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Producers' Adjustment Trajectories Resulting in Equilibrium in the Economy with Linear Consumption Sets

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Abstract

We consider the Debreu private ownership economy in which all consumption plans belong to a proper linear subspace of the commodity-price space $\mathbb{R}^l$. This geometric property of consumption sets means that there is a dependency between quantities of some commodities in all consumption plans. Competitive mechanism makes producers adjust their plans of action to the same dependency. It results in the mild evolution of the production sector to offer production plans which are also contained in the given subspace of $\mathbb{R}^l$. Modified production system and the initial consumption system can form an economy in equilibrium. The aim of this paper is to model gentle changes of producers' activity that give equilibrium in the Debreu economy with consumption system reduced to a proper subspace of $\mathbb{R}^l$ without considering additional costs.

Keywords: Debreu economy, equilibrium, linear sets, projections

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1 Introduction

This paper is an attempt to study gentle changes of the production sphere of the Debreu private ownership economy (see Debreu (1959)), differently from the results obtained in Radner (1972) or in Magill and Quinzii (2002). In contrast to the above, we are trying to model mild evolution of the economy in which all consumption sets are contained in a proper subspace V of the commodity - price space $\mathbb{R}^l,\ l \in \{1,\ 2, \dots\}$. This property is satisfied, among others, if the consumers are not interested in consumption of $l-k$ commodities for some $k \in \{1, 2,\ \dots, l-1\}$ (then $V = \mathbb{R}^k$) or if there is a linear dependency between the quantities of commodities in all consumption plans (then $V \neq \mathbb{R}^k$ for every $k \in \{1, 2, \dots, l-1\}$). The producers observing the actions of consumers on competitive markets, are fully aware of properties of consumers' activities. Hence they might decide to modify their technologies to adjust the quantities of commodities in production plans to the given relationships.
The analysis of changes of the production sector, is based on properties of projections. Projections have been already used in economic theorizing. For example, in Ciałowicz and Malawski (2011) or in Lipieta (2013), the natural projections have been used to construct extensions of production and consumption systems. Some examples of applications of projections to economic analysis the reader can also find in Lipieta (2012).
The paper consists of four parts. In the next section the construction of the Debreu private ownership economy is presented, the third part is devoted to the definition of producers' adjustment trajectories. The fourth section takes the modeling of such a change of the production sphere of the Debreu economy, that gives equilibrium in its modified form.

2 Model

We will study the model of the Debreu private ownership economy (see Debreu (1959), p. 75; Mas-Colell et al. (1995), p. 546) in the form of the multi - range relational system (see Adamowicz and Zbierski (1997), p.10) which includes the combination of the production and the consumption systems. The notation and definitions are borrowed from Lipieta (2010).
The linear space $\mathbb{R}^l$ ($l \in \{1, 2, \dots\}$) with the standard scalar product

$$(x \circ y) = (x_1, \dots, x_l) \circ (y_1, \dots, y_l) = \sum_{k=1}^{l} x_k \cdot y_k, \tag{1}$$

is the l - dimensional commodity-price space. Let $m,\ n \in \{1, 2, \dots\}$. It is assumed that two groups of agents: producers and consumers operate in $\mathbb{R}^l$. The producers try to maximize their profits and the consumers tend to maximize their utilities on the budget sets. If there exists a price vector $p \in \mathbb{R}^l$, such that both of them manage to realize their tasks as well as the total supply equals the total demand, then it is said that the economy is in equilibrium and vector p is called the equilibrium price vector. Let

1. $J = \{1, \ldots, n\}$ be a finite set of producers,
2. $\delta: J \ni j \mapsto Y^j \subset \mathbb{R}^l$ be a correspondence of production sets, which to every producer j assigns a production set $Y^j \subset \mathbb{R}^l$ of the producer's feasible production plans,
3. $p \in \mathbb{R}^l$ be a price vector.

Definition 2.1. If for the given price vector $p \in \mathbb{R}^l$

$$\forall_{j \in J} \quad \eta^j(p) \stackrel{\text{def}}{=} \left\{ y^{j*} \in Y^j : p \circ y^{j*} = \max\{p \circ y^j : \ y^j \in Y^j\} \right\} \neq \emptyset,$$

then

1. $\eta : J \ni j \mapsto \eta^j(p) \subset \mathbb{R}^l$ is called the correspondence of supply at given price vector p, which to every producer j assigns the set $\eta^j(p)$ of production plans maximizing his profit at price system p,
2. $\pi : J \ni j \mapsto \pi^j(p) \in \mathbb{R}$ is the maximal profit function at given price vector p, where
$$\forall_{j \in J} \quad \pi^j(p) = p \circ y^{j*} \quad \text{for} \quad y^{j*} \in \eta^j(p).$$
3. the two - range relational system
$$P = (J, \ \mathbb{R}^l; \delta, p, \eta, \pi),$$

is called the production system.
Similarly, let

1. $I = \{1, \ldots, m\}$ be a finite set of consumers,
2. $\Xi \subset \mathbb{R}^l \times \mathbb{R}^l$ be the family of all preference relations defined on the commodity space $\mathbb{R}^l$,
3. $\chi : I \ni i \mapsto X^i \subset \mathbb{R}^l$ be a correspondence of consumptions sets which to every consumer $i \in I$ assigns a consumption set X^i representing the consumer's feasible consumption plans,
4. $e : I \ni i \mapsto e^i \in X^i$ be an initial endowment mapping which to every consumer $i \in I$ assigns an initial endowment vector $e^i \in X^i$,
5. $\varepsilon \subset I \times (\mathbb{R}^l \times \mathbb{R}^l)$ be a correspondence, which to every consumer $i \in I$ assigns a preference relation $\preccurlyeq^i$ from set Ξ restricted to the consumption set X^i,

6. $p \in \mathbb{R}^l$ - a price vector.

