mathbb{R}^n;\ \Phi^T x_0 \in T^{\epsilon}_0 \times \ldots \times T^{\epsilon}_{n-1}\right\}.$$

Consequently, $x_0 \in O^{\epsilon}_{n-1} \Rightarrow \Phi^T x_0 \in T^{\epsilon}_0 \times \ldots \times T^{\epsilon}_{n-1} \Rightarrow \Phi\Phi^T x_0 \in \Phi(T^{\epsilon}_0 \times \ldots \times T^{\epsilon}_{n-1}) \Rightarrow O^{\epsilon}_{n-1} \subset (\Phi\Phi^T)^{-1}\Phi(T^{\epsilon}_0 \times \ldots \times T^{\epsilon}_{n-1})$. Since the sets T^{ϵ}_i are bounded, then O^{ϵ}_{n-1} is bounded. The asymptotic stability of A implies that $\overline{C}A^k \to 0$. Since O^{ϵ}_{n-1} is bounded then we have $\overline{C}A^{k+1}O^{\epsilon}_{n-1} \subset B_\gamma$ for k sufficiently large. By (17) we deduce that $\overline{C}A^{k+1}O^{\epsilon}_{n-1} \subset T^{\epsilon}_{k+1}$. If we choose $k \geqslant n$ sufficiently large, then $O^{\epsilon}_k \subset O^{\epsilon}_{n-1} \Rightarrow \overline{C}A^{k+1}O^{\epsilon}_k \subset \overline{C}A^{k+1}O^{\epsilon}_{n-1} \subset T^{\epsilon}_{k+1}$. This and (13) implies that $O^{\epsilon}_k \subset O^{\epsilon}_{k+1}$ and consequently $O^{\epsilon}_k = O^{\epsilon}_{k+1}$. This proves that O^{ϵ}_{∞} is finitely determined for all $\epsilon > \theta$. Finally, it follows from (17) that $0 \in T^{\epsilon}_i$, $\forall i \geqslant 1$, $\forall \epsilon > 0$, and from (13) we deduce that $0 \in O^{\epsilon}_{\infty}$, $\forall \epsilon > 0$.

An other way to determine $O_\infty(C - LP, A)$ is to choose $(C - LP, A)$ unobservable. In this case, there exists an integer t and a system coordinate such that the matrices A and $(C - LP)$ have the form

$$\widetilde{A} = \begin{pmatrix} A_1 & 0 \\ A_3 & A_2 \end{pmatrix}, \qquad \widetilde{C} = \left(\widetilde{C}_1,\ 0\right), \tag{18}$$

where $\widetilde{A} = Q^{-1}AQ$ and $\widetilde{C} = (C - LP)Q$, with $A_1 \in \mathbb{R}^{t\times t}$ and the pair $\left(\widetilde{C}_1, A_1\right)$ observable. In this case,

$$y_i - h_i = (C - LP)x_i = \widetilde{C}\widetilde{x}_i = \widetilde{C_1}\widetilde{x}_{1i},$$

where $\widetilde{x}_i = Q^{-1}x_i = \begin{pmatrix} \widetilde{x}_{1i} \\ \widetilde{x}_{2i} \end{pmatrix}$, with $\widetilde{x}_{1i} \in \mathbb{R}^t$ and

$$\begin{cases} \widetilde{x}_{i+1} &= \widetilde{A}\widetilde{x}_i + \widetilde{B}u_i \\ \widetilde{x}_0 &= Q^{-1}x_0 \end{cases} \tag{19}$$

with $\widetilde{B} = Q^{-1}B$. From (19) and (18) we deduce that $\widetilde{x}_{1(i+1)} = A_1\widetilde{x}_{1i} + \widetilde{B_1}u_i$, where $\widetilde{B} = \begin{pmatrix} \widetilde{B}_1 \\ \widetilde{B}_2 \end{pmatrix}$. Hence,

$$\begin{aligned} x_0 \in O_\infty(C - LP, A, B) &\Leftrightarrow \widetilde{x}_{01} \in O_\infty(\widetilde{C}_1, A_1, \widetilde{B}_1) \\ &\Leftrightarrow \widetilde{x}_0 \in O_\infty(\widetilde{C}_1, A_1, \widetilde{B}_1) \times \mathbb{R}^{n-t} \end{aligned}$$

which implies that $O_\infty(C - LP, A, B) = Q(O_\infty(\widetilde{C}_1, A_1, \widetilde{B}_1) \times \mathbb{R}^{n-t})$. Since the pair $(\widetilde{C}_1, A_1)$ is observable, results of the previous section can be applied to determine $O_\infty(\widetilde{C}_1, A_1, \widetilde{B}_1)$.

5. Example

We take

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \qquad C = (C_1,\ C_2), \qquad P = (p_1,\ p_2),$$

$$L = l \in \mathbb{R}, \quad M = m_{11} \in \mathbb{R}, \quad \text{and} \quad U = [-\alpha,\ \alpha].$$

In this case: $n = 2$, $m = 1$, $p = 1$, $q = 1$ and

$$H_M = \begin{pmatrix} a_{11} - m_{11} & a_{12} \\ a_{21} & a_{22} - m_{11} \end{pmatrix}.$$

We have $\det(H_M) = m_{11}^2 - (a_{11} + a_{22})m_{11} + a_{11}a_{22} - a_{12}a_{21}$.

