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10.1515/acsc-2015-0001

Archives of Control Sciences
Volume 25(LXI), 2015
No. 1*, pages 5–20*

Generalized control with compact support for systems with distributed parameters

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We propose a generalization of the Butkovskiy's method of control with compact support [1] allowing to derive exact controllability conditions and construct explicit solutions in control problems for systems with distributed parameters. The idea is the introduction of a new state function which is supported in considered bounded time interval and coincides with the original one therein. By means of techniques of the distributions theory the problem is reduced to an interpolation problem for Fourier image of unknown function or to corresponding system of integral equalities. Treating it as infinite dimensional problem of moments, its L^1, L^2 and L^∞-optimal solutions are constructed explicitly. The technique is explained for semilinear wave equation with distributed and boundary controls. Particular cases are discussed.

Key words: distributed system, controls with compact support, problem of moments, L^1, L^2 and L^∞-optimal controls, distributions.

1. Introduction

The answer to one of the most important problem of the control theory, *controllability*, i.e. ability to 'bring' the system from a given initial state into a given terminal state, often is not enough for construction of a particular control system in practice: sometimes explicit representation or at least characterization of the controls is required. Among existing techniques only a few may give answer to the question about the controllability and, at the same time, provide a method for constructing the required controls. One of those few methods is the method of A. G. Butkovskiy of control with compact support [1]. Assuming, that the control and the state functions are concentrated on some finite time-interval [2] where the control process is carried out, the Butkovskiy's method provide an efficient procedure for obtaining the system of necessary and sufficient conditions for the system exact controllability. On this way the well-known Wiener–Paley theorem [1] is used. Namely, those conditions become the system of restrictions where

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The application of this method in particular problems were done in collaboration with my friends and colleagues Am. Khurshudyan and Sh. Arakelyan, whom I heartily thankful.

To the blessed memory of innocent victims of Armenian Genocide (1915) is dedicated.

Received 7.10.2014. Revised 20.02.2015.

the control function must be determined from. In [1] two approaches are suggested for constructing the unknown controls. According to the first one the system of restrictions is reduced to interpolation problem with respect to the Fourier image of the unknown function. Interpolating it and applying Fourier inverse transform, one is able to find approximate expression of required controls. The second approach suggests to separate the real and the imaginary parts of the system of restrictions. This will lead us to an *infinite dimensional* problem of moments. Actually, by means of the Butkovskiy's method we obtain a whole class of admissible for exact controllability functions, therefore optimality conditions may be proposed.

Using the Fourier method of variables separation is also an accepted approach for solving control problems for systems with distributed parameters [1].

In this paper we propose a generalization of the Butkovskiy's method for investigating distributed systems which have not classical solutions and we have to deal with its distributional or weak solutions [2]. Unlike [1] instead of *assuming* the compactness of the state and the control functions support, we introduce a generalized (distributional) state function which is concentrated on the time-interval where the control process is carried out and write the governing system in terms of distributions. As a result, we may freely operate with it, apply the distributional Fourier transform which is justified for both regular and singular distributions [2, 3, 4]. Thereafter, applying the same procedure as above we arrive at a similar infinite dimensional system of necessary and sufficient conditions for the system exact controllability. For this purpose the Wiener–Paley–Schwartz theorem [1, 2, 3, 4] should be used. The two approaches of the controls determination hold in this case as well.

Actually, this idea is borrowed from Professor E. Kh. Grigoryan, who applied the mentioned strategy to solve the well–known problem of 'cork' [5], where the contact interaction between an elastic semi–plane and a finite inclusion terminating to the boundary of the semi–plane is investigated. To determine the tangential stresses in the contact area the author thought up a clever trick. Using the techniques of the theory of distributions, he replaced the finite inclusion by a semi–infinite piecewise–homogeneous one, the finite part of which has the same characteristics as the initial inclusion does, and the rest– as the semi–plane does. As a result he was able to obtain the unknown contact stresses explicitly.

The techniques is used previously in [6-11] to obtain explicit solutions in different control problems. Note, that a quite similar approach was used in [12].

To demonstrate the procedure we shortly consider boundary and distributed exact controllability of semilinear wave equations in a rectangle and a semi-infinite strip [6-20]. The case when the distributed and the boundary controls contains constant delay are also discussed.

The paper is organized as follows. The main concepts and notations used throughout the paper are brought in the first section. The statement of boundary and distributed exact controllability problem for general distributed system is discussed in the second section. The Butkovskiy's generalized method is outlined thereafter and two possibilities for solving the problem is proposed. Derivation of L^1, L^2 and L^∞-optimal solutions of the general problem explicitly is described in the Section 5. The main results obtained by proposed method in several particular problems are considered in the Section 6. An overall conclusion completes the presentation of the article.

2. Notations and abbreviations

$\overline{O}$ denotes the ordinary closure, ∂O– the boundary and $\mathcal{S}_O$ a closed subset of finite or infinite domain O. The set supp $\eta(x) = \overline{\{x \in \mathbb{R};\, \eta(x) = 0\}}$ denotes the support of a function $\eta : \mathbb{R} \to \mathbb{R}$. $\mathcal{D}[\cdot]$ denotes an operator and $\mathcal{D}^{\gamma}$– weak derivative. By $W^{q,p}$ we denote the Sobolev space $W^{p,q}(O) = \{\eta \in L^p(O);\, \mathcal{D}^q\eta \in L^p(O)\}$. We will use the notion $\{1;n\}$ instead of $\{1,2,\ldots,n\}$ for short. $\mathcal{F}_t[\cdot]$ denotes the operator of the distributional Fourier transform with respect to t [3]:

$$\mathcal{F}_t[\eta] = \int_{\mathbb{R}} \eta(t)\exp[i\sigma t]\,dt \equiv \overline{\eta}(\sigma),$$

