

Problem of the Moving Boundary in Continuous Casting Solved by the Analytic-Numerical Method

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Abstract

Mathematical modeling of thermal processes combined with the reversible phase transitions of type: solid phase – liquid phase leads to formulation of the parabolic or elliptic moving boundary problem. Solution of such defined problem requires, most often, to use some sophisticated numerical techniques and far advanced mathematical tools. The paper presents an analytic-numerical method, especially attractive from the engineer's point of view, applied for finding the approximate solutions of the selected class of problems which can be reduced to the one-phase solidification problem of a plate with the unknown a priori, varying in time boundary of the region in which the solution is sought. Proposed method is based on the known formalism of initial expansion of a sought function, describing the field of temperature, into the power series, some coefficients of which are determined with the aid of boundary conditions, and on the approximation of a function defining the freezing front location with the broken line, parameters of which are determined numerically. The method represents a combination of the analytical and numerical techniques and seems to be an effective and relatively easy in using tool for solving problems of considered kind.

Keywords: Application of information technology to the foundry industry, Solidification process, Numerical techniques, Moving boundary problem

1. Introduction

Designing of the technology for producing the ingots in course of continuous casting process appears to be a complicated and multistage problem. One of the most important factors determining the quality of continuous ingot, which can serve as a measure of effectiveness of the designed technology, is the field of temperature in the solidifying metal volume defined by an important parameter for the considered technology which is the location of freezing front determining the thickness of solidified layer (thickness of the ingot skin). Too fast either too slow increase of the solidified layer is unacceptable. If the skin of solidified ingot leaving the casting mould will be too thin then it can break and the liquid metal can leak which may cause a very

serious damage of the continuous casting equipment. From the other hand, too fast increase of the skin is connected with the excessive drop of temperature on its cross-section which may cause the high thermal tension leading to the ingot cracking.

Taking these facts into account we will consider the flat ingot of the rectangular cuboid shape, produced in the vertical continuous casting equipment with the constant ingot forming velocity. Dimensions of its cross-section sides $2\bar{x}$ and $2\bar{y}$ ($2\bar{x}$ – thickness of the ingot, $2\bar{y}$ – width of the ingot) satisfy condition $2\bar{x} \ll 2\bar{y}$. This assumption enables to discuss the solidifying ingot as an axisymmetrical 2-dimensional object in which the thermal processes take place in the surface of thermal symmetry (see Figure 1).

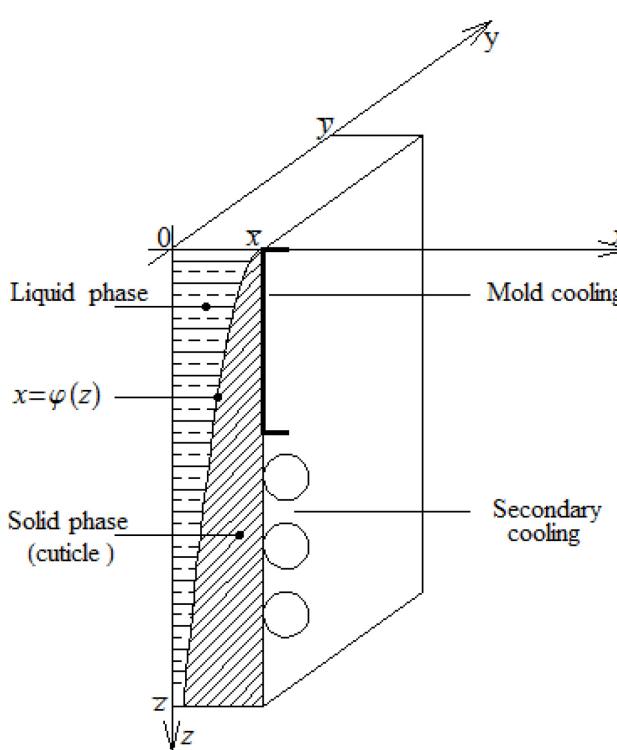


Fig. 1. The modeled area

If we assume simultaneously that the ingot is produced from the metal solidifying in constant temperature T^* and in such temperature is poured into the casting mould, then in the course of non-failure working of the continuous casting equipment the pseudo-steady temperature field in the solidified part of the ingot of length \bar{z} is generated which, in the coordinate system oriented in space like in Figure 1, is described by means of the equation

