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Design of transmultiplexer integer filters

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The designing of transmultiplexer systems relies on determining filters for the transmitter and receiver sides of multicarrier communication system. The perfect reconstruction conditions lead to the bilinear equations for FIR filter coefficients. Generally there is no way of finding all possible solutions. This paper describes methods of finding a large family of solutions. Particular attention is devoted to obtaining algorithms useful in fixed-point arithmetic needed to design the integer filters. As a result, the systems perform perfect reconstruction of signals. Additionally, a simple method is presented to transform any transmultiplexer into an unlimited number of different transmultiplexers. Finally, two examples of integer filters that meet perfect reconstruction conditions are shown. The first illustrates a FIR filter which does not require multiplications. The frequency properties of filters and signals are discussed for the second example.

Key words: fixed point arithmetic, MIMO systems, integer digital filters

1. Introduction

One of the main issues involved in telecommunications is to efficiently divide scarce bandwidth among a large number of users. Conventional multi-input and multi-output transmultiplexers combine several signals into a single one to meet this requirement. The Code-Division Multiple Access (CDMA) and the Time-Division Multiple Access (TDMA) telecommunication systems are special cases of general transmultiplexers [1]. Fig. 1 shows the scheme of a four-channel criticall sampled transmultiplexer.

Designing transmultiplexer filters is not easy, even though the necessary and sufficient conditions given in the z-domain are well known [2, 3]. Moreover, it is shown in [2] and [3] that interchanging the analysis and synthesis filters in the M-band filter bank results in a transmultiplexer system. This means that it is possible to use similar filter design methods for both systems. Recent pa-

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pers [4–7] describe the design of transmultiplexers which provide nearly perfect reconstruction.

Both, the transmitter and receiver sides are usually assumed to be equipped with FIR filters. Rounding errors are unavoidable in transmultiplexer systems, when the coefficients of filters are real numbers. If the design method generates integer coefficients [8], then all calculations are provided without divisions to omit the rounding errors. Theorems presented in this paper can be used to design such systems. Some ideas have been already published by authors in conference materials in [9–11]. The proposed algorithms are improved and a wider scope of applications is available.

The algorithms for digital filter design shown below result from the analysis of the mathematical model of transmultiplexers in the z-variable domain. The correctness of mathematical analysis requires the convergence of z-transforms for modeled digital signals. In practice, the signals are a finite sequence of samples $s(\theta)$, where $\theta = 1, \ldots, T$. Then the necessary and sufficient condition for the convergence of the transforms are the finite values of the signal samples, i.e. $s(\theta) < \infty$ for all θ. The situation is more complicated when signals of infinite duration are the subject of transformation. In such cases the convergence conditions [12] need to be fulfilled. Moreover, the convergence area is frequently limited to a subset of complex numbers.

2. Transmultiplexers

On the transmitter side (see Fig. 1), input signals are upsampled, filtered and summed to obtain a composite signal, which is then sent through a single transmission channel to all recipients. On the receiver side, the composite signal is split into channels, filtered and downsampled to restore the original input signals. A transmultiplexer achieves the perfect reconstruction if s_n^{out} are delayed only versions of s_n^{in}, where $n \in \{1, 2, \ldots, M\}$ is the signal consecutive number.

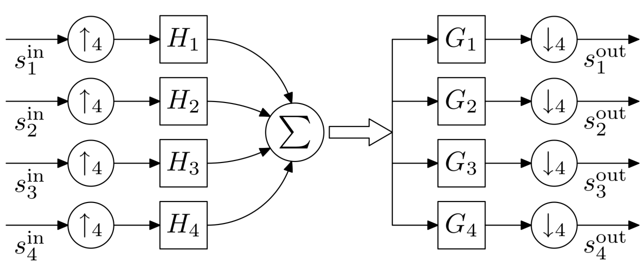

Figure 1: Transmultiplexer system

The relationship between outputs and inputs is described in [2,3] by the equation

$$\bar{s}_n^{\text{out}}(z) = \frac{1}{M}\sum_{m=1}^{M}\bar{s}_m^{\text{in}}(z)\sum_{i=0}^{M-1}H_m(w_M^i z^{1/M})G_n(w_M^i z^{1/M}), \tag{1}$$

where $w_M = \exp(-2\pi j/M)$, $\bar{s}_n(z)$ stands for the z-spectrum of signal s_n, $H_m(z)$ are composition and $G_n(z)$ are separation transfer functions. To fulfill the perfect reconstruction conditions for all signals, a set of M^2 equations

$$\sum_{i=0}^{M-1}H_m(w_M^i z^{1/M})G_n(w_M^i z^{1/M}) = z^{-\tau}M\delta_{m,n} \tag{2}$$

must be satisfied, where τ is a delay and $\delta_{m,n}$ is the Kronecker function.

Designing a transmultiplexer system relies on determining filters H_m and G_n which meet conditions (2). Finding non-trivial solutions is difficult since conditions (2) lead to non-linear equations with respect to filter coefficients. Interesting approximations were suggested in [13–26] to solve this or similar problems. Frequently, numerical procedures are used to determine filter coefficients by minimizing residuals of equations obtained from the perfect reconstruction conditions. However, the suggested algorithms give near-perfect signal reconstructions only.

