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New reachability and observability tests for positive linear discrete-time systems

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Abstract. New tests (criterions) for checking the reachability and the observability of positive linear-discrete-time systems are proposed. The tests do not need checking of rank conditions of the reachability and observability matrices of the systems. Simple sufficient conditions for the unreachability and unobservability of the systems are also established.

Key words: discrete-time, linear, positive, system, test, reachability, observability.

1. Introduction

In positive systems inputs, state variables and outputs take only nonnegative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchanges and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behaviors can be found in engineering, managements science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in monographs [1-3]. The basic tests for checking the reachability and observability of positive linear systems are Kalman type [1-3] and have the forms of the rank conditions of the reachability and observability matrices. For standard linear systems besides the well-known Kalman tests [4,5] are also very popular the Hautus tests [2, 4, 6] (of the form for reachability of systems $\dot{x} = \mathbf{A}x + \mathbf{B}u, x \in \mathbb{R}^n$, $\text{rank}[\mathbf{I}s - \mathbf{A}, \mathbf{B}] = n \vee s \in \mathbb{C}$ (the field of complex numbers)). Corresponding tests for positive linear systems, to the best knowledge of the author, do not exist.

In this short paper new tests for checking the reachability and observability of positive linear discrete-time systems will be proposed. These tests can be partly considered as some kind Hautus type tests for positive linear discrete-time systems.

Simple sufficient conditions for the unreachability and unobservability of the positive systems will be also established.

2. Reachability of positive discrete-time systems

Let $\mathbb{R}^{n\times m}$ be the set of $n \times m$ real matrices and $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. The set $n \times m$ real matrices with nonnegative entries will be denoted by $\mathbb{R}_+^{n\times m}$ and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$. The set of nonnegative integers will be denoted by $\mathbb{Z}_+$.

Consider the linear discrete-time systems

$$x_{i+1} = \mathbf{A}x_i + \mathbf{B}u_i,\ i \in \mathbb{Z}_+ \qquad (1)$$
$$y_i = \mathbf{C}x_i.$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are the state, input and output vectors and $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$.

DEFINITION 1. [3] The system (1) is called (internally) positive if $x_i \in \mathbb{R}_+^n$ and $y_i \in \mathbb{R}_+^p$, $i \in \mathbb{Z}_+$ for any $x_0 \in \mathbb{R}_+^n$ and all input sequences $u_i \in \mathbb{R}_+^m$, $i \in \mathbb{Z}_+$.

THEOREM 1. [3] The system (1) is positive if and only if

$$\mathbf{A} \in \mathbb{R}_+^{n\times n},\ \mathbf{B} \in \mathbb{R}_+^{n\times m},\ \mathbf{C} \in \mathbb{R}_+^{p\times n}. \qquad (2)$$

DEFINITION 2. [3] The positive system (1) is called reachable if for any $x_f \in \mathbb{R}_+^n$ there exists $q \in \mathbb{Z}_+$ $(q > 0)$ and an input sequence $u_i \in \mathbb{R}_+^m$, $i = 0, 1, ...q - 1$ that steers the state of system from $x_0 = 0$ to the final state x_f, i.e., $x_q = x_f$.

Let e_i be the ith column of the identity matrix $\mathbf{I}_n$ and $c > 0$.

A column $ce_i \in \mathbb{R}_+^n$ is called monomial (it has only one positive entry and its remaining entries are zero).

THEOREM 2. [3] The positive system (1) is reachable if and only if the reachability matrix $\mathbf{R}_n$ of the system

$$\mathbf{R}_n = \left[\mathbf{B}\ \mathbf{AB} \ldots \mathbf{A}^{n-1}\mathbf{B}\right] \qquad (3)$$

contains n linear independent monomial columns.

3. Main results

In this Section new tests (criterions) for checking the unreachability and reachability and unobservability and observability of the positive linear discrete-time systems will be proposed.

Let a_i, $i = 1, ..., n$ $(b_j,\ j = 1, ..., m)$ be the ith $(j$th$)$ column of the matrix $\mathbf{A}$ $(\mathbf{B})$. Let the jth column b_j be

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monomial $b_j = e_j b$. The jth column $a_j = \mathbf{A}b_j$ of the matrix $\mathbf{A}$ is called monomial column corresponding to the jth column of $\mathbf{B}$ if and only if it is monomial and linearly independent of the monomial column b_j. Let a_j be ith monomial column $a_j = e_i a$, $i \neq j$, $i = 1, ..., n$. The ith column $a_i = A(Ab_j)$ is called monomial column corresponding to the ith column of A if only it is monomial and linearly independent of the monomial columns b_j and a_j. In this way we may find a sequence of monomial linearly independent column for each monomial column of the matrix $\mathbf{B}$. We stop the procedure if the last column found by the procedure is not monomial and linearly independent from the previous monomial columns. This procedure will be called shortly the procedure of finding linearly independent monomial columns of the matrix $[\mathbf{A}, \mathbf{B}]$.

THEOREM 3. The positive system (1) is unreachable if the matrix $\mathbf{B}$ has no monomial columns or the matrix $[\mathbf{A}, \mathbf{B}]$ has less than n linearly independent monomial columns.