$$\nu \frac{\partial T}{\partial z} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right), \quad \varphi(z) < x < \bar{x}, \quad 0 < z \leq \bar{z}, \quad (1)$$

where $T = T(x, z)$ denotes the temperature, ν refers to the velocity vector coordinate in the direction of ingot forming, α is the thermal diffusivity coefficient and $\varphi(z)$ denotes a function describing the freezing front location

$$\varphi(z) = \bar{x} - \xi(z), \quad (2)$$

where $\xi(z)$ denotes the ingot skin thickness (thickness of the solidified layer) variable on the ingot length and

$$\xi(0) = 0. \quad (3)$$

In view of taken assumptions equation (1) is complemented by the boundary conditions on the freezing front

$$\lambda \text{grad } T|_{x=\varphi(z)} = \gamma \kappa v_n, \quad 0 \leq z \leq \bar{z}, \quad (4)$$

$$T|_{x=\varphi(z)} = T^*, \quad 0 \leq z \leq \bar{z}, \quad (5)$$

where n denotes the directed outside unitary vector normal to the freezing front, and by one of the conditions defined on the ingot surface

$$T|_{x=\bar{x}} = f, \quad 0 < z \leq \bar{z} \quad (6)$$

or

$$-\lambda \frac{\partial T}{\partial x}|_{x=\bar{x}} = q, \quad 0 < z \leq \bar{z} \quad (7)$$

or, relatively

$$-\lambda \frac{\partial T}{\partial x}|_{x=\bar{x}} = \alpha(T|_{x=\bar{x}} - T^\infty), \quad 0 < z \leq \bar{z}. \quad (8)$$

In the last equations symbols T^∞ , λ , ν and α denote, in turn, the ambient temperature, thermal conductivity coefficient, metal density and latent heat. Whereas, elements $f = f(z)$, $q = q(z)$, $\alpha = \alpha(z)$ and $v_n = v_n(z)$ define, respectively, temperature of the ingot surface, distribution of the heat flux, distribution of the heat transfer coefficient and coordinate of the velocity vector of the freezing front moving in direction normal to this front.

Equation (1) can be simplified if the ingot is produced from the material with the low value of thermal conductivity coefficient. It is because, in this case, the thermal conductivity in direction of the ingot forming is usually small, therefore it can be neglected [1]. Taking it into account, the term $\frac{\partial^2 T}{\partial z^2}$ in equation (1) can be ignored. In result, in place of elliptic equation (1) we receive the parabolic equation

$$\nu \frac{\partial T}{\partial z} = \alpha \frac{\partial^2 T}{\partial x^2}, \quad \varphi(z) < x < \bar{x}, \quad 0 < z \leq \bar{z}, \quad (9)$$

in which the variable z plays the role of time. Whereas, the boundary condition (4) on the freezing front takes the form

$$-\lambda \frac{\partial T}{\partial x}|_{x=\varphi(z)} = \gamma \kappa v_n, \quad 0 \leq z \leq \bar{z}. \quad (10)$$

2. Method of solution

Mathematical modeling of thermal processes combined with the reversible phase transitions of type: liquid phase – solid phase leads to the moving boundary problems which takes place in this case as well. Solving of such determined problem requires to use the appropriate numerical techniques [2-8] or approaches applying, very modern in recent times, genetic algorithms [9] or algorithms of artificial intelligence [10]. In the current paper we present the approximate analytic-numerical method, especially attractive from the engineer's point of view. Proposed method is based on the known formalism of initial expansion of the sought function, describing the field of temperature, into the power series, some number of coefficients of which is determined with the aid of boundary conditions, and on the approximation of function defining the freezing front location by means of the broken line, parameters of which are numerically determined.