It is difficult to examine the existence and uniqueness of solutions to equations obtained from conditions (2). It is known that for sufficiently large filter orders, there are many solutions. Shorter filters offer considerable advantages in computation and hardware implementation. The methods presented below make it possible to find the filter banks which strictly fulfill the perfect reconstruction conditions (2).

Under the assumption that all filters are of the FIR-type

$$H_m(z) = \sum_{i=0}^{I}h_m(i)z^{-i} \tag{3}$$

and

$$G_n(z) = \sum_{i=0}^{I}g_n(i)z^{-i}, \tag{4}$$

conditions (2) are equivalent to

$$\sum_{k=0}^{\lfloor 2I/M\rfloor} z^{-k}\sum_{l=\max\{0,Mk-I\}}^{\min\{Mk,I\}} h_m(Mk-l)\,g_n(l) = z^{-\tau}\delta_{m,n}\,, \tag{5}$$

where $\lfloor 2I/M\rfloor$ means an integer part of $2I/M$, $h_m(.)$ and $g_n(.)$ are impulse responses of filters H_m and G_n, respectively.

Equations (5) are satisfied for all $z \in C$ if and only if

$$h_m^T A_k g_n = \delta_{m,n}\, \delta_{\tau,k}\,, \tag{6}$$

where $k = 0,\ 1, \ldots, \lfloor 2I/M \rfloor$, $h_m = [h_m(0)\ \cdots\ h_m(I)]^T$, $g_n = [g_n(0)\ \cdots\ g_n(I)]^T$, $A_k = [a_k(p,q)] \in \Re^{(I+1)\times(I+1)}$ and p, q are the row and column indexes of matrices A_k. Elements of these matrices are given by the formula

$$a_k(p,q) = \begin{cases} 1 & \text{if} \quad p+q = Mk+2, \\ 0 & \text{if} \quad p+q \neq Mk+2. \end{cases}$$

For each pair of filters h_m and g_n where m, $n \in \{1,2,\ldots,M\}$, the matrix equation (6) gives $\lfloor 2I/M \rfloor + 1$ bilinear equations, since $k = 0,\ 1,\ \ldots,\ \lfloor 2I/M \rfloor$. This results in a system of $M^2(\lfloor 2I/M \rfloor + 1)$ algebraic scalar equations.

To show the equations more clearly, let us consider the case $M = 4$ and $I = 8$. Then, all equations (6) can be joined together and written as single matrix equation $H \cdot G = b_\tau$ where

$$H = \begin{bmatrix}
h_1(0) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h_1(4) & h_1(3) & h_1(2) & h_1(1) & h_1(0) & 0 & 0 & 0 & 0 \\
h_1(8) & h_1(7) & h_1(6) & h_1(5) & h_1(4) & h_1(3) & h_1(2) & h_1(1) & h_1(0) \\
0 & 0 & 0 & 0 & h_1(8) & h_1(7) & h_1(6) & h_1(5) & h_1(4) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_1(8) \\
h_2(0) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h_2(4) & h_2(3) & h_2(2) & h_2(1) & h_2(0) & 0 & 0 & 0 & 0 \\
h_2(8) & h_2(7) & h_2(6) & h_2(5) & h_2(4) & h_2(3) & h_2(2) & h_2(1) & h_2(0) \\
0 & 0 & 0 & 0 & h_1(8) & h_1(7) & h_2(6) & h_2(5) & h_2(4) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_2(8) \\
h_3(0) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h_3(4) & h_3(3) & h_3(2) & h_3(1) & h_3(0) & 0 & 0 & 0 & 0 \\
h_3(8) & h_3(7) & h_3(6) & h_3(5) & h_3(4) & h_3(3) & h_3(2) & h_3(1) & h_3(0) \\
0 & 0 & 0 & 0 & h_3(8) & h_3(7) & h_3(6) & h_3(5) & h_3(4) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_3(8) \\
h_4(0) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h_4(4) & h_4(3) & h_4(2) & h_4(1) & h_4(0) & 0 & 0 & 0 & 0 \\
h_4(8) & h_4(7) & h_4(6) & h_4(5) & h_4(4) & h_4(3) & h_4(2) & h_4(1) & h_4(0) \\
0 & 0 & 0 & 0 & h_4(8) & h_4(7) & h_4(6) & h_4(5) & h_4(4) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_4(8)
\end{bmatrix},$$

$$G = \begin{bmatrix} g_1(0) & g_2(0) & g_3(0) & g_4(0) \\ g_1(1) & g_2(1) & g_3(1) & g_4(1) \\ g_1(2) & g_2(2) & g_3(2) & g_4(2) \\ g_1(3) & g_2(3) & g_3(3) & g_4(3) \\ g_1(4) & g_2(4) & g_3(4) & g_4(4) \\ g_1(5) & g_2(5) & g_3(5) & g_4(5) \\ g_1(6) & g_2(6) & g_3(6) & g_4(6) \\ g_1(7) & g_2(7) & g_3(7) & g_4(7) \\ g_1(8) & g_2(8) & g_3(8) & g_4(8) \end{bmatrix},$$