$\sigma \in \mathbb{R}$ is the spectral parameter of the transform. The inverse distributional Fourier transform is denoted by $\mathcal{F}_t^{-1}[\cdot]$ [3]:

$$\mathcal{F}_t^{-1}[\overline{\eta}] = \frac{1}{2\pi}\int_{\mathbb{R}} \overline{\eta}(\sigma)\exp[-i\sigma t]\,d\sigma.$$

C_m^n are the binomial coefficients.

$$\theta(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0, \end{cases} \qquad \operatorname{sign} t = \begin{cases} 1, & t > 0, \\ -1, & t < 0, \end{cases} \qquad \delta(t) = \begin{cases} 0, & t \neq 0, \\ \infty, & t = 0, \end{cases} \qquad \chi_O(t) = \begin{cases} 1, & t \in O, \\ 0, & t \notin O, \end{cases}$$

are the Heaviside's unit step, the sign function, the Dirac's delta and the characteristic functions.

3. The general problem

Suppose we have to solve a control problem with a fix end-point for an abstract differential equation

$$\mathcal{D}[w] = f(x,t), \;\; (x,t) \in O \times \mathbb{R}^+, \tag{1}$$

subjected to boundary conditions

$$\mathcal{B}[w] = u_b(t), \;\; (x,t) \in \partial O \times \mathbb{R}^+. \tag{2}$$

Here $w : O \times \mathbb{R}^+ \to \mathbb{R}$ is the state function, $f : \mathcal{S}_O \times \mathbb{R}^+ \to \mathbb{R}$ is the right hand side. $\mathcal{D} : W^{p,q} \to L^1_{loc}$ is the state operator, $\mathcal{B} : W^{p,q} \to L^2$ is the operator of the boundary conditions: they both are supposed to have *stationary* or coordinate dependent coefficients.

The initial state of the system is supposed to be known:

$$\mathcal{D}^{\iota} w(x,t)\big|_{t=0} = w_0^{\iota}(x),\ \ \iota \in \{1;n-1\},\ x \in \overline{\mathcal{O}}, \tag{3}$$

n is the order of the highest time-derivative of the state function in (1).

The aim of the control problem is the ensuring of terminal condition

$$\mathcal{D}^{\iota} w(x,t)\big|_{t=T} = w_T^{\iota}(x),\ \ \iota \in \{1;n-1\},\ x \in \overline{\mathcal{O}}, \tag{4}$$

by means of an appropriate choice of control function. Functions (3) and (4) are chosen from appropriate Sobolev spaces (see [13] for details).

The control may be implemented either by means of the boundary function u_b or the right hand side f. In the first case the problem will be the explicit description of admissible boundary controls $u_b \in L^2[0,T]$. In the second case, supposing that $f(x,t) = \nu(x)u_d(t)$, where $0 < \nu \in L^1_{loc}(\mathcal{S}_O)$ describes the distribution of controls, we will seek admissible controls $u_d \in L^1[0,T]$.

Surely, those admissible controls are non-unique [1] and, therefore, question of finding an optimal control minimizing a particular cost functional is raised. Our attention will be paid to three particular functionals:

$$\kappa_1[u] = ||u||_{L^1[0,T]} \equiv \int_0^T |u(t)|dt, \tag{5a}$$

$$\kappa_2[u] = ||u||^2_{L^2[0,T]} \equiv \int_0^T u^2(t)dt, \tag{5b}$$

or

$$\kappa_\infty[u] = ||u||_{L^\infty[0,T]} \equiv \max_{t\in[0,T]} |u(t)|. \tag{5c}$$

The corresponding controls we shall call L^1, L^2 and L^∞-optimal controls.

4. Solution of the problem

The solution technique proposed in this paper is mainly based on the Butkovskiy's method of control with compact support and uses the idea of [5]. We introduce operator $\mathcal{A}_{[0,T]}[\cdot]$ which puts any ordinary function $\eta : [0,T] \to \mathbb{R}$ into the correspondence with compactly supported function $\eta_1 : \mathbb{R} \to \mathbb{R}$ concentrated on $[0,T]$:

$$\eta_1(t) \equiv \mathcal{A}_{[0,T]}[\eta] = \begin{cases} \eta(t), & t \in [0,T], \\ 0, & t \notin [0,T]. \end{cases}$$

The explicit form of $\mathcal{A}_{[0,T]}[\cdot]$ may be expressed in several manners, for instance by the characteristic function of $[0,T]$:

$$\mathcal{A}_{[0,T]}[\eta] = \chi_{[0,T]}(t)\eta(t), \ \ t \in \mathbb{R}.$$

According to [5] we may represent it by the Heaviside unit step function

$$\mathcal{A}_{[0,T]}[\eta] = [\theta(t) - \theta(t-T)]\,\eta(t), \ \ t \in \mathbb{R}.$$

In the sense of distributions [4, 2] we may differentiate η_1 arbitrary times.