As we have previously mentioned, method of solving such formulated problem is based, in the first step, on the proper presentation of the function representing the expected solution in the form of power series, similarly as it was done in papers [11-16]. In considered case the series is of the following form

$$T(x,z) = \sum_{i=0}^{\infty} \frac{A_i(z)(x - \bar{x} + \xi(z))^i}{i!}, \quad (11)$$

where $A_i(z)$ denote the unknown, dependent on variable z , functional coefficients. In case of the elliptic problem we determine these coefficients by using equation (1), condition (5) and the transformed condition (4) which can be written in form

$$\lambda \left(\frac{\partial T}{\partial x} \Big|_{x=\varphi(z)} - \varphi'(z) \frac{\partial T}{\partial z} \Big|_{x=\varphi(z)} \right) = \gamma \kappa v \varphi'(z), \quad 0 \leq z \leq \bar{z}, \quad (12)$$

Whereas, in case of the parabolic problem we use equation (9) and conditions (5) and (10) on the freezing front.

From the assumed form (11) of the sought solution and from relation (2) it results that

$$\frac{\partial T}{\partial x} = \sum_{i=0}^{\infty} \frac{A_{i+1}(z)(x - \bar{x} + \xi(z))^i}{i!}, \quad (13)$$

$$\frac{\partial T}{\partial z} = \sum_{i=0}^{\infty} (A'_i(z) + A_{i+1}(z)\xi'(z)) \frac{(x - \bar{x} + \xi(z))^i}{i!}, \quad (14)$$

$$\frac{\partial^2 T}{\partial x^2} = \sum_{i=0}^{\infty} \frac{A_{i+2}(z)(x - \bar{x} + \xi(z))^i}{i!}, \quad (15)$$

$$\frac{\partial^2 T}{\partial z^2} = \sum_{i=0}^{\infty} (A''_i(z) + 2A'_{i+1}(z)\xi'(z) + A_{i+1}(z)\xi''(z) + A_{i+2}(z)(\xi'(z))^2) \frac{(x - \bar{x} + \xi(z))^i}{i!}. \quad (16)$$

By substituting the properly received formulas into equation (1) we obtain

$$\begin{aligned} v \sum_{i=0}^{\infty} (A'_i(z) + A_{i+1}(z)\xi'(z)) \frac{(x - \bar{x} + \xi(z))^i}{i!} &= \\ &= a \sum_{i=0}^{\infty} (A_{i+2}(z) + A''_i(z) + 2A'_{i+1}(z)\xi'(z) + \\ &+ A_{i+1}(z)\xi''(z) + A_{i+2}(z)(\xi'(z))^2) \frac{(x - \bar{x} + \xi(z))^i}{i!}. \end{aligned} \quad (17)$$

Whereas, by substituting the same formulas into equation (9) we get the relation

$$\begin{aligned} v \sum_{i=0}^{\infty} (A'_i(z) + A_{i+1}(z)\xi'(z)) \frac{(x - \bar{x} + \xi(z))^i}{i!} &= \\ &= a \sum_{i=0}^{\infty} A_{i+2}(z) \frac{(x - \bar{x} + \xi(z))^i}{i!}. \end{aligned} \quad (18)$$

Comparing the terms situated on the both sides of equations (17) and (18) preceding expressions $\frac{(x - \bar{x} + \xi(z))^i}{i!}$, $i = 0, 1, 2, \dots$, we have

$$\begin{aligned} a(A''_i(z) + 2A'_{i+1}(z)\xi'(z) + A_{i+1}(z)\xi''(z) + \\ + A_{i+2}(z)(\xi'(z))^2) &= \\ &= v(A'_i(z) + A_{i+1}(z)\xi'(z)), \quad i = 0, 1, 2, \dots \end{aligned} \quad (19)$$

for the elliptic problem and

$$v(A'_i(z) + A_{i+1}(z)\xi'(z)) = aA_{i+2}(z), \quad i = 0, 1, 2, \dots \quad (20)$$

for the parabolic problem.

