10.1515/acsc-2015-0018

Archives of Control Sciences
Volume 25(LXI), 2015
No. 3, *pages 279–288*

Lyapunov matrices approach to the parametric optimization of time-delay systems

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In the paper a Lyapunov matrices approach to the parametric optimization problem of time-delay systems with a P-controller is presented. The value of integral quadratic performance index of quality is equal to the value of Lyapunov functional for the initial function of the time-delay system. The Lyapunov functional is determined by means of the Lyapunov matrix.

Key words: time-delay system, Lyapunov matrix, Lyapunov functional.

1. Introduction

There are papers which concerned parametric optimization problems for time-delay systems, see for instance [2,3,4]. The performance index of quality in the parametric optimization problem is calculated by means of quadratic Lyapunov functionals. The method of calculation of the Lyapunov functional was presented by Repin [7]. This method is used to obtain analytical formulas for the Lyapunov functional coefficients to calculate the Lyapunov functional value for the initial function of the time-delay system. This value is equal to the value of integral quadratic performance index of quality. In last years a method of determination of the Lyapunov functional by means of Lyapunov matrices is very popular, see for example [5]. In the paper a Lyapunov matrices approach to the parametric optimization problem of time-delay systems is presented.

2. Preliminaries

Let us consider a time-delay system

$$\begin{cases} \frac{dx(t)}{dt} = \sum_{j=0}^{m} A_j x(t-h_j) \\ x(\theta) = \varphi(\theta) \end{cases} \tag{1}$$

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Received 04.02.2015.

for $t \geqslant 0$, $\theta \in [-h,0]$, where $x(t) \in \mathbb{R}^n$, $A_j \in \mathbb{R}^{n\times n}$, $0 = h_0 < h_1 < ... < h_m = h$, function $\varphi \in PC([-h,0],\mathbb{R}^n)$ - the space of piece-wise continuous vector valued functions defined on the segment $[-h,0]$ with the uniform norm $\| \varphi \|_{PC} = \sup_{\theta\in[-h,0]} \| \varphi(\theta) \|$.

Let $x(t;\varphi)$ be the solution of system (1) with the initial function φ. The initial condition holds $x(\theta;\varphi) = \varphi(\theta)$ for $\theta \in [-h,0]$.

Definition 1 *$K(t)$ is the **fundamental matrix** of system (1) if it satisfies the matrix equation*

$$\frac{d}{dt}K(t) = \sum_{j=0}^{m} A_j K(t-h_j)$$

for $t \geqslant 0$ and the following initial condition $K(0) = I_{n\times n}$ and $K(t) = 0_{n\times n}$ for $t < 0$ where $I_{n\times n}$ is the identity $n\times n$ matrix and $0_{n\times n}$ is the zero $n\times n$ matrix.

Theorem 1 *(Bellman and Cooke [1]). Let $K(t)$ be the fundamental matrix of system (1), then for $t \geqslant 0$*

$$x(t,\varphi) = K(t)\varphi(0) + \sum_{j=1}^{m} \int_{-h_j}^{0} K(t-h_j-\theta)A_j\varphi(\theta)d\theta. \tag{2}$$

Definition 2 *The function $x_t(\varphi) : [-h,0] \to \mathbb{R}^n$ is called a **shifted restriction** of $x(\cdot;\varphi)$ to an interval $[t-h,t]$ and is defined by a formula*

$$x_t(\varphi)(\theta) := x(t+\theta;\varphi) \tag{3}$$

for $t \geqslant 0$ and $\theta \in [-h,0]$.

Definition 3 *The trivial solution of (1) is said to be **stable** if for any $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\| \varphi \|_{PC} \leqslant \delta \Rightarrow \| x(t;\varphi) \| \leqslant \varepsilon$$

for every $t \geqslant 0$.

Definition 4 *The trivial solution of (1) is said to be **asymptotically stable** if it is stable and $x(t;\varphi) \to 0$ as $t \to \infty$*

Definition 5 *The trivial solution of (1) is said to be **exponentially stable** if there exist $M \geqslant 1$ and $\sigma > 0$ such that for every solution $x(t;\varphi)$ of the system with initial function $\varphi \in PC([-h,0],\mathbb{R}^n)$ the following condition holds*

$$\| x(t;\varphi) \| \leqslant M \| \varphi \|_{PC}\, e^{-\sigma t}$$

for every $t \geqslant 0$.