It should be noted that the expenditures of every consumer i cannot be greater than the value

$$w^i = p \circ e^i. \tag{2}$$

Definition 2.2. If for the given price vector $p \in \mathbb{R}^l$ and for every $i \in I$,

$$\beta^i(p) \overset{\text{def}}{=} \{x \in X^i : p \circ x \leq w^i\} \neq \emptyset \tag{3}$$

$$\varphi^i(p) \overset{\text{def}}{=} \{x^{i*} \in \beta^i(p) : \quad \forall_{x^i \in \ \beta^i(p)} \ \ x^i \preccurlyeq^i x^{i*}\} \neq \emptyset \tag{4}$$

then

1. $\beta : I \ni i \mapsto \beta^i(p) \subset \mathbb{R}^l$ is the correspondence of budget sets at the given price vector p, which to every consumer i assigns his set of budget constrains $\beta^i(p) \subset \chi(i)$ at the price system p and the initial endowment e^i,
2. $\varphi : I \ni i \mapsto \varphi^i(p) \subset X^i$ is the demand correspondence at the given price vector p, which to every consumer i assigns his consumption plans maximizing his preference on the budget set $\beta^i(p)$,
3. the three-range relational system

$$C = (I, \ \mathbb{R}^l, \Xi; \ \chi, e, \varepsilon, p, \beta, \varphi)$$

is called the consumption system.

Let $p \in \mathbb{R}^l$ be a price vector, P – a production system and C – a consumption system in the same space $\mathbb{R}^l$. Suppose that the mapping $\theta : I \times J \mapsto [0,1]$ satisfying

$$\forall_{j \in J} \sum_{i=1}^{m} \ \theta(i,j) = 1 \tag{5}$$

is given. It is assumed that every consumer shares in the producers' profits. Number $\theta(i,j)$ indicates that part of the profit of producer j which is owned by consumer i. In this situation the value w^i in (2) is changed by the rule

$$w^i = p \circ e^i + \sum_{j=1}^{n} \theta(i,j) \cdot \pi^j(p). \tag{6}$$

Let

$$\omega = e(1) + \cdots + e(m) \in \mathbb{R}^l. \tag{7}$$

If for every $i \in I$ and w^i given by (6) sets $\beta^i(p)$ and $\varphi^i(p)$ are not empty (see (6) and (4)), then the following definition is formulated:
Definition 2.3. The relational system

$$E = (P, C, \theta, \omega)$$

is called the Debreu private ownership economy (in short: the Debreu economy). The vector (7) is called the total endowment of the economy E.
It is well known that the Debreu economy operates as follows. Let a price vector $p \in \mathbb{R}^l$ be given. Every producer j chooses a production plan $y^{j*} \in \eta^j(p) \subset Y^j$ maximizing his profit at the price system p. Maximal profit of each producer is divided among all consumers according to function θ (see (5)). Now the expenditures of every consumer i cannot be greater than value w^i (see (6)). Every consumer i chooses his consumptions plans $x^{i*} \in \varphi^i(p) \subset X^i$ maximizing his preference on the budget set $\beta^i(p)$. If

$$x^* - y^* = \omega, \tag{8}$$

where $x^* = x^{1*} + \cdots + x^{m*}$ and $y^* = y^{1*} + \cdots + y^{n*}$, then the economy is in equilibrium and the vector p is the equilibrium price vector. Consequently, the sequence

$$(x^{1*}, \ldots, \; x^{m*}, \; y^{1*}, \ldots, y^{n*}, p) \in \left(\mathbb{R}^l\right)^{m+n+1} \tag{9}$$

is called the state of equilibrium in the Debreu economy.

3 Changing production in the Debreu private ownership economy

Let $E = (P, C, \theta, \omega)$ be the Debreu economy. Assume that there exists a proper subspace V of the commodity - price space $\mathbb{R}^l$ such that

$$\forall_{i \in I} \; X^i \subset V \tag{10}$$

(see also Lipieta (2010)). The consumption system, in which condition (10) is satisfied, will be called the consumption system reduced to subspace V (see Lipieta (2012)). Similarly, the production system, in which condition

$$\forall_{j \in J} \; Y^j \subset V$$

is satisfied, will be called the production system reduced to subspace V (see Lipieta (2012)). The sets satisfying condition (10) are called the linear sets (see for example Moore (2007), p. 161), hence the Debreu economy in which consumption sets are linear is called the economy with linear consumption sets. In the same way the economy with linear production sets is defined.
It is well known that if V is a subspace of dimension $l - k$ ($k \in \{1, 2, \ldots, l-1\}$)

of space $\mathbb{R}^l$, then there exist linearly independent vectors $g^1, \ldots, g^k \in \mathbb{R}^l$, $(g^s = (g^s_1, \ldots, g^s_l), \quad s \in \{1, 2, \ldots, k\})$ such that

$$V = \bigcap_{s=1}^{k} ker\tilde{g}^s \tag{11}$$

where the mapping

$$\tilde{g}^s \colon \mathbb{R}^l \ni (x_1, \ldots, x_l) \mapsto g^s_1 x_1 + \cdots + g^s_l x_l \in \mathbb{R} \tag{12}$$