In case of the elliptic problem, conditions (5), (12) and (2) imply that

$$A_0(z) = T^*, \quad (21)$$

$$A_1(z) = -\frac{v \kappa v \xi'(z)}{\lambda((\xi'(z))^2 + 1)}. \quad (22)$$

Having coefficients $A_0(z)$ and $A_1(z)$ we can, by using formula (19), determine the remaining coefficients $A_i(z)$, $i = 2, 3, 4, \dots$. We get

$$A_{i+2}(z) = \frac{v(A'_i(z) + A_{i+1}(z)\xi'(z)) - a(A''_i(z) + 2A'_{i+1}(z)\xi'(z) + A_{i+1}(z)\xi''(z))}{a(1 + (\xi'(z))^2)}, \quad (23)$$

$$i = 0, 1, 2, \dots$$

From the obtained formulas for coefficients $A_i(z)$, $i = 0, 1, 2, \dots$, we can conclude that all the coefficients $A_i(z)$, $i = 1, 2, 3, \dots$, except coefficient $A_0(z)$, depend on the still unknown function $\xi(z)$, its derivatives and powers of those derivatives. One can try to determine analytically this function by using one of conditions (6), (7) or (8). In particular, for condition (6) we receive

$$\sum_{i=0}^{\infty} A_i(z) \frac{(\xi(z))^i}{i!} = f(z). \quad (24)$$

However, equation (24) is so much complicated that determination of function $\xi(z)$ with the aid of this equation is possible only in case of its certain simplification. In particular, by taking only two first terms of the series in relation (24) we obtain differential equation of the form

$$T^* - \frac{v \kappa v \xi'(z) \xi(z)}{\lambda((\xi'(z))^2 + 1)} = f(z). \quad (25)$$

Since function $\xi(z)$ is increasing by assumption, after simple transformations it results from relation (25) that

$$\xi'(z) = \frac{v \kappa v \xi(z) + \sqrt{(\gamma \kappa v \xi(z))^2 - 4 \lambda^2 (f(z) - T^*)^2}}{2 \lambda (T^* - f(z))}. \quad (26)$$

Unfortunately, an analytic solution of equation (26) is not possible for arbitrarily given function $f(z)$.

Problem of determining function $\xi(z)$ is even more complicated if we consider the boundary conditions of second and third kind ((7) and (8), respectively). For finding analytic solution in these cases, similar necessary simplifications must be made.

For example, considering condition (7) of the second kind we get the equation

$$-\lambda \sum_{i=0}^{\infty} \frac{A_{i+1}(z)(\xi(z))^i}{i!} = q(z). \quad (27)$$

By taking only one term (the first one) of the series in relation (27) we obtain

$$\frac{\gamma \kappa v \xi'(z)}{(\xi'(z))^2 + 1} = q(z). \quad (28)$$

Hence, after simple transformations the following equation results

$$\xi'(z) = \frac{\gamma \kappa v + \sqrt{(\gamma \kappa v)^2 - 4(q(z))^2}}{2q(z)}. \quad (29)$$

Equation (29), similarly as equation (26), will have the explicit solution only if function $q(z)$ will have the appropriate form.