$b_\tau = [b_\tau(p,q)] \in \Re^{20\times 4}$ and their elements

$$b_\tau(p,q) = \begin{cases} 4 & \text{if} \quad p = \tau + 1 + 5(q-1) \\ 0 & \text{if} \quad p \neq \tau + 1 + 5(q-1). \end{cases}$$

3. Solutions of Bilinear Equations

It is easy to verify that for $\tau = 0$ or $\tau \geqslant \lfloor 2I/M \rfloor$ the set of conditions (6) is contradictory. Therefore, admissible delays must satisfy the inequalities

$$0 < \tau < \left\lfloor \frac{2I}{M} \right\rfloor. \tag{7}$$

This means that a certain delay is unavoidable and the maximal admissible delay is the increasing function of the filters range I and the decreasing function of the number of channels M. Here, we shall consider the case where $I = N \cdot M$ for arbitrarily integer value $N > 0$. For this case, we obtain $M^2(2N+1)$ equations (6) and $k = 0, \ldots, 2N$. For case $k = 0$, we obtain

$$h_m^T A_0 \, g_n = 0 \;\Leftrightarrow\; h_m(0)\, g_n(0) = 0, \tag{8}$$

while for case $k = 2N$, we have conditions

$$h_m^T A_{2N} \, g_n = 0 \;\Leftrightarrow\; h_m(N \cdot M)\, g_n(N \cdot M) = 0. \tag{9}$$

To satisfy the above conditions, we can take

$$g_1(0) = g_2(0) = \ldots = g_M(0) = 0 \tag{10}$$

and

$$h_1(I) = h_2(I) = \ldots = h_M(I) = 0, \tag{11}$$

which does not limit our general considerations. The opposite case is a symmetric solution, since the role of filters H_m and G_n in (2) can be swapped; this means filters H_m can be composition or separation filters and then G_n plays the opposite role.

Let us introduce notation

$$H_p^c = \begin{bmatrix} h_1(pM-1) & \cdots & h_1(pM-M) \\ h_2(pM-1) & \cdots & h_2(pM-M) \\ \vdots & \ddots & \vdots \\ h_M(pM-1) & \cdots & h_M(pM-M) \end{bmatrix}, \tag{12}$$

$$G_p^s = \begin{bmatrix} g_1(pM-M+1) & \cdots & g_M(pM-M+1) \\ g_1(pM-M+2) & \cdots & g_M(pM-M+2) \\ \vdots & \ddots & \vdots \\ g_1(pM) & \cdots & g_M(pM) \end{bmatrix}, \tag{13}$$

where $p = 1, \ldots, N$. Taking into account (10)–(13), the set of equations (6) takes the form

$$\begin{cases} H_1^c G_1^s = \delta_{\tau,1} E \\ H_2^c G_1^s + H_1^c G_2^s = \delta_{\tau,2} E \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ H_N^c G_1^s + H_{N-1}^c G_2^s + \ldots + H_1^c G_N^s = \delta_{\tau,N} E \\ H_N^c G_2^s + H_{N-1}^c G_3^s + \ldots + H_2^c G_N^s = \delta_{\tau,N+1} E \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ H_N^c G_{N-1}^s + H_{N-1}^c G_N^s = \delta_{\tau,2N-2} E \\ H_N^c G_N^s = \delta_{\tau,2N-1} E, \end{cases} \tag{14}$$

where E is the identity matrix. Equations (14) can be written in short as

$$\sum_{\substack{m,n=1,\ldots,N \\ m+n=k+1}} H_m^c G_n^s = \delta_{\tau,k} E \tag{15}$$

for cases $k = 1, \ldots, 2N-1$. This is the equivalent form to the conditions known in the $z-$domain [2, 3].

To design filters, we need to find matrices $H_n^c, G_n^s \in \Re^{M \times M}$ (where $n = 1, \ldots, N$) which are the solution of the set of equations (15). Some simple solutions are well known; for example, each multiplexer is the simplest transmultiplexer system, where $N = 1$ and $H_1^c = G_1^s = E$.

Theorem 1 *Let* $\tau \in \{1,\ldots,N-1\}$ *for arbitrary* $N \geqslant 3$. *Assuming* $\det H_1^c \neq 0$ *and*

$$\left((H_1^c)^{-1}H_2^c\right)^N = 0 \tag{16}$$

the matrices that meet conditions

$$\begin{cases} G_p^s = 0 \quad \mathit{for} \quad 1 \leqslant p \leqslant N \quad \mathit{and} \quad \tau+1 \neq p \neq \tau \\ G_\tau^s = (H_1^c)^{-1} \\ G_{\tau+1}^s = -(H_1^c)^{-1}H_2^c(H_1^c)^{-1} \end{cases} \tag{17}$$

and

$$H_k^c = H_2^c\left((H_1^c)^{-1}H_2^c\right)^{k-2} = H_1^c\left((H_1^c)^{-1}H_2^c\right)^{k-1} \tag{18}$$

for $k = 3,\ldots,N$ *are solutions of* (15). *The last condition can also be written as* $H_k^c = H_2^c(H_1^c)^{-1}H_{k-1}^c$.