Let $\Delta = (a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})$. If $\Delta < 0$ then $\det(H_M) \neq 0$, $\forall m_{11} \in \mathbb{R}$ and the equation $PA = MP$ has only $P = 0$ as solution. Suppose that $\Delta \geqslant 0$ and choose M such that $\det(H_M) = 0$, i.e., $m_{11} = (a_{11} + a_{22} \pm \sqrt{\Delta})/2$. In this case the equation $PA = MP$ is equivalent to $(a_{11} - m_{11})p_1 + a_{21}p_2 = 0$, or equivalently

$$\begin{cases} p_2 = \dfrac{(m_{11} - a_{11})p_1}{a_{21}} & \text{if} \quad a_{21} \neq 0, \\ p_1 = 0,\ p_2 \in \mathbb{R} & \text{if} \quad m_{11} \neq a_{11}, \quad a_{21} = 0, \\ P \in \mathbb{R}^2 & \text{if} \quad a_{21} = 0, \quad m_{11} = a_{11}\,. \end{cases}$$

We choose L and P such that $PA = MP$, $(C - LP, A)$ observable and $\|C - LP\|_\infty$ as small as possible.

Remark 3 *From the expression of $C - LP$ (we suppose that $a_{21} \neq 0$)*

$$C - LP = \left(c_1 - lp_1,\ c_2 - \frac{m_{11} - a_{11}}{a_{21}} lp_1 \right)$$

we see that if $c_1 = \dfrac{a_{21}c_2}{m_{11} - a_{11}}$ where $m_{11} = (a_{11} + a_{22} \pm \sqrt{\Delta})/2$ then there exists L and P such that $C - LP = 0$. In this case we have $y_i = h_i$, $\forall i \geqslant 0$, $\forall u \in \mathcal{U}$.

Numerical simulation

Let $A = \begin{pmatrix} 4/9 & -1/18 \\ -1/9 & 7/18 \end{pmatrix}$ be an asymptotically stable matrix, $B = \begin{pmatrix} -0.1 \\ 0.1 \end{pmatrix}$, $C = (-20,\ 10)$, $\alpha = 0.5$. In this case we have $c_1 = \dfrac{a_{21}c_2}{m_{11} - a_{11}}$ with

$m_{11} = (a_{11} + a_{22} + \sqrt{\Delta})/2$. Then it follows from remark 3 that we can find L and P such that $C - LP = 0$, the reduced system produces the same output for every $x_0 \in \mathbb{R}^2$ and every $u \in \mathcal{U}$. Let $C = (-19.999,\ 10)$ then application of the algorithm described above with $m_{11} = (a_{11} + a_{22} + \sqrt{\Delta})/2$, $L = 1$, $p_1 = c_1 + 0.0001$, $p_2 = \dfrac{(m_{11} - a_{11})p_1}{a_{21}}$ shows that for $\epsilon = 0.0001$ we have $O^{\epsilon}_{\infty} = O^{\epsilon}_{2}$. The graphical representation of O^{ϵ}_{2} is given by Fig. 1.

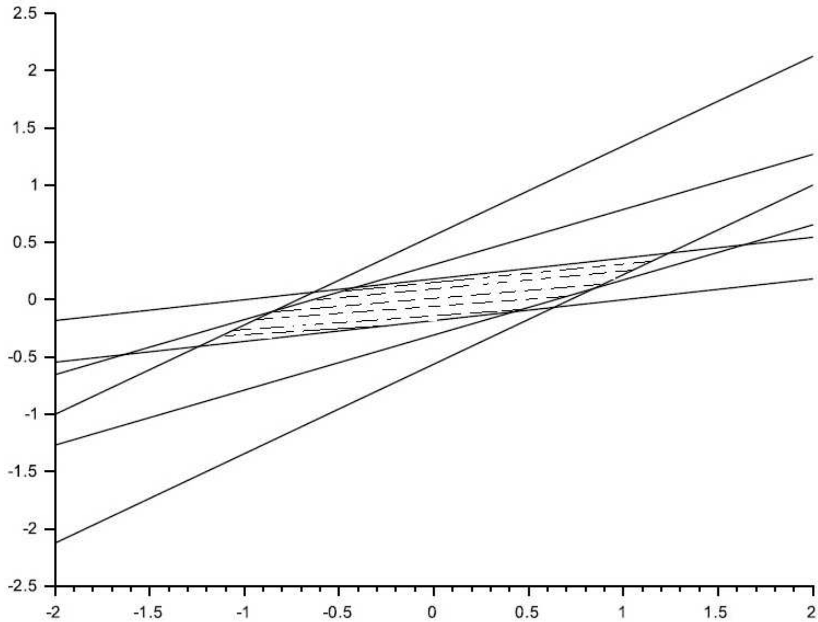

Figure 1: The ϵ-admissible set

6. Conclusion

To resolve the model order reduction problem, we have developed an original method which takes into account the initial state. Indeed, we have investigated a low dimensional system that produces, with a tolerance index ϵ, the same output than the original one when the initial state belongs to a set called ϵ-admissible set. Results of existence and steps of determining the low dimensional system parameters are described. We have characterized the ϵ-admissible set by a finite number of inequalities. We have given an algorithm for determining O^{ϵ}_{∞}. This algorithm is practical since it uses only linear programming problems. Result of convergence is also given.

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