Similar relations can be received while considering the parabolic equation. In this case, conditions (5), (10) and (2) imply that

$$A_0(z) = T^*, \quad (30)$$

$$A_1(z) = -\frac{\gamma \kappa v}{\lambda} \xi'(z). \quad (31)$$

Analogically as in previous case, by having the coefficients $A_0(z)$ and $A_1(z)$ we can calculate the remaining coefficients $A_i(z)$, $i = 2, 3, 4, \dots$, by using formula (20). We obtain

$$A_{i+2}(z) = \frac{v}{a} (A_i'(z) + A_{i+1}(z) \xi'(z)), \quad i = 0, 1, 2, \dots \quad (32)$$

Also in here, all the coefficients $A_i(z)$, $i = 1, 2, 3, \dots$, except coefficient $A_0(z)$, depend on the still unknown function $\xi(z)$, its derivatives and powers of those derivatives.

Whereas the equations connecting function $\xi(z)$ with the functions defining the heat transfer on the ingot surface remain unchanged with accuracy to coefficients $A_i(z)$, $i = 1, 2, 3, \dots$. In particular we get

$$\sum_{i=0}^{\infty} A_i(z) \frac{(\xi(z))^i}{i!} = f(z) \quad (33)$$

for boundary condition (6) of the first kind,

$$-\lambda \sum_{i=0}^{\infty} \frac{A_{i+1}(z)(\xi(z))^i}{i!} = q(z) \quad (34)$$

for boundary condition (7) of the second kind and

$$-\lambda \sum_{i=0}^{\infty} \frac{A_{i+1}(z)(\xi(z))^i}{i!} = \alpha(z) \left(\sum_{i=0}^{\infty} \frac{A_i(z)(\xi(z))^i}{i!} - T^* \right) \quad (35)$$

for boundary condition (8) of the third kind.

Similarly as in case of the elliptic problem, the received equations are so much complicated that the analytic determination

of function $\xi(z)$ by using these equations is possible only with some limitations and by applying appropriate simplifications.

In particular, by taking only three first terms in the series from relation (33) we get

$$A_0(z) + A_1(z)\xi(z) + \frac{1}{2}A_2(z)\xi^2(z) = f(z). \quad (36)$$

Hence, by using formulas (30)-(32) we obtain the following differential equation

$$\begin{aligned} &T^* - \\ &\frac{\gamma \kappa v}{\lambda} \xi'(z) \xi(z) - \\ &\frac{\gamma \kappa v^2}{2a\lambda} (\xi'(z)\xi(z))^2 = \\ &f(z). \end{aligned} \quad (37)$$

By solving this equation, on the assumption of satisfying condition (3), finally we receive

$$\xi(z) = \sqrt{\frac{2}{\gamma \kappa v} \int_0^z \left(\sqrt{(ay\kappa)^2 + 2a\lambda y\kappa(T^* - f(\tau))} - \frac{a}{v} \right) d\tau}. \quad (38)$$

In spite of the fact that in some cases, like for example when $T = T^0 = \text{constans}$, $0 \leq z \leq \bar{z}$, we can receive from equation (38) very simple formula

$$\xi(z) = \sqrt{\frac{2}{\gamma \kappa v} \left(\sqrt{(ay\kappa)^2 + 2a\lambda y\kappa(T^* - T^0)} - \frac{a}{v} \right) z}. \quad (39)$$

in general case relation (38) is not very suitable since we are not always able to calculate the integral appearing in this relation.

Approximate solution of this problem, as well as of the other problems, can be obtained with no difficulties if we assume that function defining the freezing front location is approximated by the broken line (Figure 2), it means

$$\varphi(z) = \bar{x} - \bar{\xi}(z), \quad (40)$$

where

$$\tilde{\xi}(z) = x_j + m_j(z - z_j), \quad z \in (z_j, z_{j+1}), \quad j = 0, 1, 2, \dots \quad (41)$$

and we have

$$x_0 = 0, \quad z_{j+1} > z_j, \quad j = 0, 1, 2, \dots \quad (42)$$

and

$$x_0 = 0, \quad x_j = m_{j-1}(z_j - z_{j-1}) + x_{j-1}, \quad j = 1, 2, 3, \dots \quad (43)$$

where parameters $m_j, j = 0, 1, 2, \dots$, will be determined numerically.