To solve the control problem described in the last section we suggest to apply $\mathcal{A}_{[0,T]}[\cdot]$ to governing system (1), (2). First of all, we will be able to write it in the sense of distributions justifying all operators on the way of determining admissible controls. Taking into account the Leibniz rule of differentiation, we have

$$\mathcal{D}^n\eta_1(t) = \sum_{\iota=1}^{n} C_\iota^n \mathcal{D}^{n-\iota}\,[\theta(t) - \theta(t-T)]\,\mathcal{D}^\iota\eta(t), \ \ t \in \mathbb{R}.$$

From the other hand side, in the sense of distributions $\mathcal{D}\theta(t) = \delta(t)$, therefore

$$\mathcal{D}^n\eta_1(t) = [\theta(t) - \theta(t-T)]\,\mathcal{D}^n\eta(t) + \sum_{\iota=1}^{n-1} C_\iota^n \mathcal{D}^{n-\iota-1}\,[\delta(t) - \delta(t-T)]\,\mathcal{D}^\iota\eta(t), \ \ t \in \mathbb{R}.$$

Using the filtering property of the Dirac's delta [4, 2, 3], the second term at the right hand side is well-defined by means of initial and terminal data (3), (4). For instance, when $n = 1$

$$\frac{\partial w_1}{\partial t} = [\theta(t) - \theta(t-T)]\,\frac{\partial w}{\partial t} + w_0^1(x)\delta(t) + w_T^1(x)\delta(t-T), \ \ t \in \mathbb{R},$$

and when $n = 2$–

$$\frac{\partial^2 w_1}{\partial t^2} =$$

$$[\theta(t) - \theta(t-T)]\,\frac{\partial^2 w}{\partial t^2} + w_0^1(x)\delta'(t) - w_T^1(x)\delta'(t-T) + w_0^2(x)\delta(t) - w_T^2(x)\delta(t-T), \ \ t \in \mathbb{R}.$$

Thus, from (1), (2) we will have

$$\mathcal{D}[w_1] = f_1(x,t) - W(x,t), \ \ (x,t) \in O \times \mathbb{R}, \tag{6}$$

$$\mathcal{B}[w_1] = u_{1b}(t), \ \ (x,t) \in \partial O \times \mathbb{R}, \tag{7}$$

where $W(x,t)$ contains the initial and the terminal data of the system derived in the mentioned manner. $w_1(x,t) = \mathcal{A}_{[0,T]}[w]$ will be called generalized state function. Obviously, the generalized state function is concentrated on $[0,T]$ where the control process is carried out and it coincides with the ordinary state function w therein.

Now, we may apply the distributional Fourier transform to (6) and (7). As a result we will obtain a Cauchy problem with respect to Fourier image of the generalized state function:

$$\overline{\mathcal{D}}[\overline{w}_1] = \overline{f}_1(x,\sigma) - \overline{W}(x,\sigma), \;\; (x,\sigma) \in O \times \mathbb{R}, \tag{8}$$

$$\overline{\mathcal{B}}[\overline{w}_1] = \overline{u}_{1b}(\sigma), \;\; (x,\sigma) \in \partial O \times \mathbb{R}, \tag{9}$$

and we will be able to proceed as is suggested in [1] and is done in [7-11].

According to Wiener–Paley–Schwartz theorem [1-4] $\overline{w}_1(x,\sigma+i\varsigma)$, $\varsigma \in \mathbb{R}$, i.e. the continuation of the distributional Fourier transform of the generalized state function in the whole complex plane, is an entire function. Suppose, we have derived the dependence $\overline{w}_1 = \overline{w}_1(x,\sigma,\overline{u}(\sigma))$, where u_1 denotes any of functions u_{1d} or u_{1b}. Extending that dependence to the whole complex plane and providing that $\overline{w}_1$ must not have any singularities there, we have to equate the numerator of all fractions existing in it to zero in the complex roots of its denominator. It will provide us a system of restrictions as follows:

$$\overline{u}_1(\sigma_k + i\varsigma_k) = \mathcal{M}_k, \;\; k \in \mathbb{K}, \tag{10}$$

where the constants $\mathcal{M}_k$ depend on parameters of the system (8), (9). It tuns out, that [1, 10, 7, 8, 9, 11] usually $\mathbb{K} = \mathbb{N}$. After separating the real and the imaginary parts, this system will be equivalent to

$$\int_{\mathbb{R}} u_1(t)\cos(\sigma_k t)\exp[-\varsigma_k t]\,dt = \mathcal{M}_{1k}, \;\; \int_{\mathbb{R}} u_1(t)\sin(\sigma_k t)\exp[-\varsigma_k t]\,dt = \mathcal{M}_{2k}, \;\; k \in \mathbb{K}. \tag{11}$$

Actually, both integrals are taken in $[0,T]$ where the unknown function is concentrated on.

Thus, as long as the conditions (10) or equivalently (11) are satisfied by a control function u_1 (boundary or distributed) the system (1)–(4) is exact controllable, which means that the set of admissible controls consists of measurable functions satisfying (10) or (11). Using the system (10) or (11) to construct the set of admissible for the problem (1)–(4) has its privileges. (10), for instance, may be used as interpolation conditions in nodes $\sigma_k + i\varsigma_k$ to interpolate the function $\overline{u}_1$. The real part of the function $\mathcal{F}_t^{-1}[\overline{u}_1]$ will form the set of admissible controls. Being a system of linear Fredholm integral equations of the first kind and therefore solved by efficient numerical methods [21], (11) may be treated as a problem of moments [1,7-11]. In [1] the general form of admissible L^p, $1 \leqslant p \leqslant \infty$, controls is derived for linear problem of moments, as well as necessary and sufficient conditions of their existence are obtained.

Among others, one of the privileges of this approach is that with explicit representation of admissible controls we simultaneously derive also existence conditions for them.

5. L^1, L^2 and L^∞-optimal solutions of (11)

In this section we bring L^1, L^2 and L^∞-optimal solutions of system (11) when $\mathbb{K}$ is a set of finite power, since it is proved in [1], that the case $\mathbb{K} = \mathbb{N}$ is resolvable if its all truncated finite parts are resolvable. The questions concerning convergence of solution of the truncated part to the solution of the infinite system are studied in traditional manners (see [1] and the references therein).