is, for every $s \in \{1, \ldots, k\}$, linear and continuous, $ker\tilde{g}^s = (\tilde{g}^s)^{-1}(0)$.
Observe that, for the given subspace V, there are infinitely many linearly independent functionals $\tilde{g}^1, \ldots, \tilde{g}^k$ satisfying condition (11).
We discuss two cases in which the assumption (10) makes sense. Firstly, if at least two commodities are complementary. Recall (see for example Varian (1999), p.112) that two commodities are complementary if an increase in the price of the first commodity causes a decrease in the demand for the second commodity. The agents' activities in the Debreu economy do not directly influence on changes of the prices of commodities. Hence, the above definition of complementary commodities cannot be adopted in the Debreu economy. The complementary commodities (also called the complements) are goods consumed together (see Varian (1999), p.112). So their quantities, noticeable in consumption plans, are approximately proportional. The above dependency leads us to the definition of complementary commodities in the Debreu economy. We say, similarly as in Lipieta (2010), that commodities $a, b \in \{1, \ldots, l\}, \ a \neq b$ are complementary in the Debreu economy if

$$\exists_{c>0} \ \forall_{i \in I} \ \forall_{x^i \in X^i} \ \ x^i_a = c \cdot x^i_b.$$

As a consequence, there exists a functional

$$\tilde{g} \colon \mathbb{R}^l \ni (x_1, \ldots, x_l) \mapsto x_a - c{\cdot}x_b \in \mathbb{R}$$

such that

$$\forall_{i \in I} X^i \subset ker\tilde{g}.$$

Generally, we say that if at least two commodities are complementary in the Debreu economy, then there exists a proper subspace V of the commodity-price space $\mathbb{R}^l$ such that condition (10) is satisfied. Without loss of generality, we assume that V is of the form (11) where linearly independent functionals $\tilde{g}^1, \ldots, \tilde{g}^k$ (see (12)) have exactly two coordinates different from zero. Secondly, if there exists a commodity $a \in \{1, 2, \ldots, l\}$ - the producers' output, that is neither the input nor the output for consumers. It could be, for instance, the air pollution. Then,

$$\forall_{i \in I} X^i \subset V \stackrel{\text{def}}{=} ker\tilde{g}$$

where

$$\tilde{g}\colon \mathbb{R}^l \ni (x_1, \ldots, x_l) \mapsto x_a \in \mathbb{R}.$$

Producers want or are forced to change their productive activity because of some reasons. It can be establishing new legal requirements e.g. the reduction of CO_2 emission into atmosphere, elimination of some harmful commodities from producers' plans, implementation new profitable technologies (innovations), keeping the constant dependency between quantities of some outcomes because of the demand structure. The reasons as above can be reflected in consumers' plans even if some of commodities are used only in producers' activities.

Further, the procedure of such a modification of production sets will be presented that the modified production sets will be reduced to the subspace V.

Let $V \subset \mathbb{R}^l$ be a subspace of the form (11) with linearly independent functionals $\tilde{g}^1, \ldots, \tilde{g}^k$ of the form (12). Consider vectors $q^1, \ldots, q^k \in \mathbb{R}^l$, a solution of the system of equations:

$$\tilde{g}^s(q^r) = \delta^{sr} \quad \text{for} \quad s, r \in \{1, \ldots, k\}, \tag{13}$$

where

$$\begin{cases} 1, & s = r \\ 0, & s \neq r \end{cases}$$

is Kronecker delta. Let mapping $Q : \mathbb{R}^l \times [0,1] \to \mathbb{R}^l$ be of the form

$$Q(x,t) = x - t \cdot \sum_{s=1}^{k} \tilde{g}^s(x) \cdot q^s. \tag{14}$$

We say that vectors $q^1, \ldots, q^k \in \mathbb{R}^l$ determine the mapping Q. Let us notice, that mapping Q is continuous. Consequently, for every fixed $t \in [0,1]$, mapping $Q(\cdot, t) : \mathbb{R}^l \to \mathbb{R}^l$ is the linear and continuous operator. Moreover, if $t = 1$ then operator $Q(\cdot, 1) : \mathbb{R}^l \to V$,

$$Q(x,1) = x - \sum_{s=1}^{k} \tilde{g}^s(x) \cdot q^s \tag{15}$$

is the linear and continuous projection from $\mathbb{R}^l$ into V determined by vectors $q^1, \ldots, q^k$. It should be noted that

$$\forall_{v \in V} \ \forall_{t \in [0,1]} \quad Q(v,t) = v, \tag{16}$$

moreover, $Q(\cdot, 1) = Id(\cdot) - \sum_{s=1}^{k} \tilde{g}^s(\cdot) q^s$. The linearity and continuity of the mapping $Q(\cdot, 1)$ is the natural consequence of the properties of functionals $\tilde{g}^s$ and the identity mapping $Id : \mathbb{R}^l \to \mathbb{R}^l$.