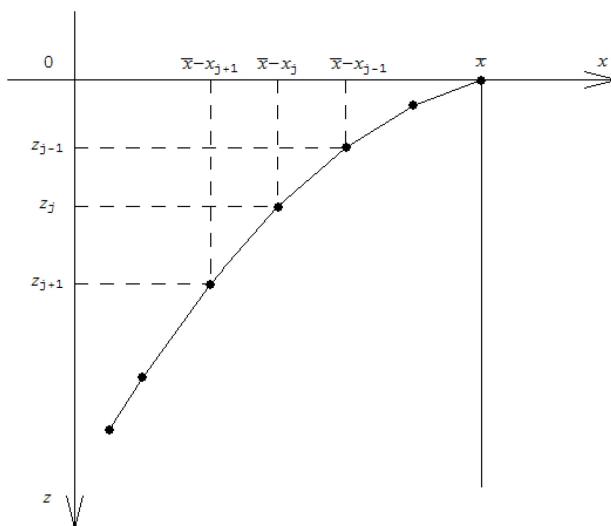


Fig. 2. Approximation of the function describing the freezing front location

Taken assumption implies that $\xi(z) = \tilde{\xi}(z)$, thus

$$\xi'(z) = m_j, \quad z \in (z_j, z_{j+1}), \quad j = 0, 1, 2, \dots \quad (44)$$

From formulas (21), (22) and relation (23) we get

$$A_0(z) = T^*, \quad z \in (z_j, z_{j+1}), \quad j = 0, 1, 2, \dots \quad (45)$$

$$A_1(z) = -\frac{\gamma \kappa v m_j}{\lambda(m_j^2 + 1)}, \quad z \in (z_j, z_{j+1}), \quad j = 0, 1, 2, \dots \quad (46)$$

$$A_{i+2}(z) = \frac{-\gamma \kappa (v m_j)^{1+i}}{\lambda^{i+1} (m_j^2 + 1)^{1+i}}, \quad z \in (z_j, z_{j+1}), \quad i = 0, 1, 2, \dots, j = 0, 1, 2, \dots \quad (47)$$

which means that function (11) describing the temperature field in the solid state can be presented in form

$$T(x, z) = T^* - \frac{\gamma \kappa a}{\lambda} \left(\exp \left(\frac{v m_j (x - \bar{x} + x_j + m_j(z - z_j))}{a(m_j^2 + 1)} \right) - 1 \right), \quad (48)$$

$$z \in (z_j, z_{j+1}), \quad j = 0, 1, 2, \dots$$

It can be easily verified that function (48) satisfies conditions (4) and (5) and its unknown elements are only parameters $m_j, j = 0, 1, 2, \dots$, for calculation of which one of the boundary conditions (6)-(8) will be used.

If we demand that for each $z_{j+1}, j = 0, 1, 2, \dots$, one of the boundary conditions (6), (7) or (8) is satisfied, we receive the equation enabling to determine the sought parameters. In particular, by applying condition (6) we have

$$f(z_{j+1}) = T^* - \frac{\gamma \kappa a}{\lambda} \left(\exp \left(\frac{v m_j (x_j + m_j(z_{j+1} - z_j))}{a(m_j^2 + 1)} \right) - 1 \right), \quad (49)$$

$$j = 0, 1, 2, \dots$$

By proceeding in similar way we can also determine the forms of function defining the freezing front location for the conditions of second (7) and third (8) kind. In particular, for condition (7) of the second kind we have

$$q(z_{j+1}) = \frac{\gamma \kappa v m_j}{m_j^2 + 1} \exp \left(\frac{v m_j (x_j + m_j(z_{j+1} - z_j))}{a(m_j^2 + 1)} \right), \quad (50)$$