As sets of admissible controls we take:

$$\mathcal{U}_1 = \{u \in L^1[0,T];\ \operatorname{supp} u \subseteq [0,T]\},$$

$$\mathcal{U}_2 = \{u \in L^2[0,T];\ \operatorname{supp} u \subseteq [0,T]\},$$

$$\mathcal{U}_\infty = \{u \in L^\infty[0,T];\ \operatorname{supp} u \subseteq [0,T],\ |u| \leqslant u_0\}.$$

The weak derivative of L^1–optimal solution of (11) has the form [7, 8, 9]

$$\mathcal{D}u^o(t) = \sum_{j=1}^{m} u_j^o \delta(t - t_j^o),\quad t \in [0,T].$$

The generalized primitive $u^o(t)$ may be represented in several manners. For example,

$$u^o(t) = \sum_{j=1}^{m} u_j^o \theta(t - t_j^o),\quad t \in [0,T],\quad \text{or} \tag{12a}$$

$$u^o(t) = \frac{1}{2}\sum_{j=1}^{m} u_j^o \operatorname{sign}(t - t_j^o),\quad t \in [0,T]. \tag{12b}$$

The intensities u_j^o are constrained by

$$\operatorname{sign} u_j^o = \operatorname{sign} h^o\left(t_j^o\right),\quad j \in \{1;m\},$$

and are determined from system

$$\sum_{j=1}^{m} u_j^o \exp\left[-\varsigma_k t_j^o\right] \cos\left(\sigma_k t_j^o\right) = \mathcal{M}_{1k},\quad \sum_{j=1}^{m} u_j^o \exp\left[-\varsigma_k t_j^o\right] \sin\left(\sigma_k t_j^o\right) = \mathcal{M}_{2k},\quad k \in \mathbb{K},$$

obtained by substituting (12a) or (12b) in (11). The moments t_j^o are determined from equality

$$\kappa_\infty[h^o] = \left[\sum_{j=1}^{m} \left|u_j^o\right|\right]^{-1}.$$

The number m of control impacts must be determined from inclusion condition $\{t_j^o\}_{j=1}^m \subset (0,T)$. Unfortunately, it is non–unique [1, 7, 8, 9].

Here

$$h^o(t) = \sum_{k=1}^{n} \exp\left[-\varsigma_k t\right]\left[l^o_{1k}\cos(\sigma_k t) + l^o_{2k}\sin(\sigma_k t)\right], \quad t \in [0,T],$$

and the optimal coefficients l^o_{1k}, l^o_{2k} are determined from the following problem of conditional extrema:

$$h^o\left(t^o_j\right) \xrightarrow[l_{1k},l_{2k}]{} \min, \quad \text{when} \ \sum_{k=1}^{n}\left[l_{1k}\mathcal{M}_{1k} + l_{2k}\mathcal{M}_{2k}\right] = 1.$$

At this, the solution $u^o(t)$ of truncated system (11) exists if and only if $\kappa_\infty[h^o] \neq 0$. Then, for solvability of infinite system, according to [1] we have

Theorem 1 *L^1–optimal solution $u^o(t)$ (12a) or equivalent (12b) of the infinite system (11) exists if and only if*

$$\sum_{j=1}^{m}\left|u^o_j\right| \neq 0$$

for all $n \in \mathbb{N}$.

In other words, at least one of the intensities u^o_j must differ from zero for any natural n. The proof may be found in [1].

In the case of L^2–optimal solution of finite system (11) we have [8]

$$u^o(t) = \frac{1}{\kappa_2^2\left[h^o\right]}\sum_{k=1}^{n}\exp[-\varsigma_k t]\left[l^o_{1k}\cos(\sigma_k t) + l^o_{2k}\sin(\sigma_k t)\right], \ t \in [0,T], \tag{13}$$

where the coefficients l^o_{pk}, $p \in \{1;2\}$, are determined from system of linear algebraic equations

$$\mathbf{J}\mathbf{L}^o = \mathbf{M}, \tag{14}$$

where $\mathbf{L}^o = (l^o_{11} \ldots l^o_{1n}\ l^o_{21} \ldots l^o_{2n})^{\mathrm{T}}$, $\mathbf{M} = (\mathcal{M}_{11} \ldots \mathcal{M}_{1n}\ \mathcal{M}_{21} \ldots \mathcal{M}_{2n})^{\mathrm{T}}$, the upper index T denotes transposition,

$$\mathbf{J} = \begin{pmatrix} J_{11}^{+} & J_{12}^{+} & \dots & J_{1n}^{+} & J_{11} & J_{12} & \dots & J_{1n} \\ J_{21}^{+} & J_{22}^{+} & \dots & J_{2n}^{+} & J_{21} & J_{22} & \dots & J_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ J_{n1}^{+} & J_{n1}^{+} & \dots & J_{nn}^{+} & J_{n1} & J_{n2} & \dots & J_{nn} \\ J_{11} & J_{21} & \dots & J_{1n} & J_{11}^{-} & J_{12}^{-} & \dots & J_{1n}^{-} \\ J_{12} & J_{22} & \dots & J_{2n} & J_{21}^{-} & J_{22}^{-} & \dots & J_{2n}^{-} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ J_{1n} & J_{2n} & \dots & J_{nn} & J_{n1}^{-} & J_{n2}^{-} & \dots & J_{nn}^{-} \end{pmatrix},$$

$$J_{jk}^{\pm} = \int_0^T \begin{pmatrix} \cos(\sigma_j t)\cos(\sigma_k t) \\ \sin(\sigma_j t)\sin(\sigma_k t) \end{pmatrix} \exp\left[-(\varsigma_j + \varsigma_k)t\right] dt, \tag{15}$$

$$J_{jk} = \int_0^T \cos(\sigma_j t)\sin(\sigma_k t)\exp\left[-(\varsigma_j + \varsigma_k)t\right] dt.$$