Consider the Debreu economy $E = (P, C, \theta, \omega)$ satisfying condition (10) with a subspace $V \subset \mathbb{R}^l$ of the form (11) with linearly independent functionals $\tilde{g}^1, \ldots, \tilde{g}^k$ of the form (12). In this situation the following is true:

Theorem 3.1. There exists a mapping $Q : \mathbb{R}^l \times [0,1] \mapsto \mathbb{R}^l$ of the form (14) such that

1. for every $t \in [0,1]$, the vector $Q(y^{j*}, t)$ maximizes, at price p, the profit of every producer j on the modified production set
$$Q\left(Y^j, t\right) = \{Q(y^j, t) \in \mathbb{R}^l : y^j \in Y^j\}, \tag{17}$$
2. for every $t \in [0,1]$, the vector $Q\left(x^{i*}, t\right)$ maximizes, at price p, the preference of consumer i on the set
$$Q\left(\beta^i\left(p\right), t\right) = \{Q\left(x^i, t\right) : \ x^i \in \ \beta^i\left(p\right)\}.$$

Proof. Note that if $p \in V^T \overset{\text{def}}{=} \left\{x \in \mathbb{R}^l : \ \forall_{v \in V} : \ x \circ v = 0\right\}$, then for every mapping Q of the form (14), determined by vectors $q^1, \ldots, q^k \in \mathbb{R}^l$ obtained by (13), the following is true:
$$\forall_{t \in [0,1]} \ \ \forall_{v \in V} \ p \circ v = p \circ Q\left(v, t\right) = 0$$
By the above, if $p \in V^T$, then every mapping of the form (14) satisfies the thesis of the theorem. Let us notice that linearly independent vectors $g^1, \ldots, g^k$ (see (12)) belong to V^T. Hence, if $p \notin V^T$, then vectors $g^1, \ldots, g^k$, p are also linearly independent. Vectors $q^1, \ldots, q^k \in \mathbb{R}^l$ calculated by (13) and satisfying additionally
$$p \circ q^s = 0 \quad \text{for} \quad s \in \{1, \ldots, k\}, \tag{18}$$
determine the mapping Q by the thesis of the theorem.

Let us notice that every mapping of the form (14) does not change the consumption sector of the Debreu economy satisfying (10). Keeping in mind assumption (10) and property (16),
$$\forall_{i \in I} \ \forall_{t \in [0,1]} \ Q\left(X^i, t\right) = X^i.$$
By the above, we get that
$$\forall_{i \in I} \ \ \forall_{t \in [0,1]} \ \ Q\left(\beta^i\left(p\right), t\right) = \beta^i\left(p\right) \text{ and } \forall_{i \in I} \ \ \forall_{t \in [0,1]} \ \ Q\left(x^{i*}, t\right) = x^{i*}.$$
□

It is easy to see that, if mapping Q is obtained by theorem 3.1 as well as $Q(y^{j*}, t) \in Y^j$ for $y^{j*} \in Y^j$and $j \in J$, then vector $Q(y^{j*}, t)$ also maximizes at price p, the profit of producer j on the production set Y^j. Hence, every mapping Q of the form (14) obtained by the thesis of theorem 3.1 will be called the producers' adjustment trajectory.

Let $t \in [0,1]$, $p \in \mathbb{R}^l$. Hereafter, unless otherwise stated, we assume that $E = (P, C, \theta, \omega)$ is the Debreu economy satisfying condition (10) with a subspace $V \subset \mathbb{R}^l$ of the form (11) with linearly independent functionals $\tilde{g}^1, \ldots, \tilde{g}^k$ of the form (12). Consider vectors $q^1, \ldots, q^k \in \mathbb{R}^l$ calculated by (13) and satisfying additionally (18) if $p \notin V^T$. Then mapping Q determined by vectors $q^1, \ldots, q^k \in \mathbb{R}^l$ (see (14)), is the producers' adjustment trajectory (see theorem 3.1). Now we can assume the following definition:
Definition 3.2. The two-range relational system

$$P_t(q^1, \ldots, q^k) = \left(J,\ \mathbb{R}^l; y_t, p, \eta_t,\ \pi_t\right) \tag{19}$$

where:

1. $y_t : \ J \ni j \mapsto Q(Y^j, t) \subset \mathbb{R}^l$ is the correspondence of production sets, which to every producer j assigns the image of production set Y^j by mapping $Q\left(\cdot, t\right)$,

2. $p \in \mathbb{R}^l$ is the price vector in economy E,

3. $\eta_t \ : \ J \ni j \mapsto \eta_t^j\left(p\right) \subset \mathbb{R}^l$ is the correspondence of supply, which to every producer j assigns set $\eta_t^j\left(p\right)$ of production plans maximizing his profit at the price system p on the set $Q(Y^j, t)$, where

$$\forall_{j \in J} \ \ \eta_t^j\left(p\right) \overset{def}{=} \left\{Q\left(y^{j*}, t\right) : \ p \circ y^{j*} = \max\left\{p \circ y^j \ : \ y^j \in Y^j\right\}\ \right\}.$$

4. $\pi_t : J \ni j \mapsto p \circ Q\left(y^{j*}, t\right) \in \mathbb{R}$ is the maximal profit function and $y^{j*} \in \eta^j(p)$ for every $j \in J$,

is called the modification of the production system $P = (J,\ \mathbb{R}^l; \delta, p, \eta, \pi)$ at time t, determined by vectors $q^1, \ldots, q^k$.