$$j = 0, 1, 2, \dots$$

whereas, for condition (8) of the third kind we get

$$\begin{aligned} \frac{\gamma \kappa v m_j}{m_j^2 + 1} \exp \left(\frac{v m_j (x_j + m_j(z_{j+1} - z_j))}{a(m_j^2 + 1)} \right) = \\ = \alpha(z_{j+1}) \left(T^* - T^{\infty} - \frac{\gamma \kappa a}{\lambda} \times \right. \\ \left. \times \left(\exp \left(\frac{v m_j (x_j + m_j(z_{j+1} - z_j))}{a(m_j^2 + 1)} \right) - 1 \right) \right), \quad (51) \end{aligned}$$

$$j = 0, 1, 2, \dots$$

For each of the conditions (6)-(8) parameters $m_j, j = 0, 1, 2, \dots$, are described by means of equations which cannot be solved analytically. Equations (49)-(51) can be solved by applying one of the many methods for approximate determination of the roots of nonlinear equations.

3. Example

In paper [16] the usefulness of described method is showed on theoretical examples in which the reconstructed, variable on the ingot length, thickness of the ingot skin $\xi(z)$ was defined by linear or nonlinear relation. Now let us present the real example.

Let us assume that in the vertical continuous casting equipment a plate of thickness $2\bar{x} = 0.2$ [m] is casted. Material of the plate is characterized by the following parameters: density $\gamma = 7000$ [kg/m³], thermal conductivity coefficient $\lambda = 25$ [W/mK], latent heat $\kappa = 247$ [kJ/kg], solidification temperature $T^* = 1500$ [K], ambient temperature $T^{\infty} = 320$ [K], velocity vector coordinate in the direction of ingot forming $v = 0.6$ [m/min] and the heat transfer with environment is defined by means of boundary condition of the third kind (8), where

$$\alpha(z) = \begin{cases} 1500, & 0 \leq z \leq 0.8, \\ 1000, & 0.8 < z \leq 1.4, \\ 800, & 1.4 < z \leq 3.4, \\ 500, & 3.4 < z \leq 7.4, \\ 200, & 7.4 < z \leq 17. \end{cases} \quad (52)$$

Moreover, let us assume that we consider the process of ingot solidification until the moment in which the ingot reaches the length $z^* = vt^* = 17$ [m].

Another important parameter is the discretization density of variable z . In this paper we take that considered interval $(0, z^*)$ is evenly divided into m sections of length $\Delta z = z^*/m$ which means $z_j = j\Delta z$, $j = 0, 1, 2, \dots, m$.

The main object examined in testing calculations is the precision of reconstruction of function $\xi(z)$ describing the thickness of solidified layer variable in time. In considered example we investigate the elliptic and parabolic problem, as well as, in order to compare the results, the method presented in paper [15]. Reconstruction of function $\xi(z)$ obtained by applying the elliptic and parabolic problem, as well as the method presented in [15], is displayed in Figure 3. One can observe that all the reconstructions almost cover.

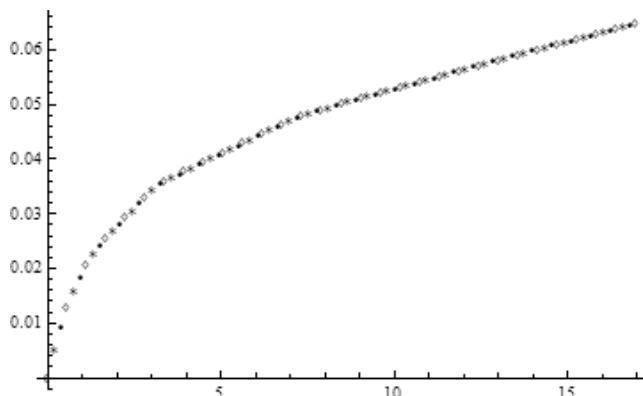


Fig. 3. Reconstruction of function $\xi(z)$, describing the thickness of solidified layer for the third kind boundary condition obtained in the elliptic problem (◊), parabolic problem (*) and method from paper [15] (●)