3. A Lyapunov-Krasovskii functional

Given a symmetric positive definite matrix $W \in \mathbb{R}^{n\times n}$. We are looking for a functional $v : PC([-h,0],\mathbb{R}^n) \to \mathbb{R}$ such that along the solutions of system (1) the following equality holds

$$\frac{d}{dt}v(x_t(\varphi)) = -x^T(t;\varphi)Wx(t;\varphi) \tag{4}$$

for $t \geqslant 0$, where $x(t;\varphi)$ is a solution of system (1), with the initial function $\varphi \in PC([-h,0],\mathbb{R}^n)$, given by (2).

We assume that system (1) is asymptotically stable and integrates both sides of (4) from zero to infinity. We obtain

$$v(\varphi) = \int_0^\infty x^T(t;\varphi)Wx(t;\varphi)dt. \tag{5}$$

Taking into account (2) we calculate the integral of the right-hand side of (5)

$$\int_0^\infty x^T(t;\varphi)Wx(t;\varphi)dt = \varphi^T(0)\int_0^\infty K^T(t)WK(t)dt\varphi(0)+$$

$$+\sum_{j=1}^{m}\int_{-h_j}^{0} 2\varphi^T(0)\int_0^\infty K^T(t)WK(t-h_j-\theta)dtA_j\varphi(\theta)d\theta+$$

$$+\sum_{j=1}^{m}\sum_{k=1}^{m}\int_{-h_j}^{0}\varphi^T(\theta)A_j^T\int_{-h_k}^{0}\int_0^\infty K^T(t-h_j-\theta)WK(t-h_k-\eta)dtA_k\varphi(\eta)d\eta d\theta. \tag{6}$$

There holds a relation

$$\int_0^\infty K^T(t-h_j-\theta)WK(t-h_k-\eta)dt = \int_{-h_j-\theta}^{\infty} K^T(\varsigma)WK(\varsigma+h_j-h_k+\theta-\eta)d\varsigma =$$

$$= \int_0^\infty K^T(\varsigma)WK(\varsigma+h_j-h_k+\theta-\eta)d\varsigma.$$

We introduce **a Lyapunov matrix**

$$U(\xi) = \int_0^\infty K^T(t) W K(t+\xi) dt. \tag{7}$$

Using the Lyapunov matrix (7) we attain a formula for the functional $v(\varphi)$

$$v(\varphi) = \varphi^T(0)U(0)\varphi(0) + 2\varphi^T(0) \sum_{j=1}^{m} \int_{-h_j}^{0} U(-\theta - h_j) A_j \varphi(\theta) d\theta +$$

$$+ \sum_{j=1}^{m} \sum_{k=1}^{m} \int_{-h_j}^{0} \varphi^T(\theta) A_j^T \int_{-h_k}^{0} U(h_j - h_k + \theta - \eta) A_k \varphi(\eta) d\eta d\theta. \tag{8}$$

Corollary 1 *The Lyapunov matrix* (7) *satisfies the following properties* [5]*:*

Dynamic property

$$\frac{d}{d\xi} U(\xi) = \sum_{j=0}^{m} U(\xi - h_j) A_j \tag{9}$$

for $\xi \geqslant 0$.

Symmetry property

$$U(-\xi) = U^T(\xi) \tag{10}$$

for $\xi \geqslant 0$.

Algebraic property

$$\sum_{j=0}^{m} [U(-h_j) A_j + A_j^T U(h_j)] = -W. \tag{11}$$

Formulas (9), (10), (11) enable us to calculate the Lyapunov matrix (7).

Definition 6 *A functional* $v : PC([-h,0],\mathbb{R}^n) \to \mathbb{R}_+$ *is called* ***a Lyapunov-Krasovskii functional*** *for* (1) *if it has the following properties: there exist* $\alpha_1, \alpha_2 > 0$ *and* $\beta > 0$ *such that*

$$\alpha_1 \parallel x(t;\varphi) \parallel^2 \leqslant v(x_t(\varphi)) \leqslant \alpha_2 \parallel x_t(\varphi) \parallel_{PC}^2$$

and

$$\frac{d}{dt} v(x_t(\varphi)) \leqslant -\beta \parallel x(t;\varphi) \parallel^2 .$$

Theorem 2 *(Krasovskii* [6]*) System* (1) *is exponentially stable if there exists a Lyapunov-Krasowskii functional for* (1)*.*

4. A Lyapunov matrix for a system with one delay

Let us consider a system

$$\begin{cases} \frac{dx(t)}{dt} = A_0 x(t) + A_1 x(t-h) \\ x(\theta) = \varphi(\theta) \end{cases} \tag{12}$$

for $t \geqslant 0$ and $\theta \in [-h,0]$. Where $A_0, A_1 \in \mathbb{R}^{n\times n}$ and $\varphi \in PC([-h,0],\mathbb{R}^n)$, $0 < h \in \mathbb{R}$.