Proof. Assuming $G_p^s = 0$ for $1 \leqslant p \leqslant N$ and $\tau + 1 \neq p \neq \tau$, set (14) obtains form

$$\begin{cases} H_1^c G_\tau^s = E \\ H_2^c G_\tau^s + H_1^c G_{\tau+1}^s = 0 \\ H_3^c G_\tau^s + H_2^c G_{\tau+1}^s = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ H_N^c G_\tau^s + H_{N-1}^c G_{\tau+1}^s = 0 \\ H_N^c G_{\tau+1}^s = 0 \end{cases} \tag{19}$$

which leads to equations (17) and the last equation is equivalent to (16). $\square$

The assumptions of Theorem 1 give the set of equations (17) and (18) which are much simpler to solve than (14). However, it can be difficult to find matrices H_1^c and H_2^c, which fulfill (16) for large values of M.

Now, let us consider the special case of the solutions given by Theorem 1. Let us denote by $X_k \in Z^{M\times M}$ the matrix whose elements satisfy the relationship

$$X_k(p,q) = \delta_{q-p,k}\,. \tag{20}$$

From this definition, it follows that $X_k \neq 0$ if and only if $k = 0,\ldots,M-1$. It is easy to verify that $X_1^k = X_k$ and $X_1^M = X_M = 0$, where the upper indexes mean the exponents.

Let us assign by $P \in \Re^{M\times M}$ a certain nonsingular matrix. Then we have $(PX_1P^{-1})^k = PX_kP^{-1}$ and $(PX_1P^{-1})^k = 0$ for all $k \geqslant M$.

Theorem 2 *Let $\tau \in \{1, \ldots, N-1\}$ and $N \geqslant M+1$. Assuming $\det H_1^c \neq 0$, the matrices that meet conditions*

$$\begin{cases} G_p^s = 0 \quad \text{for} \quad 1 \leqslant p \leqslant N \quad \text{and} \quad \tau+1 \neq p \neq \tau \\ G_\tau^s = (H_1^c)^{-1} \\ G_{\tau+1}^s = -PX_1P^{-1}(H_1^c)^{-1} \end{cases} \tag{21}$$

and

$$H_k^c = H_1^c P X_{k-1} P^{-1} \tag{22}$$

for $2 \leqslant k \leqslant M$ are solutions of set (15).

Proof. Let us take $H_2^c = H_1^c P X_1 P^{-1}$ and let us use Theorem 1. We have

$$G_{\tau+1}^s = -(H_1^c)^{-1} H_2^c (H_1^c)^{-1} = -PX_1P^{-1}(H_1^c)^{-1} \tag{23}$$

and

$$H_k^c = H_1^c \left((H_1^c)^{-1} H_2^c\right)^{k-1} = H_1^c \left((H_1^c)^{-1} H_1^c P X_1 P^{-1}\right)^{k-1} = H_1^c P X_{k-1} P^{-1}. \tag{24}$$

It remains to be shown that condition (16) is satisfied. We obtain

$$\left((H_1^c)^{-1} H_2^c\right)^N = \left((H_1^c)^{-1} H_1^c P X_1 P^{-1}\right)^N = P X_N P^{-1} = 0, \tag{25}$$

because $N \geqslant M+1$. We see that the assumptions given in Theorem 2 meet the conditions mentioned in Theorem 1. Hence, they are solutions of (14). □

Theorem 2 suggests a simple method of filter designing. Some coefficients of composition filters (i.e. H_1^c) must be assumed. The relationships given in Theorem 2 make it possible to compute separation filters and the remaining parts of composition filters. It also is possible to proceed in the opposite direction; some parts of the separation filters can be assumed and the parameters of the composition filters can be computed from the relationships given in Theorem 2. Both these cases give open-loop computation schemes, which can be immediately applied to design transmultiplexer filters. To obtain the integer values, it is sufficient to take integer elements in matrices P and H_1^c and meet conditions $|\det P| = |\det H_1^c| = 1$. Theorem 1 gives more complicated conditions, which need closed-loop computations.

Theorem 3 *Let D_p be arbitrary matrices with suitable dimensions for $p = 1, 2, \ldots, N$. Matrices*

$$H_p^c = \begin{bmatrix} E\delta_{p,1} & 0 \\ D_p & E\delta_{p,\tau} \end{bmatrix} \tag{26}$$

and

$$G_p^s = \begin{bmatrix} E\delta_{p,\tau} & 0 \\ -D_p & E\delta_{p,1} \end{bmatrix} \tag{27}$$

satisfy the set of equations (15).