The economy

$$E_t\left(q^1, \ldots, q^k\right) = \left(P_t\left(q^1, \ldots, q^k\right), C, \theta, \omega\right) \tag{20}$$

is called the modification of economy E, at time t, determined by vectors $q^1, \ldots, q^k$. It is easy to check that

$$\forall_{j \in J} \ \ \eta_t^j\left(p\right) = \left\{Q\left(y^{j*}, t\right) : p \circ Q\left(y^{j*}, t\right) = \max\left\{p \circ Q\left(y^j, t\right) : \ \ y^j \in Y^j\right\}\ \right\},$$

as well as production system (19), besides price system p, is the image of the production system P of economy E, by the mapping $Q\left(\cdot, t\right)$. By the results of theorem 3.1, we get that at every $t \in [0,1]$, the economy $E_t\left(q^1, \ldots, q^k\right)$ is the Debreu economy with consumption system reduced to subspace V. Moreover
Lemma 3.3. If sequence

$$\left(x^{1*}, \ldots,\ x^{m*},\ y^{1*}, \ldots,\ y^{n*}, p\right) \in \left(\mathbb{R}^l\right)^{m+n+1}$$

is the state of equilibrium in economy E, then sequence

$$(x^{1*}, \ldots, x^{m*},\ Q(y^{1*}, t), \ldots, Q(y^{n*}, t), p) \in (\mathbb{R}^l)^{m+n+1},$$

at every $t \in [0, 1]$, will be the state of equilibrium in the Debreu economy $E_t\left(q^1, \ldots, q^k\right)$.

Proof. Let $t \in [0, 1]$ be given. By theorem 3.1, the vector $Q(y^{j*}, t)$ maximizes at the price p the profit of producer j on the production set $Q(Y^j, t)$ and vector x^{i*} maximizes at the price p the preference of consumer i on the budget set $\beta^i(p)$. By the definition of the initial endowment mapping, the definition of the total endowment (7) as well as condition (8), the following may be easily inferred:

$$y^* = x^* - \omega \in V. \tag{21}$$

The linearity of the mapping $Q(\cdot, t)$ implies that

$$y^* = Q\left(y^{1*}, t\right) + \cdots + Q(y^{n*}, t), \tag{22}$$

which gives the result. □

In the real economies, the changes of the production sector can be forced by the market as well as can be driven by a person or an institution. Equilibrium in the economy can be one of the result of changes in the production sector.

Assuming that variable t means time, the mapping Q determined by vectors $q^1,\ \ldots,\ q^k$ lets us put the systems of Debreu economies $\{E_t\left(q^1, \ldots, q^k\right)\ : 0 \le t \le 1\}$ "in motion". If there is equilibrium in the economy $E = (P, C, \theta, \omega) = E_0\left(q^1, \ldots, q^k\right)$, then at every $t \in (0, 1]$ there is equilibrium in economy $E_t\left(q^1, \ldots, q^k\right)$. Moreover, the economy $E_1(q^1,\ \ldots,\ q^k)$, is the Debreu economy in which the consumption and production systems are reduced to V. If at least one producer does not follow the trajectory that others do, then generally, despite particular cases, the equilibrium will not exist even at point $t = 1$. Summarizing, the potential producers' disagreement in the choice of the adjustment trajectory or exclusion even one producer from the transformation of the form (14) may cause disequilibrium in the economy at point $t = 1$.

4 Modifications of the production sector of the Debreu economy with linear consumption sets resulting in equilibrium

Now let us focus on modeling some kind of producers' adjustment trajectories. In fact, we show that if equilibrium does not exist in the Debreu economy with linear consumption sets, then the choice of the proper producers' adjustment trajectory gives the opportunities to reach equilibrium in the modified form of the economy.

In this part of the paper we consider the Debreu economy $E = (P, C, \theta, \omega)$ satisfying condition (10) with a subspace $V \subset \mathbb{R}^l$ of the form (11) with linearly independent functionals $\tilde{g}^1, \ \ldots, \ \tilde{g}^k$ of the form (12). Let vectors $q^1, \ldots, q^k$ calculated by (13) satisfy additionally (18) if $p \notin V^T$. Assume that, at given price vector $p \in \mathbb{R}^l$, every producer $j \in J$ realizes plan y^{j*} maximizing his profit on the set Y^j and every consumer $i \in I$ realizes plan x^{i*} maximizing his utility on the budget set $\beta^i(p)$ (see (3)) in which value w^i is calculated by (6). Let

$$z = x^* - y^* - \omega, \tag{23}$$

where $x^* = \sum_{i=1}^m x^{i*}, \ y^* = \sum_{j=1}^n y^{j*}$ and vector ω is the total endowment (see (7)) in the economy E. Assume additionally that condition

$$p \circ z = 0 \tag{24}$$

is satisfied. Let us notice that the equality (24) means that Walras Law is fulfilled. The further analysis is based on the observation that, under the above assumptions, the sequence (9) is the state of equilibrium in economy E, at price vector p if, and only if, $z = 0$ (see (8), (23)).

Theorem 4.1. If $z \notin V$ and $p \circ z = 0$, then there exists a mapping Q of the form (14) with vectors $q^1, \ldots, q^k$ by (13), such that the sequence

$$(x^{1*}, \ldots, \ x^{m*}, \ Q(y^{1*}, 1), \ldots, Q(y^{n*}, 1), \ p) \in \left(\mathbb{R}^l\right)^{m+n+1} \tag{25}$$

is the state of equilibrium at price p in economy $E_1(q^1, \ldots, q^k)$.

Proof. Note that if $z \in \mathbb{R}^l \backslash V$ (see (23)) then $z \neq 0$. Let $t \in (0, 1]$. Without loss of generality we assume that there exists $i_0 \in I = \{1, \ldots, m\}$ such that $\tilde{g}^i(z) \neq 0$ for $i \in \{1, \ldots, i_0\}$, and $\tilde{g}^i(z) = 0$ for $i \in \{i_0 + 1, \ldots, m\}$ if $i_0 < m$. Now we define functionals $\hat{g}^1, \ldots, \ \hat{g}^k : \mathbb{R}^l \to \mathbb{R}$ by the rule

$$\hat{g}^1(x) = \frac{\tilde{g}^1(x)}{\tilde{g}^1(z)},$$

$$\hat{g}^i(x) = \frac{\tilde{g}^i(x)}{\tilde{g}^i(z)} - \frac{\tilde{g}^1(x)}{\tilde{g}^1(z)},$$

for $i \in \{2, \ldots, i_0\}$ (if $1 < i_0 \leq m$) and

$$\hat{g}^i(x) = \tilde{g}^i(x)$$

for $i \in \{i_0 + 1, \ldots, m\}$ (if $i_0 < m$), $x \in \mathbb{R}^l$.