Proof. Note that if the matrix $\mathbf{B}$ has no monomial columns then the matrix (3) does not contain n monomial columns for any matrix $\mathbf{A}$. If the matrix has at least one monomial columns but the matrix $[\mathbf{A}, \mathbf{B}]$ has less than n linearly independent monomial columns then from the procedure of finding linearly independent monomial columns it follows that the matrix (3) has less than n linearly independent monomial columns.

Example 1. The positive system (1) with

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & a_0 \\ 1 & 0 & a_1 \\ 0 & 0 & a_2 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}, \ (a_i \geqslant 0, \ i = 0, 1, 2) \quad (4)$$

is unreachable for any values of the coefficients a_i, $i = 0, 1, 2$, since the matrix

$$[\mathbf{A} \quad \mathbf{B}] = \begin{bmatrix} 0 & 0 & a_0 & 0 & 1 \\ 1 & 0 & a_1 & 1 & 0 \\ 0 & 0 & a_2 & 0 & 2 \end{bmatrix} \quad (5)$$

has less then three linearly independent monomial columns for any values of a_i, $i = 0, 1, 2$.

THEOREM 4. The positive system (1) is reachable if and only if using the procedure to the matrix $[\mathbf{A}, \mathbf{B}]$ it is possible to find n linearly independent monomial columns.

Proof. Taking into account that $\mathbf{A}^k\mathbf{B} = \mathbf{A}\left(\mathbf{A}^{k-1}\mathbf{B}\right)$ for $k = 1, 2, ..., n-1$ and the procedure, it is easy to show that the matrix (3) has n linearly independent monomial columns if and only if is possible to find n linearly independent monomial columns using the procedure to the matrix $[\mathbf{A}, \mathbf{B}]$.

Example 2. The positive system (1) with

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & a_0 \\ 1 & 0 & a_1 \\ 0 & 1 & a_2 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ (a_i \geqslant 0, \ i = 0, 1, 2) \quad (6)$$

is reachable for any values of the coefficients a_i, $i = 0, 1, 2$, since the matrix

$$[\mathbf{A} \quad \mathbf{B}] = \begin{bmatrix} 0 & 0 & a_0 & 1 \\ 1 & 0 & a_1 & 0 \\ 0 & 1 & a_2 & 0 \end{bmatrix}$$

has three linearly independent monomial columns for any values of $a_i \geqslant 0$, $i = 0, 1, 2$.

Example 3. It will be shown that the positive system (1) with

$$\mathbf{A} = \begin{bmatrix} a_0 & 0 & 0 \\ a_1 & 1 & 0 \\ a_2 & 0 & 1 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \ (a_i \geqslant 0, \ i = 0, 1, 2) \quad (7)$$

is reachable if and only if $a_0 = a_1 = 0$ and $a_2 > 0$.

Note that the necessary condition for reachability of the positive system (1) with (7) is satisfied for any a_i, $i = 0, 1, 2$ since the matrix

$$[\mathbf{A} \quad \mathbf{B}] = \begin{bmatrix} a_0 & 0 & 0 & 1 & 0 \\ a_1 & 1 & 0 & 0 & 1 \\ a_2 & 0 & 1 & 0 & 0 \end{bmatrix} \quad (8)$$

has three linearly independent monomial columns for all values of $a_i \geqslant 0$, $i = 0, 1, 2$.

By Theorem 4 the matrix (8) has three linearly independent monomial columns if and only if to the first monomial column of the matrix $\mathbf{B}$ corresponds linearly independent monomial column and this happens for $a_0 = a_1 = 0$ and $a_2 > 0$. The same result we obtain using Theorem 2.

The observability is a dual notion to the reachability.

DEFINITION 3. [1,3] The positive system (1) is called observable if there exists $q \in \mathbb{Z}_+$, $q > 0$ such that knowing the input sequence $u_i \in \mathbb{R}_+^m$, $i = 0, 1, ..., q-1$ and the corresponding output sequence $y_i \in \mathbb{R}_+^p$, $i = 0, 1, ..., q-1$ it is possible to find the unknown initial state $x_0 \in \mathbb{R}_+^n$ of the system.

THEOREM 5. The positive system (1) is observable if and only if the observability matrix $\mathbf{O}_n$ of the system

$$\mathbf{O}_n = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \quad (9)$$

contains n linearly independent monomial rows.

The proof of the Theorem 5 is dual to the proof of Theorem 2 [1,3].

The above procedure of finding linearly independent monomial columns of the matrix $[\mathbf{A}, \mathbf{B}]$ can be also applied for linearly independent monomial rows of the matrix $\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}$ by interchanging columns by rows.

THEOREM 6. The positive system (1) is unobservable if the matrix $\mathbf{C}$ has no monomial rows or the matrix $\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}$ has less than n linearly independent monomial rows.

Proof of this theorem is dual to the proof of Theorem 3.

THEOREM 7. The positive system (1) is observable if and only if using the (dual) procedure to the matrix $\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}$ it is possible to find n linearly independent monomial rows.

Proof of this theorem is dual to the proof of Theorem 4.

4. Concluding remarks

New tests (craterious) for checking the reachability and observability of positive linear discrete-time linear systems have been proposed. The tests can be considered as some kind Hautus-type tests for positive linear systems. The tests do not need checking of rank conditions of the reachability and observability matrices of the systems. Simple sufficient conditions for unreachability and unobservability of the positive linear discrete-time systems have been also established. Effectiveness of the tests has been demonstrated on numerical examples. Extension of these tests for positive linear continuous-time systems is an open problem.

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