A system of equations (9), (10), (11) takes the form

$$\frac{d}{d\xi}U(\xi) = U(\xi)A_0 + U(\xi-h)A_1 \tag{13}$$

$$U(-\xi) = U^T(\xi) \tag{14}$$

$$U(0)A_0 + U(-h)A_1 + A_0^T U(0) + A_1^T U(h) = -W \tag{15}$$

for $\xi \in [0,h]$.

Formula (14) implies

$$U(\xi-h) = U^T(h-\xi) = Z(\xi). \tag{16}$$

We compute the derivative of $Z(\xi)$

$$\frac{d}{d\xi}Z(\xi) = \frac{d}{d\xi}U^T(h-\xi) = -A_0^T U^T(h-\xi) - A_1^T U^T(-\xi) =$$

$$= -A_0^T Z(\xi) - A_1^T U(\xi). \tag{17}$$

We have received a set of ordinary differential equations

$$\begin{cases} \frac{d}{d\xi}U(\xi) = U(\xi)A_0 + Z(\xi)A_1 \\ \frac{d}{d\xi}Z(\xi) = -A_0^T Z(\xi) - A_1^T U(\xi) \end{cases} \tag{18}$$

for $\xi \in [0,h]$ with initial condition $U(0)$, $Z(0)$.

Formula (16) implies

$$U(-h) = U^T(h) = Z(0). \tag{19}$$

Taking (19) into account equation (15) takes a form

$$U(0)A_0+Z(0)A_1+A_0^TU(0)+A_1^TZ^T(0)=-W. \tag{20}$$

Using the Kronecker product we can express (18) in a form

$$\begin{bmatrix} \frac{d}{d\xi}\text{col}\,U(\xi) \\ \frac{d}{d\xi}\text{col}\,Z(\xi) \end{bmatrix} = \begin{bmatrix} A_0^T\otimes I & A_1^T\otimes I \\ -I\otimes A_1^T & -I\otimes A_0^T \end{bmatrix} \begin{bmatrix} \text{col}\,U(\xi) \\ \text{col}\,Z(\xi) \end{bmatrix} \tag{21}$$

for $\xi\in[0,h]$ with initial condition col $U(0)$, col $Z(0)$.

Formula (20) can be expressed

$$(A_0^T\otimes I+I\otimes A_0^T)\text{col}\,U(0)+(A_1^T\otimes I)\text{col}\,Z(0)+(I\otimes A_1^T)\text{col}\,Z^T(0)=-\text{col}\,W. \tag{22}$$

Solution of equation (21) is given

$$\begin{bmatrix} \text{col}\,U(\xi) \\ \text{col}\,Z(\xi) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(\xi) & \Phi_{12}(\xi) \\ \Phi_{21}(\xi) & \Phi_{22}(\xi) \end{bmatrix} \begin{bmatrix} \text{col}\,U(0) \\ \text{col}\,Z(0) \end{bmatrix} \tag{23}$$

where a matrix $\Phi(\xi)=\begin{bmatrix} \Phi_{11}(\xi) & \Phi_{12}(\xi) \\ \Phi_{21}(\xi) & \Phi_{22}(\xi) \end{bmatrix}$ is a fundamental matrix of system (21).

We determine the initial conditions col $U(0)$, col $Z(0)$. The term (16) implies $U(h)=Z^T(0)$ and $Z(h)=U^T(0)=U(0)$.

From (23) we obtain

$$\text{col}\,U(h)=\text{col}\,Z^T(0)=\Phi_{11}(h)\text{col}\,U(0)+\Phi_{12}(h)\text{col}\,Z(0) \tag{24}$$

$$\text{col}\,Z(h)=\text{col}\,U(0)=\Phi_{21}(h)\text{col}\,U(0)+\Phi_{22}(h)\text{col}\,Z(0). \tag{25}$$

We put (24) into (22) and reshape (25). In this way we attain a set of algebraic equations which enables us to calculate the initial conditions of (23).