Necessary and sufficient conditions for exact controllability in this case gives the following

Theorem 2 *L^2–optimal solution $u^o(t)$ (13) of infinite system (11) exists if and only if*

$$\begin{aligned} \kappa_2^2 = \sum_{k=1}^{n} (l_{1k}^o)^2 J_{kk}^{+} + 2\sum_{j=1}^{n-1} \sum_{k=j+1}^{n} l_{1j}^o \left(l_{1k}^o J_{jk}^{+} + l_{2k}^o J_{jk} \right) + \\ + \sum_{k=1}^{n} (l_{2k}^o)^2 J_{kk}^{-} + 2\sum_{j=1}^{n-1} \sum_{k=j+1}^{n} l_{2j}^o \left(l_{2k}^o J_{jk} + l_{2k}^o J_{jk}^{-} \right) \end{aligned} \tag{16}$$

is positive for all $n \in \mathbb{N}$.

Here the selfadjointness of $L^2[0,T]$ is taken into account.

If all the roots $z_k,\ k \in \mathbb{N}$, are real, i.e. $\varsigma_k = 0,\ k \in \mathbb{N}$, instead of formulas (13)–(16) the following should be used: optimal control (13) reads as

$$u^o(t) = \frac{1}{\kappa_2^2\left[h^o\right]} \sum_{k=1}^{n} \left[l_{1k}^o \cos(\sigma_k t) + l_{2k}^o \sin(\sigma_k t) \right],\ t \in [0,T],$$

system (14) is separated into two independent systems with respect to coefficients $l_{pk}^o,\ p \in \{1;2\}$, correspondingly:

$$\mathbf{J}^{\pm}\mathbf{L}_p^o = \mathbf{M}_p,\ p \in \{1;2\},$$

$$\mathbf{L}_p^o = (l_{p1}^o \dots l_{pn}^o)^T,\ \mathbf{M}_p = (\mathcal{M}_{p1} \dots \mathcal{M}_{pn})^T,\ \mathbf{J}^{\pm} = \{J_{jk}^{\pm}\}_{j,k=1}^n,$$

$$J_{jk}^{\pm} = \int_0^T \begin{pmatrix} \cos(\sigma_j t)\cos(\sigma_k t) \\ \sin(\sigma_j t)\sin(\sigma_k t) \end{pmatrix} dt,$$

and

$$\kappa_2^2 = \sum_{k=1}^n (l_{1k}^o)^2 J_{kk}^+ + 2\sum_{j=1}^{n-1} l_{1j}^o \sum_{k=j+1}^n l_{1k}^o J_{jk}^+ \sum_{k=1}^n (l_{2k}^o)^2 J_{kk}^- + 2\sum_{j=1}^{n-1} l_{2j}^o \sum_{k=j+1}^n l_{2k}^o J_{jk}^-.$$

L^∞-optimal solution of (11) is

$$u^o(t) = u_0 \cdot \operatorname{sign} h^o(t),\ \ t \in [0,T]. \tag{17}$$

The solvability of (11) in this case is provided by the following theorem.

Theorem 3 *L^∞–optimal solution $u^o(t)$ (17) of infinite system (11) exists if and only if*

$$\kappa_1[h^o] = \frac{1}{u_0} \leqslant \sum_{k=1}^n \sqrt{\left(l_{1k}^o\right)^2 + \left(l_{2k}^o\right)^2}\,\frac{1-\exp[-\varsigma_k T]}{\varsigma_k},$$

for all $n \in \mathbb{N}$.

The case when all z_k, $k \in \mathbb{N}$, are real ($\varsigma_k \to 0$) is obtained using l‘Hospital‘s rule. The theorems 1, 2 and 3, essentially, provide the constraints on initial, terminal data, boundary functions, right-hand side and other parameters of the state equation and control time T in purposes of exact controllability of the system.

6. Particular Problems

Now we will demonstrate the proposed technique to derive a required system of controllability conditions.

Suppose, that in the rectangle $[0,1]\times[0,T]$

$$\mathcal{D}[w] = \frac{\partial}{\partial x}\left[N(x)\frac{\partial w(x,t)}{\partial x}\right] - \rho(x)\frac{\partial^2 w(x,t)}{\partial t^2}, \tag{18}$$

$$\alpha_0 w(0,t) + \beta_0 \left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0} = u_b^0(t),\ \ \alpha_1 w(1,t) + \beta_1 \left.\frac{\partial w(x,t)}{\partial x}\right|_{x=1} = u_b^1(t),\ \ t \in [0,T],$$

$$w(x,0) = w_0(x),\ \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0} = w_0^1(x),\ \ w(x,T) = w_T(x),\ \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=T} = w_T^1(x),\ \ x \in [0,1].$$

This system describes the forced vibrations of an elastic non–homogeneous semi–infinite string with density $\rho(x)$ and tensile force $N(x)$. It describes other real–life phenomena as well. In purposes of well-posedness of weak solution in $W^{1,2}([0,1]\times[0,T])$ [2] we suppose that $0 < N \in C^1[0,1]$ and $0 < \rho \in C[0,1]$, u_b^0, $u_b^1 \in L^2[0,T]$, $w_0 \in W^{1,2}[0,1]$, $w_0^1 \in W^{1,1}[0,1]$. Moreover, the physical considerations lead to additional restriction $N(x) \sim \rho^{-1}(x)$, even in limiting cases $\rho(x) \to 0+$ and $\rho(x) \to +\infty$.