It is easy to check that, the functionals $\hat{g}^1, \ \ldots, \hat{g}^k$ are linearly independent,

$$V = \bigcap_{s=1}^{k} \ker \hat{g}^s \tag{26}$$

and

$$\hat{g}^1(z) = 1,\ \hat{g}^2(z) = \cdots = \hat{g}^k(z) = 0.$$

Now, the reasoning goes in the same way as in the proof of theorem 3.1. Put $q^1 = z$. If $p \notin V^T$, then we calculate vectors $q^r \in \mathbb{R}^l$, for $r \in \{2, \ldots, k\}$, by the system of equations:

$$\begin{cases} \hat{g}^s(x) = \delta^{sr} \\ p \circ x = 0 \end{cases} \quad s \in \{1, \ldots, k\}, \quad r \in \{2, \ldots, k\}. \tag{27}$$

If $p \in V^T$, then every vector $q^r \in \mathbb{R}^l$, $r \in \{2, \ldots, k\}$, is calculated by:

$$\hat{g}^s(q^r) = \delta^{sr} \text{ for } s \in \{1, \ldots, k\}, \quad r \in \{2, \ldots, k\} \tag{28}$$

Mapping $Q : \mathbb{R}^l \times [0,1] \to \mathbb{R}^l$ of the form

$$Q(x,t) = x - t \cdot \sum_{s=1}^{k} \hat{g}^s(x) \cdot q^s, \tag{29}$$

satisfies the thesis of theorem 3.1.
Notice that if $p \in \mathbb{R}^l$ satisfies (24) as well as the mapping Q defined in (29) is determined by vectors $q^1 = z$, $q^2, \ldots, q^k$ obtained by (27) or (28) respectively, then

$$Q(z,1) = 0 \text{ and } Q(z,1) = x^* - Q\left(y^{1*}, 1\right) - \cdots - Q(y^{n*}, 1),$$

which means that there is the state of equilibrium of the form (25) at price p in the Debreu economy $E_1(q^1,\ \ldots,\ q^k\ )$. □
Notice that the economy $E_1(q^1,\ \ldots,\ q^k)$, obtained in theorem 4.1 is the Debreu economy in which the consumption and production systems are reduced to subspace V.
Theorem 4.2. If $z \in V \backslash \{0\}$, $p \circ z = 0$ as well as either vectors $x^* - \omega,\ y^*$ are linearly independent or $x^* - \omega = 0$, then there exists a mapping $Q : \mathbb{R}^l \times [0,1] \to \tilde{V} \subset V$ such that the sequence

$$(x^{1*}, \ldots,\ x^{m*},\ Q(y^{1*}, 1), \ldots, Q(y^{n*}, 1),\ p) \in (\mathbb{R}^l)^{m+n+1}$$

is the state of equilibrium in economy $\tilde{E} = (P_1(q^1, \ldots, q^k, z), C, \theta, \omega)$.
Proof. If $x^* - \omega$, y^* are linearly independent then $dimV \geq 2$ and $k \leq l-2$. If $x^* - \omega = 0$, then $dimV \geq 1$ and $k \leq l-1$. Recall, that subspace V is of the form (11) with linearly independent functionals $\tilde{g}^1, \ldots, \tilde{g}^k$. Consider vectors $q^1, \ldots, q^k \in \mathbb{R}^l$ calculated by (13) and satisfying additionally (18) if $p \notin V^T$.
Assume first that $x^* - \omega$, y^* are linearly independent. Then vectors z, $x^* - \omega$, $q^1, \ldots, q^k$ are also linearly independent. Hence there exists a vector $g \in \mathbb{R}^l$ such that functional $\tilde{g}$ of the form (12) satisfies

$$\begin{cases} \tilde{g}(z) = 1 \\ \tilde{g}(x^* - \omega) = 0 \quad \text{for } s \in \{1, \ldots, k\} \\ \tilde{g}(q^s) = 0 \end{cases}$$

Functionals $\tilde{g}^1, \ldots, \tilde{g}^k,\ \tilde{g}$ are linearly independent. Put $\tilde{V} = V \cap ker\tilde{g}$. Notice, that $x^* - \omega \in \tilde{V} \subset V$. Let $Q : \mathbb{R}^l \times [0,1] \to \tilde{V}$

$$Q\,(x,t) \stackrel{\text{def}}{=} x - t \cdot \sum_{s=1}^{k} \tilde{g}^s(x) \cdot q^s - t \cdot \tilde{g}(x) \cdot z. \tag{30}$$