4. Conclusions

The paper presents the approximate analytic-numerical method of solving the selected kind of problems which can be reduced to the one-phase solidification problem of a plate with the unknown a priori, varying in time boundary of the region in which the solution is sought. Proposed method is based on the expansion of the sought function, describing the temperature field, into the power series, some coefficients of which are determined by using the boundary conditions, and on the approximation of function defining the freezing front location with the broken line, parameters of which are determined numerically. Numerical

verification of elaborated method confirms its efficiency in solving the elliptic as well as the parabolic problem.

References

- [1] Samojlowič, I.A. (1977). Formirowanie slitka, *Metallurgia*. Moskwa (in russian).
- [2] Piasecka Belkhayat, A. (2008). Numerical modeling of solidification process using interval boundary element method. *Archives of Foundry Engineering*. 8(4), 171-176.
- [3] Mochnacki, B. & Suchy, J.S. (1995). *Numerical methods in computations of foundry processes*, Kraków: PFTA.
- [4] Majchrzak, E., Mochnacki, B., Dziewoński, M. & Jasiński, M. (2008). Identification of boundary heat flux on the continuous casting surface. *Archives of Foundry Engineering*. 8(4), 105-110.
- [5] Mendakiewicz, J., Piasecka, Belkhayat, A & Szopa, R. (2000). Modeling of the Stefan Problem Using the BEM. *Solidification of Metals and Alloys*. 2(44), 223-228.
- [6] Mochnacki, B. & Pawlak, E. (2007). Identification of boundary condition on the contact surface of continuous casting mould. *Archives of Foundry Engineering*. 7(4), 202-206.
- [7] Majchrzak, E. & Witek, H. (1994). Analysis of complex casting solidification using combined boundary element. *Solidification of Metals and Alloys*. 19, 193-201.
- [8] Hetmaniok, E., Slota, D., Witula, R. & Zielonka, A. (2011). Comparison of the Adomian decomposition method and the variational iteration method in solving the moving boundary problem. *Computers and Mathematics with Applications*. 61, 1931-1934. DOI:10.1016/j.camwa.2010.07.050.
- [9] Slota, D. (2011). *Solving Inverse Problems of Solidification with the Use of Genetic Algorithm*. Gliwice, Silesian University of Technology Press (in Polish).
- [10] Hetmaniok, E., Slota, D., Zielonka, A. & Witula, R. (2012). Comparison of ABC and ACO Algorithms Applied for Solving the Inverse Heat Conduction Problem. *Lecture Notes in Computer Science*. 7269, 249-257. DOI: 10.1007/978-3-642-29353-5_29.
- [11] Ockendon, J.R. & Hodgkins, W.R. (1975). *Moving Boundary Problems in Heat Flow and Diffusion*, Oxford: Clarendon Press.
- [12] Wilson, D.G., Solomon, A.D. & Boggs, P.T. (1978). *Moving Boundary Problems*, New York: Academic Press.
- [13] Crank, J. (1984). *Free and Moving Boundary Problems*, Oxford : Clarendon Press.
- [14] Hetmaniok, E. & Pleszczyński, M. (2011). Analytical method of determining the freezing front location. *Zeszyty Naukowe Politechniki Śląskiej Matematyka Stosowana*. 1, 121-136.
- [15] Grzymkowski, R., Hetmaniok, E. & Pleszczyński, M. (2011). Analytic-numerical method of determining the freezing front location. *Archives of Foundry Engineering*. 11(3), 75-80.
- [16] Hetmaniok, E. & Pleszczyński, M. (2012). Application of an analytic-numerical method in solving the problem with a moving boundary. *Zeszyty Naukowe Politechniki Śląskiej Matematyka Stosowana*. 1, (in press).