Then

$$W(x,t) = \rho(x)\left[w_0(x)\delta'(t) + w_0^1(x)\delta(t) - w_T(x)\delta'(t-T) - w_T^1(x)\delta(t-T)\right].$$

The general solution of the corresponding boundary value problem (8), (9) in this case reads as

$$\overline{w}(x,\sigma) = h_1(\sigma)\exp[i\lambda(x,\sigma)] + h_2(\sigma)\exp[-i\lambda(x,\sigma)] + \Lambda(x,\sigma), \ (x,\sigma)\in[0,1]\times[0,T], \quad (19)$$

where

$$\Lambda(x,\sigma) = \int_0^x \mathcal{K}(x,\xi,\sigma)\left[\overline{f}_1(\xi,\sigma) - \overline{W}(\xi,\sigma)\right]d\xi, \quad \mathcal{K}(x,\xi,\sigma) = \frac{\sin[\lambda(x,\sigma)-\lambda(\xi,\sigma)]}{\lambda'(\xi,\sigma)},$$

$$h_p(\sigma) = \frac{\Delta_p(\sigma)}{\Delta(\sigma)}, \quad p \in \{1;2\},$$

$$\Delta(\sigma) = \begin{vmatrix} [\alpha_0 + i\beta_0\lambda'(0,\sigma)]\exp[i\lambda(0,\sigma)] & [\alpha_0 - i\beta_0\lambda'(0,\sigma)]\exp[-i\lambda(0,\sigma)] \\ [\alpha_1 + i\beta_1\lambda'(1,\sigma)]\exp[i\lambda(1,\sigma)] & [\alpha_1 - i\beta_1\lambda'(1,\sigma)]\exp[-i\lambda(1,\sigma)] \end{vmatrix},$$

$$\Delta_1(\sigma) = \begin{vmatrix} \overline{u}_{b1}^0(\sigma) & [\alpha_0 - i\beta_0\lambda'(0,\sigma)]\exp[-i\lambda(0,\sigma)] \\ \overline{u}_{b1}^{11}(\sigma) & [\alpha_1 - i\beta_1\lambda'(1,\sigma)]\exp[-i\lambda(1,\sigma)] \end{vmatrix},$$

$$\Delta_2(\sigma) = \begin{vmatrix} [\alpha_0 + i\beta_0\lambda'(0,\sigma)]\exp[i\lambda(0,\sigma)] & \overline{u}_{b1}^0(\sigma) \\ [\alpha_1 + i\beta_1\lambda'(1,\sigma)]\exp[i\lambda(1,\sigma)] & \overline{u}_{b1}^{11}(\sigma) \end{vmatrix},$$

and

$$\overline{u}_{b1}^{11}(\sigma) = \overline{u}_{b1}^1(\sigma) - \alpha_1\Lambda(1,\sigma) - \beta_1\Lambda'(1,\sigma).$$

Above $\lambda = \lambda(x,\sigma)$ is determined from Riccati differential equation

$$N(x)\frac{\partial\nu}{\partial x} + \nu^2 + \sigma^2\gamma^2 = 0, \ \gamma^2 = N(x)\rho(x), \ (x,\sigma) \in [0,1]\times\mathbb{R}, \quad (20)$$

by the relation

$$i\lambda(x,\sigma) = \int_0^x \frac{\nu(\xi,\sigma)}{N(\xi)}d\xi. \quad (21)$$

In [23] exact solution of Riccati equation (20) for various $N(x)$ are given.

According to proposed technique we have to expand (19) in the whole $\mathbb{C}$ and ensure it to be entire there. It is easy to prove, that for locally measurable f, $\Lambda(x,\sigma+i\varsigma)$ is entire, therefore $\overline{w}(x,\sigma+i\varsigma)$, $p\in\{1;2\}$, is entire if and only if $h_p(\sigma+i\varsigma)$ are simultaneously entire. Expanding the main and the auxiliary determinants brought above in the whole $\mathbb{C}$ and equating to zero the numerators of those fractions in complex roots of the denominators, we will obtain the necessary system of restrictions like (10) or (11) for unknown controls.

Particularly, if we seek a boundary control $u_{b1}^0\in\mathcal{U}_2$ the corresponding constants $\mathcal{M}_k$ will have the form

$$\mathcal{M}_k=\overline{u}_{b1}^{11}(z_k)\frac{\alpha_0-i\beta_0\lambda'(0,z_k)}{\alpha_1-i\beta_1\lambda'(1,z_k)}\exp\left[i(\lambda(1,z_k)-\lambda(0,z_k))\right].$$

Seeking a boundary control $u_{b1}^1\in\mathcal{U}_2$, we will arrive at

$$\mathcal{M}_k=\overline{u}_{b1}^{0}(z_k)\frac{\alpha_1-i\beta_1\lambda'(1,z_k)}{\alpha_0-i\beta_0\lambda'(0,z_k)}\exp\left[-i(\lambda(1,z_k)-\lambda(0,z_k))\right]+ \\ +\alpha_1\Lambda(1,z_k)+\beta_1\Lambda'(1,z_k).$$

Finally, seeking a distributed control $u_{d1}\in\mathcal{U}_1$, we will obtain

$$\mathcal{M}_k=\left[\overline{u}_{b1}^1(z_k)+\alpha_1\mathcal{R}[\overline{g}]\Big|_{x=1}+\beta_1\frac{\partial\mathcal{R}[\overline{g}]}{\partial x}\bigg|_{x=1}-\right. \\ \left.-\overline{u}_{b1}^0(z_k)\frac{\alpha_1-i\beta_1\lambda'(1,z_k)}{\alpha_0-i\beta_0\lambda'(0,z_k)}\exp\left[-i(\lambda(1,z_k)-\lambda(0,z_k))\right]\right]\left[\alpha_1\mathcal{R}[v]\Big|_{x=1}+\beta_1\frac{\partial\mathcal{R}[v]}{\partial x}\bigg|_{x=1}\right]^{-1},$$

$$\mathcal{R}[\eta]=\int_0^x\mathcal{K}(x,\xi,z_k)\eta(\xi)d\xi.$$

The points z_k are the roots of expansion of the main determinant $\Delta(\sigma)$ in the whole complex plane.