$$\begin{aligned} &[A_0^T\otimes I+I\otimes A_0^T+(I\otimes A_1^T)\Phi_{11}(h)]\text{col}\,U(0) \\ &+[A_1^T\otimes I+(I\otimes A_1^T)\Phi_{12}(h)]\text{col}\,Z(0)=-\text{col}\,W \end{aligned} \tag{26}$$

$$[I-\Phi_{21}(h)]\text{col}\,U(0)-\Phi_{22}(h)\text{col}\,Z(0)=0. \tag{27}$$

5. Formulation of the parametric optimization problem

Let us consider a time-delay system with a P-controller

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + Bu(t-h) \\ u(t) = -Kx(t) \\ x(\theta) = \varphi(\theta) \end{cases} \tag{28}$$

for $t \geqslant 0$ and $\theta \in [-h, 0]$. Where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $K \in \mathbb{R}^{p\times n}$ is a P-controller gain and $\varphi \in PC([-h,0],\mathbb{R}^n)$, $0 < h \in \mathbb{R}$.

System (28) can be written in an equivalent form

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) - BKx(t-h) \\ x(\theta) = \varphi(\theta). \end{cases} \tag{29}$$

In parametric optimization problem will be used the performance index of quality

$$J = \int_0^\infty x^T(t;\varphi)Wx(t;\varphi)dt \tag{30}$$

where $W \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix and $x(t;\varphi)$ is a solution of (29) for initial function φ.

Problem 1 *Determine the matrix $K \in \mathbb{R}^{p\times n}$ whose minimize an integral quadratic performance index of quality* (30).

According to (5) the value of the performance index of quality (30) is equal to the value of the functional (8) for initial function φ. To calculate the value of the functional (8) we need a Lyapunov matrix $U(\xi)$. To obtain a Lyapunov matrix $U(\xi)$ we solve a system of differential equations (21) and a set of algebraic equations (26) and (27) whose take a form

$$\begin{bmatrix} \frac{d}{d\xi}\mathrm{col}\,U(\xi) \\ \frac{d}{d\xi}\mathrm{col}\,Z(\xi) \end{bmatrix} = \begin{bmatrix} A^T \otimes I & -K^TB^T \otimes I \\ I \otimes K^TB^T & -I \otimes A^T \end{bmatrix} \begin{bmatrix} \mathrm{col}\,U(\xi) \\ \mathrm{col}\,Z(\xi) \end{bmatrix} \tag{31}$$

$$\begin{aligned} &[A^T \otimes I + I \otimes A^T - (I \otimes K^TB^T)\Psi_{11}(h)]\mathrm{col}\,U(0) \\ &-[K^TB^T \otimes I + (I \otimes K^TB^T)\Psi_{12}(h)]\mathrm{col}\,Z(0) = -\mathrm{col}\,W \end{aligned} \tag{32}$$

$$[I - \Psi_{21}(h)]\mathrm{col}\,U(0) - \Psi_{22}(h)\mathrm{col}\,Z(0) = 0 \tag{33}$$

where $\Psi(\xi) = \begin{bmatrix} \Psi_{11}(\xi) & \Psi_{12}(\xi) \\ \Psi_{21}(\xi) & \Psi_{22}(\xi) \end{bmatrix}$ is the fundamental matrix of system (31).

6. Example

Let us consider an inertial system with a delay and a P-controller

$$\begin{cases} \frac{dx(t)}{dt} = -\frac{1}{T}x(t) + \frac{k_0}{T}u(t-h) \\ u(t) = -kx(t) \\ x(0) = x_0 \\ x(\theta) = 0 \end{cases} \tag{34}$$

$t \geqslant 0$, $x(t) \in \mathbb{R}$, $\theta \in [-h, 0)$, k, k_0, T, $x_0 \in \mathbb{R}$, $h \geqslant 0$. The parameter k_0 is a gain of a plant, k is a gain of a P-controller, T is a system time constant, x_0 is an initial state of a system.

One can reshape an equation (34) to a form

$$\begin{cases} \frac{dx(t)}{dt} = -\frac{1}{T}x(t) - \frac{k_0 k}{T}x(t-h) \\ x(0) = x_o \\ x(\theta) = 0 \end{cases} \tag{35}$$

for $t \geqslant 0$ and $\theta \in [-h, 0)$. The initial function φ is given

$$\varphi(\theta) = \begin{cases} x_0 & for\, \theta = 0 \\ 0 & for\, \theta \in [-h, 0). \end{cases} \tag{36}$$

In parametric optimization problem we use the performance index

$$J = \int_0^\infty wx^2(t;\varphi)dt \tag{37}$$

where $w > 0$ and $x(t;\varphi)$ is a solution of (35) for initial function (36).