Proof.

$$\sum_{\substack{m,n=1,\ldots,N\\ m+n=k+1}} H_m^c G_n^s = \sum_{\substack{m,n=1,\ldots,N\\ m+n=k+1}} \begin{bmatrix} E\delta_{m,1} & 0 \\ D_m & E\delta_{m,\tau} \end{bmatrix} \begin{bmatrix} E\delta_{n,\tau} & 0 \\ -D_n & E\delta_{n,1} \end{bmatrix}$$

$$= \sum_{\substack{m,n=1,\ldots,N\\ m+n=k+1}} \begin{bmatrix} E\delta_{m,1}\delta_{n,\tau} & 0 \\ D_m\delta_{n,\tau} - \delta_{m,\tau}D_n & E\delta_{m,\tau}\delta_{n,1} \end{bmatrix}$$

$$= \begin{bmatrix} E\delta_{\tau,k} & 0 \\ D_{k+1-\tau} - D_{k+1-\tau} & E\delta_{\tau,k} \end{bmatrix} = E\delta_{\tau,k}. \tag{28}$$

Theorem 3 gives an algorithm which makes it possible to design a wide class of transmultiplexers. The integer filters are obtained when D_p consists of integer elements. The next theorem shows that each pair of non-singular matrices transforms an optional transmultiplexer system into another transmultiplexer system.

Theorem 4 *Let $P, Q \in \Re^{M\times M}$ be arbitrary non-singular matrices. If matrices $H_1^c, \ldots, H_N^c$, $G_1^s, \ldots, G_N^s \in \Re^{M\times M}$ satisfy the set of conditions* (15)*, then matrices $\widehat{H}_p^c = PH_p^cQ$ and $\widehat{G}_p^s = Q^{-1}G_p^sP^{-1}$ for $p \in \{1,\ldots,N\}$ satisfy the set of conditions* (15) *as well.*

Proof. For a fixed $k \in \{1,\ldots,2N-1\}$, we verify that the adequate equation taken from set (15) is satisfied

$$\sum_{\substack{m,n=1,\ldots,N\\ m+n=k+1}} \widehat{H}_m^c \widehat{G}_n^s = \sum_{\substack{m,n=1,\ldots,N\\ m+n=k+1}} PH_m^cQQ^{-1}G_n^sP^{-1}$$

$$= P\left(\sum_{\substack{m,n=1,\ldots,N\\ m+n=k+1}} H_m^c G_n^s\right)P^{-1} = P\delta_{\tau,k}EP^{-1} = \delta_{\tau,k}E. \tag{29}$$

Theorem 4 makes it possible to create an unlimited number of transmultiplexers from each transmultiplexer. To preserve the integer filters, it is sufficient to take integer elements in P and Q and fulfill conditions $|\det Q| = |\det P| = 1$.

4. Simulation examples

The algorithms described above enable us to design filters H_m and G_n in such a way that transmultiplexers achieve perfect reconstruction, i.e. each signal s_i^{out} is a delayed and at most amplified version of s_i^{in}. Examples of such integer filters are presented below.

4.1. Example 1

As a simple example, let us consider a four-channel transmultiplexer which has composition filters consisting on integer coefficients:

$$h_1 = [1], \qquad h_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad h_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad h_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

According to (15) we obtain separation filters:

$$g_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \qquad g_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \qquad g_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad g_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It is easy to verify that these filters fulfill conditions (15) if $N = 1$. There are no multiplications of signal values by filter coefficients because they are equal to 1 or -1. This characteristic feature makes transmultiplexers easily applicable to hardware implementation in reprogrammable electronic systems.

4.2. Example 2

Let us design filters with integer coefficients when the transmultiplexer consists of eight channels. For $\tau = 1$ and $N = 2$ let us assume matrices:

$$D_1 = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & -1 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & -1 & 1 & 0 \\ -1 & 1 & -2 & 1 & 1 & 1 & 0 & -1 \\ 2 & 0 & 1 & 0 & 0 & 1 & -1 & 1 \\ 1 & 1 & -2 & 2 & 1 & 1 & -1 & 0 \\ -1 & 0 & -2 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 1 & 1 & 0 & -1 \\ -2 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$Q = \begin{bmatrix} 2 & 0 & 0 & -1 & 1 & -1 & 1 & 0 \\ 2 & -1 & 0 & 0 & 0 & -2 & 1 & 1 \\ 1 & -1 & 1 & 0 & -1 & -1 & 0 & 1 \\ -1 & 0 & 0 & 2 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -2 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \end{bmatrix}.$$

From Theorem 3 and Theorem 4 we obtain the impulse responses of composition filters:

$$\begin{aligned}
h_1 &= [-2\ 1\ 0\ 2\ -2\ 0\ 2\ 0\ 1\ -1\ 0\ -2\ 1\ 1\ -1\ -1]^T, \\
h_2 &= [-1\ -1\ 2\ -1\ 4\ -1\ 0\ -4\ 1\ 4\ -5\ 3\ -1\ -2\ -1\ 6]^T, \\
h_3 &= [6\ 2\ -7\ -3\ 1\ 1\ -5\ 8\ -1\ 0\ 1\ 1\ -1\ 0\ 1]^T, \\
h_4 &= [2\ 0\ -2\ -3\ 7\ -3\ -1\ 0\ -1\ 3\ -2\ 4\ -2\ -2\ 1\ 4]^T, \\
h_5 &= [-4\ -2\ 5\ 0\ 3\ -2\ 4\ -7\ 1\ 2\ -3\ 1\ 0\ -1\ -1\ 3]^T, \\
h_6 &= [1\ -1\ 2\ -1\ 3\ 0\ 0\ -3\ 1\ 4\ -5\ 3\ -1\ -2\ -1\ 6]^T, \\
h_7 &= [-5\ -2\ 6\ 2\ -1\ 0\ 4\ -7\ 1\ 0\ -1\ -1\ 1\ 0\ -1]^T, \\
h_8 &= [2\ 2\ -4\ 0\ -3\ 1\ -1\ 5\ 1\ -2\ 1\ -3\ 1\ 2\ -1\ -2]^T.
\end{aligned}$$