Then

$$Q\,(z,1) = 0, Q\,(x^* - \omega, 1) = x^* - \omega.$$

If $p \in V^T$ then, for every $i \in I$ and $x^i \in X^i$

$$0 = p \circ x^i.$$

Hence $\beta^i\,(p) = X^i$ for every $i \in I$. Moreover, $p \circ Q\,\left(y^j, 1\right) = 0$ for every $j \in J$. Consequently $Q\,\left(\beta^i\,(p)\,,\ 1\right) = \beta^i\,(p)$. As a result, the sequence (25) is the state of equilibrium in economy $\tilde{E} = (P_1(q^1, \ldots, q^k, z), C, \theta, \omega)$.
If $p \notin V^T$then for the mapping of the form (30) the following is valid

$$\forall_{t\in[0,1]}\ \forall_{x\in\mathbb{R}^l}\ p \circ Q\,(x,t) = p \circ x.$$

Consequently, for every $j \in J$, vector $Q(y^{j*}, 1)$ maximizes the profit of producer j on the set $Q(Y^{j*}, 1)$ as well as the set $\beta^i\,(p)$ is not changed for every $i \in I$. Moreover

$$0 = Q\,(z,1) = x^* - \omega - Q(y^*\ ,1).$$

Mapping $Q\,(\cdot, 1)$ is the linear and continuous projection from $\mathbb{R}^l$ into $\tilde{V} \subset V$. Hence, by (24) as well as by the choice of the vectors $q^1, \ldots,\ q^k$, the sequence of the form (25) is the state of equilibrium in the Debreu economy $\tilde{E} = (P_1(q^1, \ldots, q^k, z), C, \theta, \omega)$.
If $x^* - \omega = 0$, then vectors $z,\ q^1, \ldots, q^k$ are linearly independent. There exists a vector $g \in \mathbb{R}^l$ such that functional $\tilde{g}$ of the form (12) satisfies

$$\begin{cases} \tilde{g}\,(z) = 1, \\ \tilde{g}\,(q^s) = 0, \end{cases}$$

for every $s \in \{1, \ldots, k\}$. As above, functionals $\tilde{g}^1, \ldots, \tilde{g}^k,\ \tilde{g}$ are linearly independent. Let $Q : \mathbb{R}^l \times [0,1] \to \tilde{V} \stackrel{\text{def}}{=} V \cap ker\tilde{g}$ be of the form

$$Q\,(x,t) \stackrel{\text{def}}{=} x - t \cdot \sum_{s=1}^{k} \tilde{g}^s(x) \cdot q^s - t \cdot \tilde{g}(x) \cdot z. \tag{31}$$

Then

$$Q\,(z,1) = Q\,(x^* - \omega, 1) = 0.$$

The rest of the proof goes in the same way. $\square$
Notice that the economy $\tilde{E} = (P_1(q^1, \ldots, q^k, z), C, \theta, \omega)$, obtained in theorem 4.2, is

the Debreu economy in which the consumption system is reduced to subspace V as well as the production system is reduced to subspace $\tilde{V} \subset V$. Hence we can say that the production and consumption systems of economy $\tilde{E} = (P_1(q^1, \ldots, q^k, z), C, \theta, \omega)$ are reduced to subspace V.

Theorem 4.3. If $z \in V \backslash \{0\}$, $p \circ z = 0$, $x^* - \omega \neq 0$ as well as vectors $x^* - \omega$, y^* are linearly dependent, then for every $j \in J$ there exists set of technologies $\check{Y}^j$ such that there exists equilibrium, at price system p, in the modified form of economy E where production sets are replaced with $\check{Y}^j$.

Proof. If $z \neq 0$ then there exists a real number $k \neq 1$ such that

$$x^* - \omega = ky^*. \tag{32}$$

Note that if $x^* - \omega \neq 0$ then $k \neq 0$. By (24)

$$p \circ y^* = 0. \tag{33}$$

Consider mapping $Q : \mathbb{R}^l \times [0, 1] \to V$ of the form (14), determined by vectors $q^1, \ldots, q^k \in \mathbb{R}^l$, calculated by (13), satisfying additionally (18) if $p \notin V^T$. The mapping Q satisfies the assumptions of theorem 3.1. Hence $Q(x^* - \omega) = x^* - \omega$ and

$$Q\left(Y^j + \frac{k-1}{n} y^*, 1\right) = Q\left(Y^j, 1\right) + \frac{k-1}{n} y^* \subset V.$$

Notice, that by (33) for every $j \in J$ and $y^j \in Y^j$

$$p \circ \left(Q(y^j, 1) + \frac{k-1}{n} y^*\right) = p \circ y^j + \frac{k-1}{n}(p \circ y^*) = p \circ y^j$$

which implies that

$$p \circ y^{j*} = \max\left\{p \circ \check{y}^j : \check{y}^j \in Q\left(Y^j + \frac{k-1}{n} y^*, 1\right)\right\}$$

for every $j \in J$. Hence the vector

$$\check{y}^{j*} \stackrel{\text{def}}{=} Q\left(y^{j*}, 1\right) + \frac{k-1}{n} y^*$$

maximizes the profit of producer j on the set $Q\left(Y^j + \frac{k-1}{n} y^*, 1\right)$. Replacing, for every $j \in J$, the set Y^j with the set $Q\left(Y^j + \frac{k-1}{n} y^*, 1\right)$ will not change the consumers' budget sets. Moreover

$$x^* - \omega - \sum_{j=1}^{n} \check{y}^{j*} = x^* - \omega - \sum_{j=1}^{m} Q\left(y^{j*}, 1\right) - (k-1)y^*. \tag{34}$$

By the linearity of the mapping $Q(\cdot, 1)$ we get that

$$\sum_{j=1}^{n} Q\left(y^{j*}, 1\right) = Q\left(\sum_{j=1}^{n} y^{j*}, 1\right).$$