If one needs the controlled motion of the string, he has to apply Fourier inverse distributional transform to (19):

$$w(x,t)=\mathcal{F}_t^{-1}[\overline{w}]= \\ =\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[h_1(\sigma)\exp\left[i\lambda(x,\sigma)\right]+h_2(\sigma)\exp\left[-i\lambda(x,\sigma)\right]+\Lambda(x,\sigma)\right]\exp\left[-i\sigma t\right]d\sigma,$$

$$(x,t)\in[0,1]\times[0,T].$$

It is easy to check that

$$\mathrm{Re}\,\overline{w}(x,-\sigma)=\mathrm{Re}\,\overline{w}(x,\sigma),\quad \mathrm{Im}\,\overline{w}(x,-\sigma)=-\mathrm{Im}\,\overline{w}(x,\sigma),$$

therefore its Fourier inverse transform is a real valued function.

$$w(x,t) = \frac{1}{2\pi}\int_0^\infty [\mathrm{Re}\,\overline{w}(x,\sigma)\cos(\sigma t) + \mathrm{Im}\,\overline{w}(x,\sigma)\sin(\sigma t)]\,d\sigma,$$

$$(x,t) \in [0,1]\times[0,T].$$

As the extension of the integrand in the whole complex plane has no singularities, the integral is well defined.

For example, when $N(x) = e^{ax}$, $\gamma^2 = 1$, equation (20) has exact solution in terms of Bessel's functions [23]. Then

$$\Delta(\sigma) = \frac{\sin^{2\alpha}[|\sigma|\exp[-a]] - \sin^{2\alpha}|\sigma|}{\sin^{\alpha}[|\sigma|\exp[-a]]\sin^{\alpha}|\sigma|},\;\; \alpha = \frac{1}{a^3},\;\; \Delta_1(\sigma) = \Lambda(1,\sigma).$$

Repeating the procedure described above we will obtain

$$\sigma_k = \frac{2\pi k}{1+\exp[-a]},\;\; k \in \mathbb{N},$$

i.e. all the roots of equation $\Delta(z) = 0$ are real ($\varsigma_k = 0$), which leads to trigonometric problem of moments. In limiting case, when $a \to 0$, we have $\sigma_k = \pi k$, $k \in \mathbb{N}$, which matches with results of [1].

One of the privileges of the technique is that it allows to find controllability conditions also in problems on non-compact domains. Suppose, that the equation (18) holds in strip $\mathbb{R}^+ \times [0,T]$, at this the behavior of the string at infinity is given: $w(x \to +\infty, t) = w_\infty(t)$. A Dirichlet boundary control $u \in \mathcal{U}_1$ is required to be found ensuring given terminal data. Then

$$\mathcal{M}_k = \mathcal{M}_{1k} + i\mathcal{M}_{2k} = \overline{w}_\infty(z_k) + (-1)^k\Lambda(0,z_k),$$

where z_k are the complex roots of equation $\lambda(0,z_k) = \pi k$: $\lambda(x,\sigma)$ is still defined by (21).

For the non-homogeneity considered above

$$i\lambda(x,\sigma) = \alpha\ln|\cos[|\sigma|\exp[-ax]]|,\;\; (x,\sigma) \in \mathbb{R}^+ \times \mathbb{R},$$

and therefore

$$z_k = \sigma_k = \frac{\pi k}{\alpha},\;\; k \in \mathbb{N}.$$

The Butkovskiy's generalized method provides a powerful tool to deal with control problems with constant delays. Suppose, that the control is carried out by distributed or boundary controls with constant delay τ. We will deal with (18) when $f(x,t) \equiv 0$ and Dirichlet boundary conditions are given

$$w(0,t) \equiv 0,\; w(1,t) = u_b(t-\tau),\; t \in [0,T],$$

and when

$$f(x,t) = u_d(t-\tau)v(x),$$

and mixed boundary conditions are given:

$$w(0,t) = 0, \quad \left.\frac{\partial w(x,t)}{\partial x}\right|_{x=1} = w_{b1}(t), \quad t \in [0,T].$$

Then, repeating the procedure described above we will respectively obtain

$$\Delta(\sigma) = -2i\sin[\lambda(1,\sigma)-\lambda(0,\sigma)], \quad \Delta_1(\sigma) = [\Lambda(1,\sigma) - \exp[i\sigma\tau]\,\overline{u}_{b1}(\sigma)]\exp[-i\lambda(0,\sigma)],$$

in the first case, and

$$\Delta(\sigma) = -2\lambda'(1,\sigma)\cos[\lambda(1,\sigma)-\lambda(0,\sigma)], \quad \Delta_1(\sigma) = [\Lambda'(1,\sigma) - \overline{u}_{b1}(\sigma)]\exp[-i\lambda(0,\sigma)],$$

in the second case. Correspondingly

$$\mathcal{M}_k = \Lambda(1,z_k)\exp[-iz_k\tau], \; k \in \mathbb{N},$$

as long as

$$\lambda(1,z_k) - \lambda(0,z_k) = \pi k, \quad k \in \mathbb{N},$$

and

$$\mathcal{M}_k = \left[\overline{w}_{b1}(z_k) + \left.\frac{\partial \mathcal{R}\left[\overline{W}\right]}{\partial x}\right|_{x=1}\right] \frac{\exp[-iz_k\tau]}{\left.\frac{\partial \mathcal{R}[v]}{\partial x}\right|_{x=1}}, \; k \in \mathbb{N},$$