In a similar way we obtain separation filters:

$$\begin{aligned}
g_1 &= [0\ 3\ -8\ -6\ -3\ -5\ -4\ -7\ -14\ 0\ 0\ -1\ -1\ 1\ -3\ -5\ -1]^T, \\
g_2 &= [0\ 2\ -9\ -7\ -4\ -6\ -4\ -6\ -14\ 0\ 1\ 0\ 0\ 2\ -2\ -4\ 1]^T, \\
g_3 &= [0\ 4\ -12\ -9\ -5\ -8\ -7\ -12\ -21\ 1\ -1\ -1\ 0\ 1\ -2\ -5\ -2]^T, \\
g_4 &= [0\ -5\ 14\ 11\ 6\ 10\ 5\ 10\ 23\ 0\ 0\ 1\ 1\ -1\ 3\ 5\ 1]^T, \\
g_5 &= [0\ 6\ -16\ -13\ -7\ -12\ -6\ -12\ -27\ 0\ 0\ -1\ -1\ 1\ -3\ -5\ -1]^T, \\
g_6 &= [0\ -3\ 11\ 9\ 5\ 7\ 6\ 10\ 18\ 0\ -1\ 0\ 0\ -2\ 2\ 4\ -1]^T, \\
g_7 &= [0\ -5\ 12\ 10\ 5\ 9\ 2\ 6\ 19\ 1\ -1\ 1\ 2\ -1\ 4\ 5]^T, \\
g_8 &= [0\ -5\ 13\ 10\ 5\ 8\ 6\ 12\ 22\ 0\ 0\ 1\ 1\ -1\ 3\ 5\ 1]^T.
\end{aligned}$$

This example demonstrates that the simple method enables us to design integer-to-integer perfect reconstruction filters. The amplitude characteristics of these filters are shown in Fig. 2 and the phase in Fig. 3.

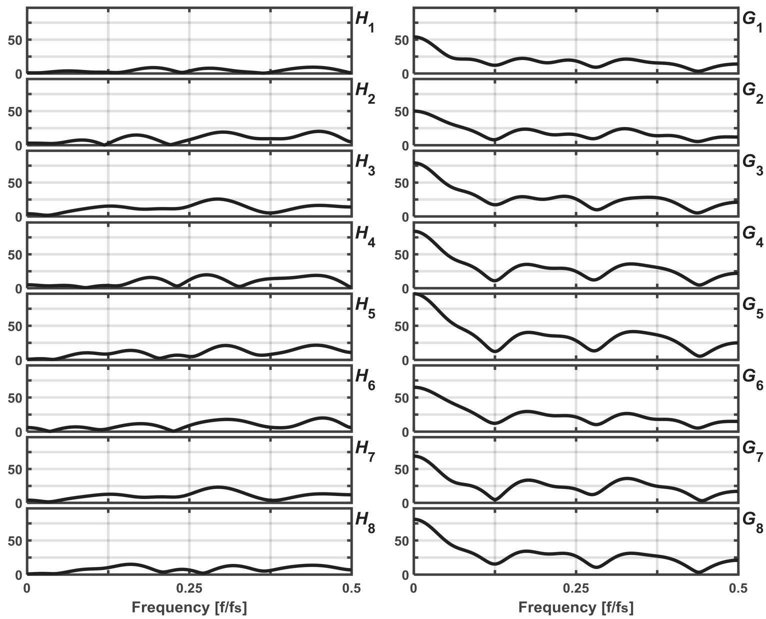

Figure 2: Amplitude characteristics of composition filters (left) and separation filters (right) for the case presented in Example 2, where f_s is a sampling frequency equal to 44 100 Hz

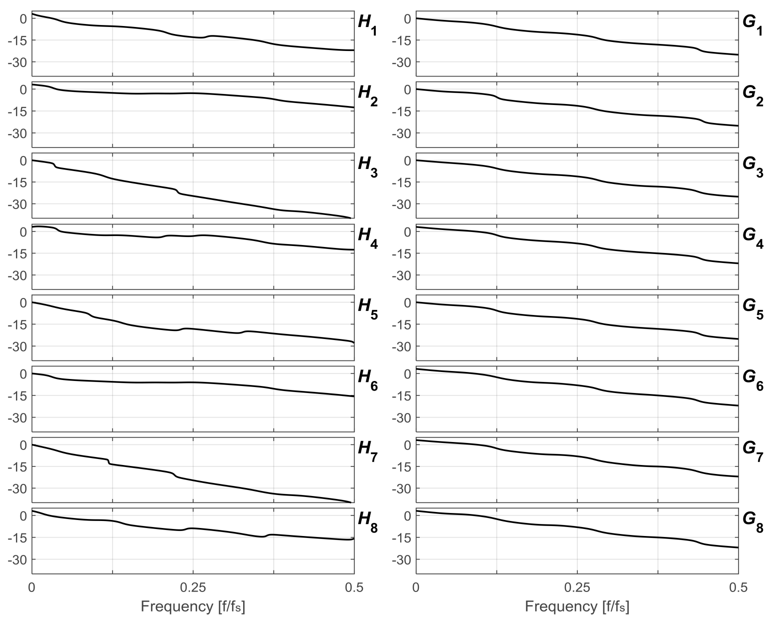

Figure 3: Phase characteristics of composition filters (left) and separation filters (right)