Moreover, since $x^* - \omega = ky^*$ and $k \neq 0$, $y^* \in V$. Taking everything into consideration,

$$Q\left(\sum_{j=1}^{n} y^{j*}, 1\right) = Q(y^*, 1) = y^*. \tag{35}$$

By (32), (34) and (35)

$$x^* - \omega - \sum_{j=1}^{n} \check{y}^{j*} = x^* - \omega - ky^* = 0.$$

By the above, the sequence

$$(x^{1*}, \ldots, x^{m*}, \ \check{y}^{1*}, \ldots, \ \check{y}^{n*}, p^*) \in \left(\mathbb{R}^l\right)^{m+n+1} \tag{36}$$

is the state of equilibrium at price p in $\check{E}$ - the modified form of economy E in which production sets are replaced with $Q\left(Y^j + \frac{k-1}{n} y^*, 1\right)$.
Now we assume that $y^* = 0$. Then by (24), it follows that

$$p \circ (x^* - \omega) = 0. \tag{37}$$

Now

$$Q\left(\left(Y^j + \frac{1}{n}(x^* - \omega)\right), \ 1\right) = Q\left(Y^j, 1\right) + \frac{1}{n}(x^* - \omega) \in V. \tag{38}$$

As above, we can check that the vector

$$\check{y}^{j*} \stackrel{\text{def}}{=} Q\left(y^{j*} + \frac{1}{n}(x^* - \omega), 1\right) \tag{39}$$

maximizes the profit of producer j on the set $Q\left(\left(Y^j + \frac{1}{n}(x^* - \omega)\right), \ 1\right)$ as well as

$$p \circ y^{j*} = \max\left\{p \circ \check{y}^j : \check{y}^j \in Q\left(\left(Y^j + \frac{1}{n}(x^* - \omega)\right), \ 1\right)\right\}. \tag{40}$$

Replacing, for every $j \in J$, the set Y^j with the set $Q\left(\left(Y^j + \frac{1}{n}(x^* - \omega)\right), \ 1\right)$ will not change the consumers' budget sets. The conditions (37) - (40) imply that sequence (36), for $\check{y}^{j*}$, $j \in J$, given by (39), is the state of equilibrium at price p in $\check{E}$ - the modified form economy E in which production sets are replaced with sets $\check{Y}^j = Q\left(\left(Y^j + \frac{1}{n}(x^* - \omega)\right), \ 1\right)$. □

Note that in the economy $\check{E}$ constructed in the proof of theorem 4.3, the production and consumption systems are also reduced to subspace V.
To sum up, if the equilibrium does not exist in the Debreu economy under the assumption that Walras Law is fulfilled (see (24)), then $z \neq 0$ (see (23)). If $z \notin V$, then it is enough to modify the feasible producers' plans according to the rules presented in the proof of theorem 4.1. The definition of the adjustment trajectory, which allows to get equilibrium in economy $E_1(q^1, \ldots, q^k)$ is presented in the proof of the theorem 4.2. If $z \in V$, then we apply theorem 4.2 or 4.3, respectively.
Let us emphasize that the aim of the paper is to present the unified and coherent description of some class of adjustment trajectories resulting in equilibrium in the Debreu economy under the assumption that Walras Law is satisfied. For given trajectory Q of the form (14), some production plans from the set $Q\left(Y^j, t\right)$ at $t \in [0, 1]$, do not have to be feasible for the given producer j. Their realizations may need some additional expenses. However, if $Q\left(y^{j*}, 1\right)$ is feasible, then it also maximizes the profit of producer j on set Y^j. Then, equilibrium can be reached without any changes of technologies. Every mapping Q of the form (14) indicates the shortest way between every y^j and $Q\left(y^j, 1\right)$, in the sense of Euclidean distance. The linearity and continuity of mappings defined in (14) make that the most properties of production sets are preserved during the discussed transformation. Moreover, every mapping of the form (14) does not disturb equilibrium and some of them indicate the paths of changes of the production sector giving equilibrium in the final form of the economy under study. Summarizing, this research indicates only the potential, future paths of development of the economy, which was initially in its equilibrium or in disequilibrium form.
In the given initial conditions, according to the rationality assumption, the producers, adjusting their plans of action, also want to minimize the costs of transformation. They are not possible to observe and to measure in the Debreu economy. However, in the real economies, we can notice that introducing small changes in the production sector usually needs less expenses than introducing bigger changes. Hence, we can assume that every producer, in his own interest, independently on the others, will choose such procedure of modification of his technologies that will modify his feasible plans of action as little as possible. The projection on the space V, closest to the identity mapping in the adequate norm, satisfies the above requirements.
If the criterion of the choice of the trajectory of changes is different, then the planner of economic life is obligated to formulate, constitute and enforce the proper rules or regulations in order to the desired transformation will be realized.

5 Conclusions

If there is no equilibrium in the Debreu economy with linear consumption sets, then the results of theorems 4.1 - 4.3 provide additional incentives to producers or to a manager of the production sector to change producers' activities. The production

plans at time $t \in (0, 1)$, changed by the rule described in theorem 3.1, do not always satisfy the desired dependency between the quantities of commodities, but production plans at point $t \in (0, 1)$ are closer (in the sense of the distance) to subspace containing the consumption sets than the production plans at point $t = 0$.
The defined producers' adjustment trajectories, without destroying of the consumption sphere and without reducing the producers' profits, indicate the path of evolution of the economy in the direction of equilibrium under the assumption that Walras Law is fulfilled. The elaboration of the evolution of the economic system when Walras Law is not valid still remains within our research plans.

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