as long as

$$\lambda(1,z_k) - \lambda(0,z_k) = \frac{2k+1}{2}\pi, \quad k \in \mathbb{N}.$$

Conclusion

We propose a generalization of Butkovskiy's method of control with compact support for systems which have not classical solutions or which state function is not compactly supported in considered time interval. The main idea is the introduction of a new compactly supported state function which coincides with the original state function in the considered time interval. This leads to transformation of governing system and inclusion of given initial and required terminal data into it. The Fourier distributional transform and the Wiener–Paley–Schwartz theorem give a unified approach for obtaining the system exact controllability necessary and sufficient conditions. The solution providing the exact controllability may be constructed either by solving an interpolation problem for the Fourier image of the unknown and applying the Fourier inverse transform or by solving an infinite dimensional linear problem of moments. It turns out, that the solution is far non-unique, hence optimality conditions are posed. The L^1, L^2 and L^∞-optimal solutions of the problem of moments are constructed.

The method is demonstrated for solving boundary and distributed control problems for one-dimensional semi-linear wave equation in finite and semi-infinite domains. The case when the controls contain constant delay is also considered. In all cases the resolving system of necessary and sufficient conditions are obtained. The procedure requires solution of a special Riccati equation, which is possible in numerous special cases.

The method may be extended for systems with special types of nonlinearities.

References

[1] A.G. BUTKOVSKIY: Methods of Control for Systems with Distributed Parameters. Nauka Publisher, Moscow, 1975, (in Russian).

[2] V.S. VLADIMIROV: Methods of the Theory of Generalized Functions. CRC Press, London–New York, 2002.

[3] A.H. ZEMANIAN: Distribution Theory and Transform analysis: An Introduction to Generalized Functions, with Applications. Dover Books on Mathematics. Dover Publications, 2010, (p. 400).

[4] P. TEODORESCU, W. KECS and A. TOMA: Distribution Theory: with Applications in Engineering and Physics. Wiley–VCH, New York, 2013.

[5] E.KH. GRIGORYAN: A solution of the problem about a finite elastic inclusion terminating to the boundary of a semi-plane. *Proc. of the Yerevan State University*, Natural Sciences, **3** (1981), 32-43.

[6] AS.ZH. KHURSHUDYAN: Generalized control with compact support of wave equation with variable coefficients. *Int. J. of Dynamics and Control*, (2015), DOI 10.1007/s40435-015-0148-3.

[7] AS.ZH. KHURSHUDYAN: On optimal boundary control of non–homogeneous string vibrations under impulsive concentrated perturbations with delay in controls. *Mathematical Bulletin of T. Shevchenko Scientific Society*, **10** (2013), 203-209.

[8] AS.ZH. KHURSHUDYAN and SH.KH. ARAKELYAN: Delaying control of non–homogeneous string forced vibrations under mixed boundary conditions. *International Siberian Conference on Control and Communication*, Krasnoyarsk, Russia, (2013), 1-5.

[9] AS.ZH. KHURSHUDYAN: On optimal boundary and distributed control of partial integro–differential equations.*Archives of Control Sciences*, **24**(1), (2014), 5-25.

[10] AS.ZH. KHURSHUDYAN: Bubnov–Galerkin procedure in control problems for bilinear systems. *Automation and Remote Control*, to appear in 2015.

[11] AM.ZH. KHURSHUDYAN and AS.ZH. KHURSHUDYAN: Optimal distribution of viscoelastic dampers under elastic finite beam subjected to moving load. *Proc. of NAS of Armenia*, **67** (2014), 56-67, (in Russian).

[12] L.V. FARDIGOLA: On controllability problems for the wave equation on a half–plane. *J. of Mathematical Physics, Analysis and Geometry*, **1**(1), (2005), 93-115.

[13] J.L. LIONS: Exact controllability, stabilization and perturbations for distributed systems. *SIAM Reviews*, **30**(1), (1988), 1-68.

[14] J. KLAMKA: Controllability of second order infinite-dimensional systems. *IMA J. of Mathematical Control and Information*, **13**(1), (1998), 79-88.

[15] J. KLAMKA: Constrained exact controllability of semilinear systems. *Systems and Control Letters*, **4**(2), (2002), 139-147.

[16] A.V. BOROVSKIKH: Boundary control formulas for inhomogeneous string. I. *Differential Equations*, **45**(1), (2007), 69-95.

[17] A.V. BOROVSKIKH: Boundary control formulas for inhomogeneous string. II. *Differential Equations*, **45**(5), (2007), 656-666.

[18] J. KLAMKA and J. WYRWAL: Controllability of second order infinite-dimensional systems. *Systems and Control Letters*, **57**(5), (2008), 386-391.

[19] K.S. KHALINA: On the Neumann boundary controllability for the non–homogeneous string on a segment. *J. of Mathematical Physics, Analysis and Geometry*, **7**(4), (2011), 333-351.

[20] K.S. KHALINA: Boundary controllability problems for the equation of oscillation of an inhomogeneous string on a semiaxis. *Ukrainian Mathematical J.*, **64**(4), (2012), 594-615.

[21] K.E. ATKINSON: A survey of numerical methods for solving nonlinear integral equations. *J. of Integral Equations and Applications*, **4** (1992), 15-46.

[22] J.T. BETTS: Practical Methods of Optimal Control and Estimation using Nonlinear Programming. 2nd ed. Philadelphia: SIAM, 2010.

[23] V.F. ZAITSEV and A.D. POLYANIN: Handbook of Exact Solutions for Ordinary Differential Equations. 2nd ed. CRC Press, Boca Radon, Florida, 2003, (p. 816).