As a result of the eight-fold upsampling procedure, eight-fold more samples are obtained, both in the time and in the frequency domains. The physical interpretation of this phenomenon can be two-fold. It can be considered that the duration of the signal does not change, but the sampling density in the time domain is reduced eight-fold. Then, the sampling density in the frequency domain does not change, but the maximum frequency increases eight-fold. This type of interpretation has been used in Fig. 5. The second type of interpretation of the upsampling procedure is based on the assumption that the sampling density in the time domain does not change. Then, with an eight-fold increase in the number of samples, the time duration of the signal increases eight-fold. In the frequency domain, this means an eight-fold decrease in the sampling density, i.e. the maximum frequency does not change. Widening the frequency band seems to be a better and natural interpretation of the upsampling procedure.

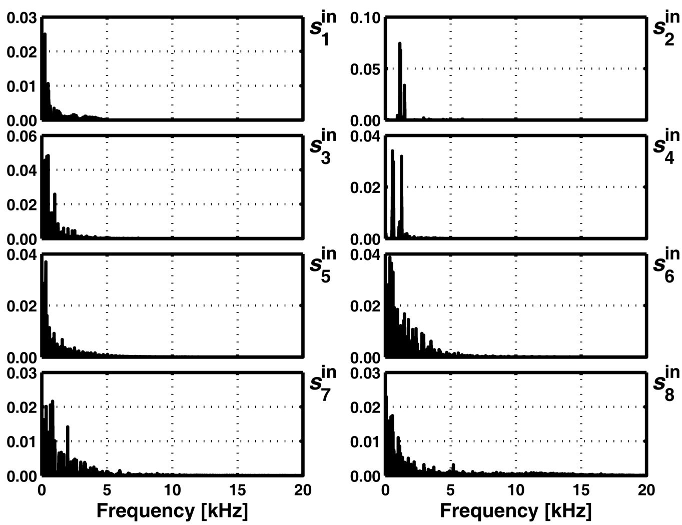

Figure 4: Amplitude spectra of the transmitted audio signals; the first is of speech and the others are of music

The distinctive properties of the spectrum of the combined signal (see Fig. 5) result from the absence of high frequencies in transmitted acoustic signals (typical examples are shown in Fig. 4) and the periodicity of discrete Fourier transform for the upsampled signals. Here, the upsampling procedure is interpreted in such a way that it reduces the sampling density eight-fold, while the time durations of both signals, before and after upsampling, are the same. Sampling densities in the frequency domain for both cases, presented in Fig. 4 and Fig. 5, are the same, although the numbers of samples are different.

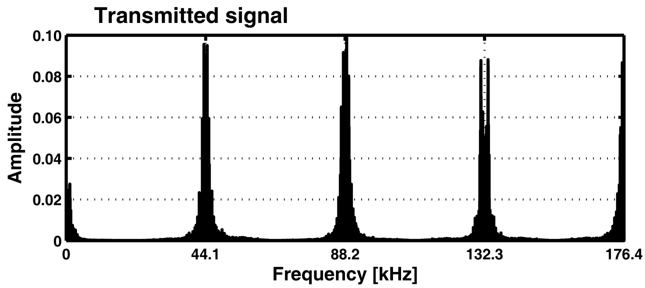

Figure 5: Spectrum of a signal obtained by transmultiplexing audio signals presented in Fig. 4

5. Conclusions

If filter coefficients are real values, then the perfect reconstruction conditions are theoretically fulfilled only. Such transmultiplexers can transmit uncoded images or audio signals satisfactorily. Software files or coded multimedia data such as MPEG files require the faultless transmission of each bit. For such cases, the use of integer-to-integer filters increases the number of applications of transmultiplexers.

The methods proposed in this paper give effective and strict algorithms, which can be easily used to design transmultiplexer filters. The suggested algorithms were obtained from the necessary conditions for a perfect reconstruction. It should be noted that the methods described here make it possible to find a wide class of solutions; however certain solutions cannot be found analytically.

References

[1] A.N. AKANSU, P. DUHAMEL, X. LIN and M. COURVILLE: Orthogonal transmultiplexers in communication: a review, *IEEE Trans. Signal Processing*, **46**(4), (1998), 979–995.

[2] P.P. VAIDYANATHAN: Multirate systems and filter banks, pp. 148–151, 259–266, NJ: PTR Prentice Hall 1993.

[3] M. VETTERLI and J. KOVACEVIC: Wavelets and subband coding, pp. 69–72, 185–187, NJ: Prentice Hall 1995.