The differential equation (31) takes a form

$$\begin{bmatrix} \frac{d}{d\xi}U(\xi) \\ \frac{d}{d\xi}Z(\xi) \end{bmatrix} = \begin{bmatrix} -\frac{1}{T} & -\frac{k_0 k}{T} \\ \frac{k_0 k}{T} & \frac{1}{T} \end{bmatrix} \begin{bmatrix} U(\xi) \\ Z(\xi) \end{bmatrix}. \tag{38}$$

The fundamental matrix of (38) is given

$$\Phi(\xi) = \begin{bmatrix} \cosh\lambda\xi - \frac{1}{\lambda T}\sinh\lambda\xi & -\frac{k_0 k}{\lambda T}\sinh\lambda\xi \\ \frac{k_0 k}{\lambda T}\sinh\lambda\xi & \cosh\lambda\xi + \frac{1}{\lambda T}\sinh\lambda\xi \end{bmatrix} \tag{39}$$

for $\xi \in [0, h]$, where

$$\lambda = \frac{1}{T}\sqrt{1 - k_0^2 k^2}. \tag{40}$$

The initial conditions for (38) are obtained from equations (32) and (33) which take a form

$$\begin{bmatrix} 2+k_0k(\cosh\lambda h-\frac{1}{\lambda T}\sinh\lambda h) & k_0k(1-\frac{k_0k}{\lambda T}\sinh\lambda h) \\ 1-\frac{k_0k}{\lambda T}\sinh\lambda h & -\cosh\lambda h-\frac{1}{\lambda T}\sinh\lambda h \end{bmatrix}\begin{bmatrix} U(0) \\ Z(0) \end{bmatrix}=\begin{bmatrix} Tw \\ 0 \end{bmatrix}. \tag{41}$$

Solving (41) we obtain

$$U(0)=\frac{\frac{Tw}{2}(\cosh\lambda h+\frac{1}{\lambda T}\sinh\lambda h)}{k_0k+\cosh\lambda h+\lambda T\sinh\lambda h} \tag{42}$$

$$Z(0)=\frac{\frac{Tw}{2}(1-\frac{k_0k}{\lambda T}\sinh\lambda h)}{k_0k+\cosh\lambda h+\lambda T\sinh\lambda h}. \tag{43}$$

The solution of (38) is given

$$U(\xi)=\frac{w}{2}\left(\frac{T\cosh\lambda h+\frac{1}{\lambda}\sinh\lambda h}{k_0k+\cosh\lambda h+\lambda T\sinh\lambda h}\cosh\lambda\xi-\frac{1}{\lambda}\sinh\lambda\xi\right) \tag{44}$$

$$\begin{aligned} Z(\xi)=&\frac{\frac{Tw}{2}}{k_0k+\cosh\lambda h+\lambda T\sinh\lambda h}\left((1-\frac{k_0k}{\lambda T}\sinh\lambda h)\cosh\lambda\xi\right. \\ &\left.+\frac{1}{\lambda T}(1+k_0k\cosh\lambda h)\sinh\lambda\xi\right). \end{aligned} \tag{45}$$

The value of the performance index (37) is equal to the value of functional (8) for $U(\xi)$ given by (44) and initial function given by (36)

$$J=\frac{\frac{Tw}{2}(\cosh\lambda h+\frac{1}{\lambda T}\sinh\lambda h)}{k_0k+\cosh\lambda h+\lambda T\sinh\lambda h}x_0^2. \tag{46}$$

We search for an optimal gain which minimize the index (46) for a given $x_0=1$, $w=1$ and $T=1$. Critical gain is a maximal admissible gain for system (35). System (35) is unstable for gains greater then critical gain. Optimization results are given in Tab. 1.

7. Conclusions

In the paper a Lyapunov matrices approach to the parametric optimization problem of time-delay systems is presented. The value of integral quadratic performance index of quality is equal to the value of Lyapunov functional for the initial function of the time-delay system. The Lyapunov functional is determined by means of the Lyapunov matrix.

Table 1: Optimization results

Delay h	*Optimal gain*	*Index value*	*Critical gain*
0.1	7.1	0.13	16.350
0.2	3.5	0.22	8.502
0.5	1.25	0.37	3.806
1	0.5	0.46	2.261
2	0.14	0.495	1.519
3	0.05	0.499	1.292

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