[4] F. CRUZ-ROLDÁN, M. BLANCO-VELASCO and J.I.G. LLORENTE: Zero-padding or cyclic prefixfor MDFT-based filter bank multicarrier communications, *Signal Procesing*, **92** (2012), 1646–1657.

[5] R.K. Soni, A. Jain and R. Saxena: A design of IFIR prototype filter for Cosine Modulated Filterbank and Transmultiplexer, *AEU International Journal of Electronics and Communications*, **67**(2), (2013), 130–135.

[6] A. Vishwakarma, A. Kumar and G.K. Singh: A prototype filter design for cosine modulated transmultiplexer using weighted constrained least squares technique, *AEU International Journal of Electronics and Communications*, **69**(6), (2015), 915–922.

[7] T.S. Bindiya and E. Elias: Meta-heuristic evolutionary algorithms for the design of optimal multiplier-less recombination filter banks, *Information Science*, **339** (2016) 31–52.

[8] S. Glesner: Finite integer computations: An algebraic foundation for their correctness, *Formal Aspects of Computing*, **18**(2), (2006), 244–262.

[9] M. Ziółko, M. Nowak and B. Ziółko: Transmultiplexer Integer-to-Integer Filter Banks, Proc. of the First IEEE & IFIP International Conference in Central Asia on Internet, the Next Generation of Mobile, Wireless and Optical Communications Networks, Bishkek 2005.

[10] B. Ziółko, M. Ziółko, M. Nowak and P. Sypka: A Suggestion of Multiple-Access Method for 4G-System, Proc. of the 47th International Symposium ELMAR-2005 on Multimedia Systems and Applications, Zadar 2005.

[11] M. Ziółko and M. Nowak: Integer-to-Integer Filters in Transmultiplexer System, Proc. of the 12th IEEE Symposium on Computers and Communications, Aveiro 2007.

[12] A.D. Poularikas (Ed.): The Transforms and Applications handbook. CRC Press, Inc. 1996, pp. 388-389.

[13] A. Mertins: Memory truncation and crosstalk cancellation in transmultiplexers, *IEEE Communications Letters*, **3**(6), (1999), 180–182.

[14] B.S. Chen and L.M. Chen: Optimal reconstruction in multirate transmultiplexer systems under channel noise: Wiener separation filtering approach, *Signal Processing*, **80** (2000), 637–657.

[15] L. Chen, K.P. Chan, T.Q. Nguyen and X.H. Dai: Algorithm for the design of transmultiplexers with time-frequency spread criteria, *Electronics Letters*, **36**(17), (2000), 1499–1500.

[16] T. Liu and T. Chen: Optimal design of multi-channel transmultiplexers, *Signal Processing*, **80**(10), (2000), 2141–2149.

[17] T. LIU and T. CHEN: Design of multichannel nonuniform transmultiplexers using general building, *IEEE Trans. Signal Processing*, **49**(1), (2001), 91–99.

[18] F. CRUZ-ROLDAN, A.M. BRAVO-SANTOS, P. MARTIN-MARTIN and R. JIMENEZ-MARTINEZ: Design of multi-channel near-perfect-reconstruction transmultiplexers using cosine-modulated filter banks, *Signal Processing* 2003;83,5:1079-91.

[19] X.M. XIE, S.C. CHAN and T.I. YUK: Design of perfect-reconstruction nonuniform filter banks with flexible rational sampling factors, *IEEE Trans. Circuits and Systems*, **52**(9), (2005), 1965–1981.

[20] Y. SHI Y, F. DING and T. CHEN: 2-norm based recursive design of transmultiplexers with designable filter lengt, *Circuits Systems Signal Processing*, **25**(4), (2006), 447–462.

[21] A. EGHBALI, H. JOHANSSON and P. LOWENBORG: A multimode transmultiplexer structure, *IEEE Trans. Circuits and Systems*, **55**(3), (2008), 279–283.

[22] D.C. CHANG and H.C. CHIU: A stabilized multichannel fast RLS algorithm for adaptive transmultiplexer receivers. Circuits Systems Signal Processing 2009;28,6:845-67.

[23] T. IHALAINEN, A. VIHOLAINEN, T.H. STITZ and M. RENFORS: Generation of Filter Bank-Based Multicarrier Waveform Using Partial Synthesis and Time Domain Interpolation, *IEEE Tansations on Circuits and Systems I-Regular Papers*, **57**(7), (2010), 1767–1778.

[24] J. CIOSMAK: Efficient algorithm for calculating nonseparable two-dimensional filter banks for transmultiplexer systems, *Przegląd Elektrotechniczny*, **87**(11), (2011), 217–220.

[25] R.K. SONI, A. JAIN and R. SAXENA: An optimized transmultiplexer using combinational window functions, *Signal Image and Video Processing*, **5**(3), (2011), 389–397.

[26] A. EGHBALI, H. JOHANSSON and P. LOWENBORG: Reconfigurable nonuniform transmultiplexers using uniform modulated filter banks, *IEEE Transactions on Circuits and Systems I-Regular Papers*, **583** (2011), 539–547.